Functional Analysis, BSM, Spring 2012

Exercise sheet: special operators

Solutions

1. a)
$$(TT^*x, x) = (T^*x, T^*x) = \|T^*x\|^2 \geq 0 \text{ and } (T^*Tx, x) = (Tx, Tx) = \|Tx\|^2 \geq 0.$$
b) If T is self-adjoint, then $T^2 = TT^*$ is positive by part a).

2. a) Right shift operator.

b) Left shift operator.

c) $T : (\alpha_1, \alpha_2, \alpha_3, \ldots) \mapsto (\alpha_1, 0, 0, \ldots)$.

3. (i) $\Rightarrow$ (ii): using the polarisation formula we get
$$(Tx, Ty) = \frac{1}{4}\sum_{k=0}^{3} i^k \|Tx + i^k Ty\|^2 = \frac{1}{4}\sum_{k=0}^{3} i^k \|T(x + i^k y)\|^2 = \frac{1}{4}\sum_{k=0}^{3} i^k \|x + i^k y\|^2 = (x, y).$$
(ii) $\Rightarrow$ (iii): for any fixed $y \in H$ we have
$$(x, y) = (Tx, Ty) = (x, T^*Ty), \text{ thus } (x, (T^*T - I)y) = 0 \text{ for all } x \in H.$$
Therefore $(T^*T - I)y = 0$. This holds for every $y \in H$, so $T^*T = I$.

(iii) $\Rightarrow$ (i):
$$\|Tx\|^2 = (Tx, Tx) = (x, T^*Tx) = (x, Ix) = (x, x) = \|x\|^2.$$

4. If T is an isometry, then $T^*T = I$ by the previous exercise. It readily follows that T^* must be surjective. It remains to prove that T^* is a partial isometry, that is,
$$\|T^*x\| = \|x\| \text{ for any } x \in (\ker T^*)^\perp.$$
We have seen that $(\ker T^*)^\perp = \mathrm{cl}(\mathrm{ran}\, T)$. Since T is an isometry, T is bounded below, so $\mathrm{ran}\, T$ is closed. Therefore $(\ker T^*)^\perp = \mathrm{ran}\, T$. It means that for any $x \in (\ker T^*)^\perp$ there exists $y \in H$ such that $x = Ty$. Then
$$\|T^*x\| = \|T^*Ty\| = \|Iy\| = \|y\| = \|Ty\| = \|x\|.$$

5. T is unitary $\Leftrightarrow T^*T = TT^* = I \Leftrightarrow T^*T = I$ and T is surjective $\Leftrightarrow T$ is an isometry and T is surjective. (We used Exercise 3 at the last step.)

6. First we notice that $(T^2)^* = (TT)^* = T^*T^*$ and $(T^*T)^* = T^*T$. We use the equality $\|S^*S\| = \|S\|^2$ for
$$S = T^2 : \|T^*T^*TT\| = \|T^2\|^2$$
and
$$S = T^*T : \|T^*TT^*T\| = \|T^*T\|^2 = \|T\|^4.$$
For a normal operator T the left-hand sides of the equations coincide, therefore the right-hand sides are the same, too, thus $\|T^2\| = \|T\|^2$ as claimed.

Second solution. If T is normal, then
$$\|T^2x\|^2 = (TTx, TTx) = (Tx, T^*TTx) = (Tx, TT^*Tx) = (T^*Tx, T^*Tx) = \|T^*Tx\|^2$$
for all $x \in H$. Therefore $\|T^2\| = \|T^*T\| = \|T\|^2$.

b) Clearly, if T is normal, then so is T^2. So we can use part a) for the operators $T, T^2, \ldots, T^{2^{k-1}}$:
$$\|T^{2^k}\| = \|T^{2^{k-1}}\|^2 = \|T^{2^{k-2}}\|^4 = \ldots = \|T\|^{2^k}.$$
c) We know that
$$r(T) = \lim_{n\to\infty} \sqrt[n]{\|T^n\|}.$$
However, for $n = 2^k$ we have $\sqrt[n]{\|T^n\|} = \|T\|$ by part b). Therefore $r(T) = \|T\|$.

d) We also know that
$$r(T) = \inf_n \sqrt[n]{\|T^n\|}.$$
Since $r(T) = \|T\|$, it follows that $\|T^n\| \geq r(T)^n = \|T\|^n$. The converse inequality $\|T^n\| \leq \|T\|^n$ holds for any operator T.

