Large dimensional sets not containing a given angle

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Abstract

We say that a set in a Euclidean space *does not contain* an angle α if the angle determined by any three points of the set is not equal to α. The goal of this paper is to construct compact sets of large Hausdorff dimension that do not contain a given angle $\alpha \in (0, \pi)$. We will construct such sets in $\mathbb{R}^n$ of Hausdorff dimension $c(\alpha)n$ with a positive $c(\alpha)$ depending only on α provided that α is different from $\pi/3$, $\pi/2$ and $2\pi/3$. This improves on an earlier construction due to several authors which gave $c(\alpha) \log n$ for the dimension.

The main result of the paper concerns the case of the angles $\pi/3$ and $2\pi/3$. We present self-similar sets in $\mathbb{R}^n$ of Hausdorff dimension $c\sqrt[3]{n}/\log n$ with the property that they do not contain the angles $\pi/3$ and $2\pi/3$.

The constructed sets avoid not only the given angle α but also a small neighbourhood of α.

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1 Introduction

The following problem was addressed in [3]: given an angle $\alpha \in (0, \pi)$ and a positive integer n, what is the maximal Hausdorff dimension $C(n, \alpha)$ of an analytic set $K \subset \mathbb{R}^n$ with the property that K does not contain the angle α, that is, for any three distinct points A, B, C of the set $\angle ABC \neq \alpha$. (If K does not need to be analytic, then one can use transfinite recursion to construct a full dimensional set not containing α [3, Theorem 3.13].) It was proved that $C(n, \alpha) \leq n - 1$ for arbitrary α, in other words, if the Hausdorff dimension of an analytic set $K \subset \mathbb{R}^n$ is greater than $n - 1$, then K contains every angle $\alpha \in (0, \pi)$.

As far as lower bounds are concerned, the line segment shows that $C(n,\alpha) \geq 1$. This was improved in [3] for angles $\alpha \neq \pi/3, \pi/2, 2\pi/3$ by proving that $C(n,\alpha) \geq c(\alpha)\log n$ where $c(\alpha) > 0$ depends only on α. It was actually shown that for any $\delta > 0$ there exists a self-similar set $K \subset \mathbb{R}^n$ of dimension $c_\delta \log n$ such that all angles contained by K are from the δ-neighbourhood of the set $\{0, \pi/3, \pi/2, 2\pi/3, \pi\}$.

In Section 2 we will improve this construction and present a self-similar set of dimension $c_\delta n$ with the same property (that is, all angles are from the δ-neighbourhood of $\{0, \pi/3, \pi/2, 2\pi/3, \pi\}$). It implies that $C(n,\alpha) \geq c(\alpha)n$ given that $\alpha \neq \pi/3, \pi/2, 2\pi/3$.

The main result of this paper is the construction of Section 3, where we present a self-similar set in $\mathbb{R}^n$ with dimension $c\sqrt[3]{n}/\log n$ that contains neither $\pi/3$, nor $2\pi/3$. (Previously, no analytic set of dimension greater than 1 was known that avoids any of these angles.) The constructed sets also avoid a small neighbourhood of $\pi/3$ and $2\pi/3$. To be more precise, for any $\delta > 0$ we prove the existence of a set (in some Euclidean space of sufficiently large dimension) which has dimension $c\delta^{-1}/\log(\delta^{-1})$ and which contains no angle in the δ-neighbourhood of $\pi/3$ and $2\pi/3$. This latter result is essentially sharp: if the dimension of K is at least $C\delta^{-1}\log(\delta^{-1})$ for some $\delta > 0$, then K must contain an angle in the δ-neighbourhood of $\pi/3$ [3, Corollary 3.7] as well in the δ-neighbourhood of $2\pi/3$ [3, Theorem 3.11]. (Throughout this paper c and C denote absolute constants but different appearances may denote different values.)

As for the angle $\pi/2$, it was proved that if the dimension of K is greater than 1, then K must contain angles arbitrarily close to $\pi/2$ [3, Theorem 3.2]. In other words, it is impossible to construct sets of dimension greater than 1 that avoid a neighbourhood of $\pi/2$. However, a recent (and yet unpublished) result of András Máthé says that there exist compact sets in $\mathbb{R}^n$ with Hausdorff dimension $n/2$ such that they do not contain the angle $\pi/2$. (His construction uses number theoretic ideas.) On the other hand, it was already known that analytic sets in $\mathbb{R}^n$ with Hausdorff dimension greater than $\lceil n/2 \rceil$ must contain $\pi/2$ [3, Theorem 2.4]. Consequently, $n/2 \leq C(n, \pi/2) \leq \lceil n/2 \rceil$.

