ACUTE SETS OF EXPONENTIALLY OPTIMAL SIZE

BALÁZS GERENCSÉR AND VIKTOR HARANGI

ABSTRACT. We present a simple construction of an acute set of size $2^{d-1}+1$ in $\mathbb{R}^d$ for any dimension d. That is, we explicitly give $2^{d-1}+1$ points in the d-dimensional Euclidean space with the property that any three points form an acute triangle. It is known that the maximal number of such points is less than 2^d. Our result significantly improves upon a recent construction, due to Dmitriy Zakharov, with size of order φ^d where $\varphi = (1+\sqrt{5})/2 \approx 1.618$ is the golden ratio.

1. INTRODUCTION

Around 1950 Erdős conjectured that given more than 2^d points in $\mathbb{R}^d$ there are three of them determining an obtuse angle. In 1962 Danzer and Grünbaum proved this conjecture [4] (their proof can also be found in [1]).

In other words, if we want to find as many points as possible with all angles being at most $\pi/2$, then we cannot do better than 2^d points in dimension d. The vertices of the d-dimensional hypercube show the existence of 2^d points with this property. However, in the hypercube many angles are actually equal to $\pi/2$. A natural question arises: what is the maximal number of points if we want all angles to be acute, that is, strictly less than $\pi/2$? A set of such points will be called an *acute set*.

The exclusion of right angles seemed to decrease the maximal number of points dramatically: Danzer and Grünbaum could only find $2d-1$ points, and they conjectured that this is the best possible. However, this was only proved for $d=2,3$. (For the non-trivial case $d=3$ see e.g. [2, 6].)

Later Erdős and Füredi used the probabilistic method (choosing random vertices of the hypercube) to prove the existence of exponentially large acute sets of size

$$\frac{1}{2}\left(\frac{2}{\sqrt{3}}\right)^d > 0.5 \cdot 1.154^d.$$

In [7] this construction was generalized and the improved bound $c \cdot 1.2^d$ was obtained.

Recently Dmitriy Zakharov (a high school student from Moscow) came up with stunning explicit constructions [9]. His approach is recursive: given an acute set, the basic idea is to double certain points (by lifting them in a particular way using an extra dimension) and then fix the newly created right angles by a small perturbation. This way he obtained d-dimensional acute sets of size

$$F_{d+2} > \left(\frac{1+\sqrt{5}}{2}\right)^d > 1.618^d, \text{ where } F_n \text{ denotes the Fibonacci sequence.}$$

2010 *Mathematics Subject Classification.* 51M04, 51M15.

Key words and phrases. acute set, acute angles, hypercube, strictly antipodal.

The first author was supported by NKFIH (National Research, Development and Innovation Office) grant PD 121107. The second author was supported by "MTA Rényi Lendület Véletlen Spektrum Kutatócsoport".

The next surprise came when a mathematics enthusiast from Ukraine (who wished to remain anonymous) constructed numerical examples of a 4-dimensional acute set of size 9 and a 5-dimensional acute set of size 17. (See `http://dxdy.ru/post1222167.html#p1222167` and `http://dxdy.ru/post1231694.html#p1231694` for these examples.) Previously, the best known lower bounds were 8 in dimension 4 and 13 in dimension 5, see [7, 9]. His idea was to start from the vertices of a $(d-1)$-dimensional hypercube and slightly modify the coordinates. Using only one extra dimension he could turn the vertex set into an acute set. Moreover, one extra point could be easily added.

Inspired by these examples we managed to make the same essential idea work in any dimension d.

Theorem 1. *There exist $2^{d-1}+1$ points in $\mathbb{R}^d$ such that any three of them form an acute triangle.*

By this we achieve the optimal exponential rate 2. Furthermore, the best known lower and upper bounds ($2^{d-1}+1$ and 2^d-1, respectively) are now within a factor 2.

Sketch of the construction. Let X denote a $(d-1)$-dimensional hypercube in $\mathbb{R}^d$, and let u be a unit vector orthogonal to X. The idea is to slightly perturb X in a way that all the right angles in X become acute. First we take a vertex x and move it a bit closer to the center of X, that is, we choose a small $a>0$ and shift x towards the center by distance a. How do the right angles (involving x) change? It is easy to see that if x is the middle point of the angle, then the angle becomes obtuse. However, if it is a non-middle point, the angle will be acute. We can get rid of the new obtuse angles by further translating x (this time in the orthogonal direction) with bu for some small $b>0$. If a and b are appropriately coupled, then all angles involving x become acute.

We want to proceed similarly for all other vertices. We want to avoid, however, disturbing the acute angles that we have created so far. To this end, we will move the subsequent vertices by a much smaller magnitude. So we take another vertex x_2, shift it towards the center by distance a_2 and then translate it with b_2u, where a_2 and b_2 are appropriately coupled and much smaller than a and b. If we continue this way (taking smaller and smaller pairs a_i, b_i), then all angles will be acute in the end. Furthermore, one more point can be easily added to this acute set: translate the center of the hypercube by cu for some large enough c.