7.* We saw earlier that the "if" part is true even in Banach spaces.

For the "only if" part, suppose that T is compact. Then the image $T(B_1(0))$ of the unit ball is totally bounded, that is, it has a finite ε-lattice for any $\varepsilon > 0$: $y_1^\varepsilon, \dots, y_{m_\varepsilon}^\varepsilon \in H$. By Y_ε we denote the linear subspace spanned by $y_1^\varepsilon, \dots, y_{m_\varepsilon}^\varepsilon$; Y_ε is finite dimensional, so it is closed. Let P_ε denote the orthogonal projcetion onto Y_ε and let $T_\varepsilon = P_\varepsilon T$. Since P_ε is of finite rank, so is T_ε. So it suffices to prove that

$$\|T - T_\varepsilon\| \le 2\varepsilon.$$

Let $x \in H$ be arbitrary with $\|x\| < 1$. Since $Tx \in T(B_1(0))$, there exists $1 \le k \le m_\varepsilon$ such that $\|Tx - y_k\| \le \varepsilon$. Since $\|P_\varepsilon\| = 1$, it follows that $\|P_\varepsilon Tx - P_\varepsilon y_k\| \le \varepsilon$. However, $P_\varepsilon y_k = y_k$, so

$$\|(T - T_\varepsilon)x\| = \|Tx - P_\varepsilon Tx\| \le \|Tx - y_k\| + \|P_\varepsilon Tx - y_k\| \le \varepsilon + \varepsilon = 2\varepsilon.$$

Since this holds for any $x \in H$ with $\|x\| < 1$, we get $\|T - T_\varepsilon\| \le 2\varepsilon$ as claimed.

8. Suppose that T is compact. The previous exercise yields that there exists finite rank operators T_n such that $\|T - T_n\| \to 0$. Then

$$\|T^* - T_n^*\| = \|(T - T_n)^*\| = \|T - T_n\| \to 0.$$

We use the fact that the adjoint of a finite rank operator also has finite rank. So T^* is the limit of finite rank operators. As such, it is compact (again by the previous exercise).

It remains to verify that if T has finite rank, then so does T^*. Using $\mathrm{cl}(\operatorname{ran} T^*) = (\ker T)^\perp$ and $H = \ker T \oplus (\ker T)^\perp$ we get

$$\mathrm{cl}(\operatorname{ran} T^*) = (\ker T)^\perp \cong H/\ker T \cong \operatorname{ran} T.$$

(Here $\cong$ means linear isomorphism.) It follows that if $\operatorname{ran} T$ is finite dimensional, then so is $\operatorname{ran} T^*$, we are done.

Second solution. Let (x_n) be any bounded sequence in H; $\|x_n\| \le M$ for all n. We need to prove that $(T^* x_n)$ has a convergent subsequence. Since T^* is a bounded operator, the sequence $(T^* x_n)$ is also bounded. Then, by compactness of T, there exists a subsequence (n_i) for which $(TT^* x_{n_i})$ is convergent and thus Cauchy. Then $(T^* x_{n_i})$ is also Cauchy:

$$\|T^* x_{n_i} - T^* x_{n_j}\|^2 = (T^*(x_{n_i} - x_{n_j}), T^*(x_{n_i} - x_{n_j})) = (TT^*(x_{n_i} - x_{n_j}), x_{n_i} - x_{n_j}) \le$$
$$\|TT^*(x_{n_i} - x_{n_j})\|\|x_{n_i} - x_{n_j}\| \le 2M\|TT^* x_{n_i} - TT^* x_{n_j}\|.$$

Since we are in a complete space, it follows that $(T^* x_{n_i})$ is convergent.