Finally, we mention a result of similar flavour. In [2, 5, 6] it was shown that for any three points in $\mathbb{R}$ or in $\mathbb{R}^2$ there exists a set of full Hausdorff dimension that contains no similar copy to the three given points. It is open whether the analogous result holds in higher dimension.

2 Avoiding general angles

In this section we construct sets with the property that any angle contained by the set is close to one of the following angles: $0, \pi/3, \pi/2, 2\pi/3, \pi$.

First we define homothetic self-similar sets and prove some simple facts about them. Let us take points $P_1, \ldots, P_m$ in some Euclidean space $\mathbb{R}^n$. We denote the convex hull of these points by K_0. For every $i = 1, \ldots, m$ we take a homothety φ_i with center P_i and scale factor $0 < q_i < 1$. Let K be the unique non-empty compact set satisfying $K = \bigcup_i \varphi_i(K)$. One can get this homothetic self-similar set K by setting

$$K_r \stackrel{\text{def}}{=} \bigcup_{i=1}^{m} \varphi_i\left(K_{r-1}\right) = \bigcup_{i_1,\ldots,i_r} \varphi_{i_1} \circ \cdots \circ \varphi_{i_r}(K_0),$$

then $K = \bigcap_{r=1}^{\infty} K_r$. We will use the following notations:

$$d_{\min} = \min\{|P_i - P_j| : i \neq j\}; d_{\max} = \max\{|P_i - P_j| : i \neq j\}; q_{\max} = \max\{q_1, \dots, q_m\}.$$

Set $\eta \overset{\text{def}}{=} q_{\max} d_{\max}/d_{\min}$. We will assume that $\eta < 1/2$ which clearly implies that the sets $\varphi_i(K_0)$ $(i = 1, \dots, m)$ are pairwise disjoint. Therefore the well-known Moran equation for the dimension s of the self-similar K holds:

$$q_1^s + \cdots + q_m^s = 1,$$

which yields that in the special case $q_1 = \cdots = q_m = q$ the dimension is

$$s = \frac{\log m}{\log(1/q)}.$$

For these sets most of the dimension notions (like Hausdorff or Minkowski dimension) coincide so from now on we will simply say dimension.

The next proposition says that the set of directions in K is *close* to the set of directions in $\{P_1, \dots, P_m\}$.

Proposition 2.1. *Suppose that $\eta = q_{\max} d_{\max}/d_{\min} < 1/2$. Then for any two distinct points $A, B \in K$ there exist $i \neq j$ such that the angle between the vectors $A - B$ and $P_i - P_j$ is less than $\pi\eta$.*

Proof. There exist unique sequences $i_1, i_2, \dots$ and $j_1, j_2, \dots$ such that

$$A \in \varphi_{i_1} \circ \cdots \circ \varphi_{i_r}(K) \text{ and } B \in \varphi_{j_1} \circ \cdots \circ \varphi_{j_r}(K)$$

for any positive integer r. Let r be the smallest index with $i_r \neq j_r$. Now let ψ be the homothety defined as $\varphi_{i_1} \circ \cdots \circ \varphi_{i_{r-1}} = \varphi_{j_1} \circ \cdots \circ \varphi_{j_{r-1}}$. Clearly $A' \overset{\text{def}}{=} \psi^{-1}(A) \in \varphi_{i_r}(K)$ and $B' \overset{\text{def}}{=} \psi^{-1}(B) \in \varphi_{j_r}(K)$. It also follows that $A' - B'$ and $A - B$ are parallel (one is a positive scalar multiple of the other).

So we can assume that A and B are in different level 1 parts of K, that is, there exist indices $i \neq j$ such that $A \in \varphi_i(K)$ and $B \in \varphi_j(K)$. Thus $|A - P_i|, |B - P_j| \leq q_{\max} d_{\max}$. Let us now translate the segment $P_i P_j$ by the vector $A - P_i$ so that P_i goes to A, and P_j goes to some point Q. Then the angle in question is equal to $\angle BAQ$. We have $|B - Q| \leq |A - P_i| + |B - P_j| \leq 2q_{\max} d_{\max}$. On the other hand, $|A - Q| = |P_i - P_j| \geq d_{\min}$. Since $\eta < 1/2$, it follows that $|B - Q| < |A - Q|$. Under this condition the angle $\angle BAQ$ is clearly at most

$$\arcsin\left(\frac{|B - Q|}{|A - Q|}\right) \leq \arcsin(2\eta) \leq \pi\eta.$$