In Section 2 we will make the above argument precise.

Related problems and corollaries. First we define the notion of *strict antipodality.*

Definition 2. Let X be a subset of $\mathbb{R}^d$. For distinct points $x,y\in X$ the pair (x,y) is said to be *strictly antipodal* if there exist parallel hyperplanes H_1 and H_2 passing through x and y (respectively) such that $X\setminus\{x,y\}$ lie strictly between H_1 and H_2.

If this holds true for hyperplanes H_1, H_2 perpendicular to the line xy, then (x,y) is called a *strict double-normal pair.*

If each pair in X is strictly antipodal, then we say that X is *a strictly antipodal set.*

It is easy to see that X is an acute set if and only if each pair in X is a strict double-normal pair. If follows that any acute set is also a strictly antipodal set, and hence our result readily implies the following lower bound on the maximal cardinality of strictly antipodal sets.

Corollary 3. *There exists a strictly antipodal set in $\mathbb{R}^d$ of cardinality $2^{d-1}+1$.*

The best earlier lower bound was $3^{\lfloor d/2\rfloor-1}-1$ due to Barvinok, Lee and Novik [3]. Note that the same upper bound 2^d-1 holds for strictly antipodal sets as well [4]. Therefore our result implies that the optimal exponential rate for strictly antipodal sets is also 2.

We also get a nearly optimal answer for another related problem. As we have seen, it is not possible for $N \ge 2^d$ points in $\mathbb{R}^d$ that all angles formed by the points are acute, or equivalently, that all pairs are strict double-normal. The following question arises: what is the maximal number of strict double-normal pairs for N points in $\mathbb{R}^d$? It is not hard to deduce from the Erdős–Stone theorem that this maximal number is asymptotically

$$\left(1-\frac{1}{k'(d)}\right)\binom{N}{2}+o(N^2) \quad \text{for some positive integer } k'(d).$$

A nice result of Andrey Kupavskii says that given a d-dimensional acute set of size m, one can construct $(m+d)$-dimensional sets for which $1-1/m$ fraction of the pairs are strict double-normal, see [8, Section 3]. In other words, $k'(m+d)\ge m$. Combining this with our result (i.e. setting $m=2^{d-1}+1$) we get the exact value $k'(2^{d-1}+d+1)=2^{d-1}+1$ for any positive integer d. (In other dimensions we almost get the optimal answer: the lower and upper bounds for k' in [8, Theorem 3] differ by at most 1.)

2. Construction of the acute set

This section contains the proof of Theorem 1. First we give 2^{d-1} points by perturbing the vertices of a hypercube, then we add an extra point.

2.1. **Perturbation of the hypercube.** Our construction is based on the following lemma.

Lemma 4. *Given a $(d-1)$-dimensional hypercube in $\mathbb{R}^d$ and $\varepsilon>0$ one can move a vertex of the hypercube by distance at most ε in a way that any angle determined by this point and two other vertices of the hypercube is acute.*

Formally, let $X\subset\mathbb{R}^d$ denote the vertex set of a $(d-1)$-dimensional hypercube and let $x\in X$ be an arbitrary vertex. Then for all $\varepsilon>0$ there exists $x'\in\mathbb{R}^d$ such that $|x-x'|\le\varepsilon$ and the angles $\angle x'yz$ and $\angle yx'z$ are acute for any distinct $y,z\in X\setminus\{x\}$.

Proof. We may assume that

$$X=\underbrace{\{0,1\}\times\cdots\times\{0,1\}}_{d-1}\times\{0\}\subset\mathbb{R}^d,$$

that is, we consider the set X of those points for which each of the first $d-1$ coordinates is 0 or 1 and the last coordinate is 0.

We also assume that the vertex we want to move is $x=(0,\dots,0,0)\in X$. We claim that the point

$$x'=(\underbrace{a,\dots,a}_{d-1},b)\in\mathbb{R}^d$$

satisfies the required properties for appropriately chosen $0<a<1$ and b.

We need to check that all angles formed by x' and two distinct vertices $y,z\in X\setminus\{x\}$ are acute.

Case 1: x' is not the middle point. To show that the angle $\angle x'yz$ is acute we need to prove that the inner product $\langle x'-y,z-y\rangle$ is positive. We can write this inner product as a coordinate-wise sum. Since y and z are distinct, there is at least one coordinate where they are different (one is 0, the other is 1). The contribution of such a coordinate to the

inner product is either a, or $1-a$, which is positive given that $0<a<1$. In the other coordinates the contribution is clearly 0.
Case 2: x' is the middle point. This time we need that the inner product $\langle y-x', z-x'\rangle$ is positive. The contribution of each of the first $d-1$ coordinates is one of the following: $-a(1-a)$, a^2, $(1-a)^2$. It follows that their total contribution is at least $-(d-1)a(1-a) > -(d-1)a$. As for the last coordinate, its contribution to the inner product is b^2. Therefore

$$\langle y-x', z-x'\rangle > b^2-(d-1)a.$$

In conclusion, all required angles are acute provided that

$$0<a<1 \text{ and } b^2 \geq (d-1)a.$$

These conditions can be easily satisfied along with $|x-x'|^2=(d-1)a^2+b^2 \leq \varepsilon^2$. ☐

By repeatedly applying the above lemma we get the following.