9. The claim follows easily from the Hilbert-Schmidt theorem. We use the notations of the theorem. It is easy to see that each λ_n is an eigenvalue with eigenvector s_n and the only other possible eigenvalue is 0. If $\lambda_n \notin [0, \infty)$ for some n, then $(Ts_n, s_n) = (\lambda_n s_n, s_n) = \lambda_n$, thus T is not positive. If $\lambda_n \in [0, \infty)$ for all n, then

$$(Tx, x) = \sum_n \lambda_n |(x, s_n)|^2 \ge 0,$$

so T is positive.

10. Since the underlying field is $\mathbb{C}$, being positive implies being normal (see Exercise 4 on "Operators on Hilbert spaces"). Therefore the previous exercise tells us that $\lambda_n \in [0, \infty)$ for all n. Set

$$Sx = \sum_n \sqrt{\lambda_n}(x, s_n)s_n \text{ for } x \in H.$$

Clearly, S is positive, compact and it satisfies $S^2 = T$.

11.* a) We have seen that the adjoint of the left shift operator L is the right shift operator R, so

$$\|L - U\| = \|(L - U)^*\| = \|R - U^*\|.$$

Notice that

$$U \text{ is unitary } \Leftrightarrow U^* \text{ is unitary } \Leftrightarrow -U^* \text{ is unitary}.$$

So it is enough to show that $\|R + U\| = 2$ for any unitary operator U. Every isometry has operator norm 1, thus

$$\|R + U\| \le \|R\| + \|U\| \le 1 + 1 = 2.$$

So we need to prove that there exists $x \in H$ with $\|x\| = 1$ such that $\|(R+U)x\|$ is arbitrarily close to 2. Since $U^{-1} = U^*$ is also an isometry, we have

$$\|(R+U)x\| = \|U^{-1}(R+U)x\| = \|U^{-1}Rx + x\|.$$

Therefore it suffices to show that 1 is in the approximate point spectrum of $U^{-1}R$. Let $V \stackrel{\text{def}}{=} U^{-1}R$; it is clearly an isometry. Also, since R is not surjective, neither is V. We claim that the approximate point spectrum of every non-surjective isometry V is the closed unit disk $\{\lambda \in \mathbb{C} : |\lambda| \le 1\}$. Since V is an isometry, it is bounded below, so $\operatorname{ran} V$ is closed. Also, $\operatorname{ran} V \neq H$, because V is not surjective. Therefore $(\operatorname{ran} V)^\perp \neq \{0\}$. Let $s_0 \in (\operatorname{ran} V)^\perp$ with $\|s_0\| = 1$ and let $s_n = V^n s_0$. Then $s_n \in \operatorname{ran} V$ for all n. Consequently, $(s_0, s_n) = 0$. However, V is an isometry, so it preserves the inner product and so does V^m:

$$(s_m, s_{m+n}) = (V^m s_0, V^m s_n) = (s_0, s_n) = 0.$$

Also,

$$(s_m, s_m) = (V^m s_0, V^m s_0) = (s_0, s_0) = 1.$$

Thus (s_n) is an orthonormal system and $Vs_n = s_{n+1}$. (Basically, V acts like a right shift on $\operatorname{cl}(\operatorname{span}\{s_0, s_1, \ldots\})$.) Now it is easy to see that the approximate point spectrum is the closed unit disk. Since $\|V\| = 1$, the spectrum cannot contain any point outside the unit disk. If $|\lambda| \le 1$, then for

$$x_N = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \lambda^{N-1-n} s_n$$

we have $\|x_N\| = 1$ and

$$\|Vx_N - \lambda x_N\| = \frac{1}{\sqrt{N}} \|s_N - \lambda^N s_0\| \le \frac{2}{\sqrt{N}},$$

which converges to 0 as $N \to \infty$. Consequently, λ is in the approximate point spectrum.