□

Corollary 2.2. *Suppose that $\eta < 1/2$. Then for any three distinct points A, B, C of K there exist indices i_1, i_2, i_3, i_4 such that*

$$|\angle ABC - \angle(P_{i_1} - P_{i_2}, P_{i_3} - P_{i_4})| < 2\pi\eta.$$

Proof. Let $A, B, C, D \in K$ with $A \neq B$ and $C \neq D$. We apply the above proposition for the vectors $A - B$ and $C - D$. It follows that there exist indices i_1, i_2, i_3, i_4 such that

$$|\angle(A - B, C - D) - \angle(P_{i_1} - P_{i_2}, P_{i_3} - P_{i_4})| < 2\pi\eta.$$

Setting $B = D$ completes the proof. □

In [3] this self-similar construction was used in the special case when the points P_i are the vertices of a regular simplex in $\mathbb{R}^n$. Then $m = n + 1$; $d_{\min} = d_{\max}$ so setting $q_1 = \cdots = q_m = q < 1/2$ yields that K has dimension $\log(n+1)/\log(1/q)$ and all angles are in the $2\pi q$-neighbourhood of the set $\{0, \pi/3, \pi/2, 2\pi/3, \pi\}$. So for any angle α not in this set there is a constant $c(\alpha)$ such that in $\mathbb{R}^n$ a set K of dimension $c(\alpha)\log(n+1)$ can be given with the property that K does not contain α as an angle.

The following simple observation enables us to do better than that, namely, we will show the existence of a set with the same property and of dimension $c(\alpha)n$. For the above construction to work, it suffices to know that the distances $|P_i - P_j|$ are approximately the same (equal with some small error δ). And there are a lot of points in a Euclidean space with this property: in 1983 Erdős and Füredi proved [1] that for any $\delta > 0$ there exist at least $(1 + c\delta^2)^n$ points in $\mathbb{R}^n$ such that the distance of any two is between 1 and $1 + \delta$. This is also a special case of the well-known lemma of Johnson and Lindenstrauss which was first published in 1984 (see Lemma 3.3 in the next section).

Now we prove the simple fact that if we have four points with each pair having approximately the same distance then the angles enclosed by the segments are close to either $\pi/3$ or $\pi/2$.

Lemma 2.3. *Suppose that the distance of any two of some given points is between* 1 *and* $1 + \delta$ *for some* $\delta > 0$. *Then the angle between two arbitrary nonzero vectors with endpoints from the given set is in the* $C\delta$*-neighbourhood of the set* $\{0, \pi/3, \pi/2, 2\pi/3, \pi\}$.

Proof. We will assume that $0 < \delta < 0.1$. If the lemma holds under this assumption, then it must also hold for arbitrary $\delta > 0$ (possibly with some larger C).

Take the endpoints of the two vectors. The set of these endpoints consists of either two, three or four points.

In the first case the two vectors coincide or they are the negative of each other. So the enclosed angle is 0 or π.

In the second case the two vectors share exactly one common endpoint which we denote by A. Let the two other endpoints be B_1, B_2 and let $\alpha = \angle B_1 A B_2$. (So the angle enclosed by the vectors is α or $\pi - \alpha$.) By the cosine law we have

$$\cos\alpha = \frac{|A - B_1|^2 + |A - B_2|^2 - |B_1 - B_2|^2}{2|A - B_1||A - B_2|}.$$

Using this and the inequalities $(1 + \delta)^2 < 1 + 3\delta$ and $1 - 3\delta < 1/(1 + 3\delta)$ we obtain that

$$\frac{1}{2} - 3\delta < \frac{(1 - 3\delta)^2}{2} \leq \frac{2 - (1 + 3\delta)}{2(1 + 3\delta)} \leq \cos\alpha \leq \frac{2(1 + 3\delta) - 1}{2} = \frac{1}{2} + 3\delta.$$

Since arccos is a Lipschitz function on the interval $[0.2, 0.8]$, it follows that $|\alpha - \pi/3| < C\delta$. Therefore, in this case the enclosed angle is in the $C\delta$-neighbourhood of $\pi/3$ or $2\pi/3$.