Proposition 5. *Given a $(d-1)$-dimensional hypercube in $\mathbb{R}^d$ and $\delta>0$ one can move each vertex of the hypercube by distance at most δ such that the resulting 2^{d-1} points form an acute set.*

Proof. Let $N := 2^{d-1}$ and let $x_1, x_2, \ldots, x_N \in X$ be an enumeration of the vertices of the hypercube (in an arbitrary order). We will apply the lemma to each x_i (with different ε_i that we will specify later) and obtain new points x'_i. We start with x_1 and apply the lemma using $\varepsilon_1 = \delta$.

Now suppose that we have already obtained $x'_1, \ldots, x'_i$ for some $1 \leq i < N$ and let S_i be the set of all triangles that are acute and whose vertices are among $x'_1, \ldots, x'_i, x_{i+1}, \ldots, x_N$. We can clearly choose $0 < \varepsilon_{i+1} \leq \varepsilon_i$ in a way that if we move each vertex of a triangle in S_i by distance at most ε_{i+1}, then we still get an acute triangle.

We claim that in the end the points $x'_1, \ldots, x'_N$ will form an acute set. We need to show that the triangle $x'_i x'_j x'_k$ is acute for any $i<j<k$. Since x_j and x_k are vertices of the hypercube, the triangle $x'_i x_j x_k$ is acute according to the lemma. Therefore this triangle is in the set S_i. Since $|x'_j - x_j| \leq \varepsilon_j \leq \varepsilon_{i+1}$ and $|x'_k - x_k| \leq \varepsilon_k \leq \varepsilon_{i+1}$, it follows that $x'_i x'_j x'_k$ is also an acute triangle. ☐

2.2. **The cherry on the cake.** Finally, we add one more point to the acute set. Let $X = \{x_1, \ldots, x_N\} = \{0,1\} \times \cdots \times \{0,1\} \times \{0\}$ be the $(d-1)$-dimensional unit hypercube and let $\{x'_1, \ldots, x'_N\}$ be the acute set obtained using Proposition 5 with some small δ. We claim that if we add $x_0 = (1/2, \ldots, 1/2, c)$ to this set, then we will still have an acute set provided that $c > \sqrt{d-1}/2$ and δ is sufficiently small.

To see this, we first note that the distance $|x_0 - x_i| = \sqrt{(d-1)/4 + c^2}$ is the same for each i. Therefore every triangle $x_0 x_i x_j$ is isosceles, and consequently the angles at x_i and x_j are automatically acute. As for the angle at x_0, it is acute if and only if $|x_i - x_j| < \sqrt{2}|x_0 - x_i|$. Since $|x_i - x_j|$ is at most $\sqrt{d-1}$ for any two vertices of the unit hypercube, this is always satisfied provided that $c > \sqrt{d-1}/2$. Now by choosing δ to be sufficiently small the triangles $x_0 x'_i x'_j$ can be arbitrarily close to the triangles $x_0 x_i x_j$, and hence they are acute as well.

References

[1] M. Aigner, G.M. Ziegler, *Proofs from THE BOOK*, 3rd ed. Springer-Verlag (2003), 79–83.
[2] H.T. Croft, On 6-point configurations in 3-space, *J. London Math. Soc.* **36** (1961), 289–306.

[3] A. Barvinok, S.J. Lee, I. Novik, Explicit constructions of centrally symmetric k-neighborly polytopes and large strictly antipodal sets, *Discrete Comput. Geom.* **49** (2013), no. 3, 429–443.
[4] L. Danzer, B. Grünbaum, Über zwei Probleme bezüglich konvexer Körper von P. Erdős und von V.L. Klee, *Math. Zeitschrift* **79** (1962), 95–99.
[5] P. Erdős, Z. Füredi, The greatest angle among n points in the d-dimensional Euclidean space, *Ann. Discrete Math.* **17** (1983), 275–283.
[6] B. Grünbaum, Strictly antipodal sets, *Israel J. Math.* **1** (1963), 5–10.
[7] V. Harangi, Acute sets in Euclidean spaces, *SIAM J. Discrete Math.* **25** (2011), no. 3, 1212–1229.
[8] A. Kupavskii, The number of double-normals in space *Discrete Comput. Geom.* **56** (2016), 711–726.
[9] D. Zakharov, Acute sets, arXiv:1707.04829, (2017), to appear in *Discrete Comput. Geom.*

MTA Alfréd Rényi Institute of Mathematics H-1053 Budapest, Reáltanoda utca 13-15; and Eötvös Loránd University, Department of Probability and Statistics H-1117 Budapest, Pázmány Péter sétány 1/c
E-mail address: gerencser.balazs@renyi.mta.hu

MTA Alfréd Rényi Institute of Mathematics H-1053 Budapest, Reáltanoda utca 13-15
E-mail address: harangi@renyi.hu