b) Let A denote the image of the closed unit ball, that is, $A = T(\overline{B}_1(0))$, and let (e_n) be the standard orthonormal basis of ℓ_2. Then $Le_{n+1} = e_n$ for $n \ge 1$, thus

$$\|L - T\| \ge \|(L-T)e_{n+1}\| = \|e_n - Te_{n+1}\| \ge d(e_n, A).$$

So it suffices to show that $d(e_n, A) \to 1$ as $n \to \infty$. Since T is compact, A is totally bounded, so it has a finite ε-lattice $L \subset A$ for any fixed $\varepsilon > 0$. Let $x \in L$. We know that

$$\sum_{n=1}^{\infty} |(x, e_n)|^2 = \|x\|^2 < \infty,$$

thus $|(x, e_n)| \to 0$ as $n \to \infty$. Since L is finite, there exists N such that $|(x, e_n)| < \varepsilon$ for all $x \in L$ and $n \ge N$. Therefore

$$1 - \varepsilon < (e_n, e_n) - |(x, e_n)| \le |(e_n - x, e_n)| \le \|e_n - x\| \|e_n\| = \|e_n - x\|,$$

whenever $x \in L$ and $n \ge N$. Since L is an ε-lattice of A, we get that

$$\|e_n - y\| > 1 - 2\varepsilon,$$

whenever $y \in A$ and $n \ge N$. Since $0 \in A$, we have $1 \ge d(e_n, A) > 1 - 2\varepsilon$, and $d(e_n, A) \to 1$ follows.

12.* If $\|x\| = \|y\| = 1$, then

$$|2(Tx, y) + 2(Ty, x)| = |(T(x+y), x+y) - (T(x-y), x-y)| \le |(T(x+y), x+y)| + |(T(x-y), x-y)| \le w(T)\left(\|x+y\|^2 + \|x-y\|^2\right) = 2w(T)\left(\|x\|^2 + \|y\|^2\right) = 4w(T).$$

For arbitrary nonzero $x, y \in H$ using the above inequality for $x/\|x\|$ and $y/\|y\|$ we get

$$|(Tx, y) + (Ty, x)| \le 2w(T)\|x\|\|y\|.$$

Plugging $y = Tx$ into this:

$$\left|\|Tx\|^2 + (T^2x, x)\right| \le 2w(T)\|x\|\|Tx\|.$$

Using this for αT instead of T, where $\alpha \in \mathbb{C}$ with $|\alpha| = 1$:

$$\big|\|Tx\|^2 + \alpha^2 (T^2 x, x)\big| \leq 2w(T)\|x\|\|Tx\|.$$

This holds for all complex α of unit length, therefore

$$\|Tx\|^2 + \big|(T^2 x, x)\big| \leq 2w(T)\|x\|\|Tx\|.$$

Consequently,

$$\big|(T^2 x, x)\big| \leq 2w(T)\|x\|\|Tx\| - \|Tx\|^2 = w(T)^2\|x\|^2 - (w(T)\|x\| - \|Tx\|)^2 \leq w(T)^2\|x\|^2,$$

which clearly implies $w(T^2) \leq w(T)^2$, we are done.

13. We proved in class that $\|S\| \leq 2w(S)$ for any operator $S \in B(H)$. Plugging $S = T^{2^k}$ and using $\|T^{2^k}\| = \|T\|^{2^k}$ (see Exercise 6):

$$\|T\|^{2^k} = \|T^{2^k}\| \leq 2w(T^{2^k}).$$

However, by the previous exercise:

$$w(T^{2^k}) \leq w(T^{2^{k-1}})^2 \leq w(T^{2^{k-2}})^4 \leq \cdots \leq w(T)^{2^k}.$$

Therefore,

$$\|T\|^{2^k} \leq 2w(T)^{2^k}, \text{ thus } \|T\| \leq \sqrt[2^k]{2}\,w(T).$$

We get $\|T\| \leq w(T)$ in the limit (as $k \to \infty$). Since the reverse inequality is trivial, we are done.