Finally, in the third case we have four distinct points A_1, A_2, B_1, B_2. Using coordinates, one can easily obtain the following formula for the inner product of the vectors $A_1 - A_2$ and $B_1 - B_2$:

$$\langle A_1 - A_2, B_1 - B_2 \rangle = \left(|A_1 - B_2|^2 + |A_2 - B_1|^2 - |A_1 - B_1|^2 - |A_2 - B_2|^2\right)/2,$$

which yields that for the angle β enclosed by $A_1 - A_2$ and $B_1 - B_2$ it holds that

$$\cos\beta = \frac{|A_1 - B_2|^2 + |A_2 - B_1|^2 - |A_1 - B_1|^2 - |A_2 - B_2|^2}{2|A_1 - A_2||B_1 - B_2|}$$

(see also in [7]). Using that each distance is between 1 and $1 + \delta$ we obtain that

$$|\cos\beta| \leq \frac{2(1+\delta)^2 - 2}{2} = 2\delta + \delta^2 \leq 3\delta.$$

It follows that $|\beta - \pi/2| < C\delta$. □

In the next theorem we put together the above results to obtain large dimensional sets with all angles close to the special angles $0, \pi/3, \pi/2, 2\pi/3, \pi$.

Theorem 2.4. *There is a $\delta_0 > 0$ such that for any $0 < \delta \leq \delta_0$ there exists a self-similar set in $\mathbb{R}^n$ of dimension at least*

$$c_\delta n = c\delta^2 \log^{-1}(1/\delta) \cdot n$$

such that the angle determined by any three points of the set is in the δ-neighbourhood of the set $\{0, \pi/3, \pi/2, 2\pi/3, \pi\}$.

Proof. Take some real number $0 < \delta \leq 1/3$. As we mentioned before Lemma 2.3, there exist $m \geq (1 + c\delta^2)^n$ points $P_1, \ldots, P_m \in \mathbb{R}^n$ such that the distance of any two of them is between 1 and $1 + \delta$. Take the homotheties with centre P_i and ratio $q_i = q = \delta$, and consider the corresponding self-similar set K. On one hand, the dimension of K is

$$\frac{\log m}{\log(1/q)} \geq \frac{n \log(1 + c\delta^2)}{\log(1/\delta)} \geq c\frac{\delta^2}{\log(1/\delta)} n.$$

On the other hand, Lemma 2.3 and Corollary 2.2 imply that any angle in our self-similar set is in the $C\delta$-neighbourhood of $\{0, \pi/3, \pi/2, 2\pi/3, \pi\}$. Changing δ to δ/C completes the proof. □

3 Avoiding angles $\pi/3$ and $2\pi/3$

Our goal in this section is to construct large dimensional sets avoiding the angles $\pi/3$ and $2\pi/3$. Again, we will use the self-similar construction described at the beginning of the previous section. The idea is to find (many) points P_i such that any angle determined by them is in a small neighbourhood of $\pi/3$ but avoids an even smaller neighbourhood of $\pi/3$.

We were inspired by the following r-colouring of the complete graph on 2^r vertices. Let $C_1, \ldots, C_r$ denote the colours and let us associate to each vertex a 0-1 sequence of length

r. Consider the edge between the vertices corresponding to the sequences $i_1, \dots, i_r$ and $j_1, \dots, j_r$. We colour this edge with C_k where k denotes the first index where the sequences differ, that is, $i_1 = j_1, \dots, i_{k-1} = j_{k-1}, i_k \neq j_k$. Let us denote this coloured graph by $\mathcal{G}_r = \mathcal{G}_r(C_1, \dots, C_r)$. This is a folklore graph colouring showing that the multicolour Ramsey number $R_r(3)$ is greater than 2^r.

One can obtain $\mathcal{G}_r$ recursively as well. Consider the colouring $\mathcal{G}_{r-1}(C_2, \dots, C_r)$, and take two copies of this coloured graph. Let the edges going between the two copies be all coloured with C_1. It is easy to see that this way we get $\mathcal{G}_r(C_1, \dots, C_r)$. This colouring clearly has the property that there is no monochromatic triangle in the graph. Moreover, every triangle has two sides with the same colour and a third side with a different colour of higher index.

The idea is to realize $\mathcal{G}_r$ geometrically in the following manner: the vertices of the graph will be represented by points of a Euclidean space and edges with the same colour will correspond to equal distances. In the sequel we will show that $\mathcal{G}_r$ can be represented in the above sense. First we prove a simple geometric fact.

Proposition 3.1. *Let m be a non-negative integer and R, l be positive real numbers with $R \le l/\sqrt{2}$. Take a $(2m+2)$-dimensional sphere $\mathcal{S}$ with radius*

$$R' \stackrel{def}{=} \sqrt{\frac{1}{4}l^2 + \frac{1}{2}R^2} \le \sqrt{\frac{1}{4}l^2 + \frac{1}{2}\left(\frac{l}{\sqrt{2}}\right)^2} = \frac{l}{\sqrt{2}}.$$

Then there exist two m-dimensional spheres $\mathcal{X}, \mathcal{Y} \subset \mathcal{S}$ with radius R such that $|X - Y| = l$ for any $X \in \mathcal{X}$ and any $Y \in \mathcal{Y}$.

Proof. We may assume that $\mathcal{S} = \{P \in \mathbb{R}^{2m+3} : |P| = R'\}$. Set $t \stackrel{\text{def}}{=} \sqrt{l^2 - 2R^2}/2$ and take the spheres

$$\mathcal{X} \stackrel{\text{def}}{=} \left\{X = (x_1, \dots, x_{m+1}, -t, 0, \dots, 0) \in \mathbb{R}^{2m+3} : x_1^2 + \dots + x_{m+1}^2 = R^2\right\},$$
$$\mathcal{Y} \stackrel{\text{def}}{=} \left\{Y = (0, \dots, 0, t, y_1, \dots, y_{m+1}) \in \mathbb{R}^{2m+3} : y_1^2 + \dots + y_{m+1}^2 = R^2\right\}.$$

For any $X \in \mathcal{X}$ we have $|X| = \sqrt{R^2 + t^2} = R'$ and thus $\mathcal{X} \subset \mathcal{S}$. Similarly, $\mathcal{Y} \subset \mathcal{S}$. On the other hand, for any $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$ it clearly holds that $|X - Y| = \sqrt{R^2 + (2t)^2 + R^2} = l$. □

Lemma 3.2. *Let $l_1 \ge l_2 \ge \ldots \ge l_r > 0$ be a decreasing sequence of positive reals. By $\mathcal{I}_r$ we denote the set of 0-1 sequences of length r. Then 2^r points $P_{i_1,\dots,i_r}$, $(i_1, \dots, i_r) \in \mathcal{I}_r$ can be given in some Euclidean space in such a way that for two distinct 0-1 sequences $(i_1, \dots, i_r) \neq (j_1, \dots, j_r)$ the distance of $P_{i_1,\dots,i_r}$ and $P_{j_1,\dots,j_r}$ is equal to l_k where k denotes the first index where the sequences differ, that is, $i_1 = j_1, \dots, i_{k-1} = j_{k-1}, i_k \neq j_k$.*

Proof. For the sake of simplicity, we say that the points $P_{i_1,\dots,i_r}$, $(i_1, \dots, i_r) \in \mathcal{I}_r$ have configuration $\mathcal{P}_r(l_1, \dots, l_r)$ if the distances between the points are as in the claim of the lemma.

We will prove by induction that there exist points with configuration $\mathcal{P}_r(l_1, \dots, l_r)$ on a $(2^r - 2)$-dimensional sphere with radius at most $l_1/\sqrt{2}$. This is clearly true for $r = 1$. Suppose that it holds for $r - 1$. The induction hypothesis applied for the distances

$l_2 \geq \ldots \geq l_r$ yields that there exist points with configuration $\mathcal{P}_{r-1}(l_2, \ldots, l_r)$ on a $(2^{r-1}-2)$-dimensional sphere with radius $R \leq l_2/\sqrt{2}$.

Since $R \leq l_2/\sqrt{2} \leq l_1/\sqrt{2}$, Proposition 3.1 implies that there is a $(2^r - 2)$-dimensional sphere $\mathcal{S}$ with some radius $R' \leq l_1/\sqrt{2}$ such that it contains two $(2^{r-1} - 2)$-dimensional spheres with common radius R such that no matter how we take one point from each sphere their distance is l_1.

We can take a copy of the configuration $\mathcal{P}_{r-1}(l_2, \ldots, l_r)$ on each of these two spheres. The union of them will clearly have configuration $\mathcal{P}_r(l_1, \ldots, l_r)$. □

Using the above lemma we now construct a large set of points with the property that any angle determined by them is in a small neighbourhood of $\pi/3$ but avoids an even smaller neighbourhood of $\pi/3$. We will need the previously mentioned Johnson-Lindenstrauss lemma.

Lemma 3.3 (Johnson-Lindenstrauss lemma [4])**.** *Suppose that m points $P_1, \ldots, P_m$ are given in some Euclidean space $\mathbb{R}^d$. For any $\delta > 0$ one can find points $P'_1, \ldots, P'_m$ in the $\lceil C \log m/\delta^2 \rceil$-dimensional Euclidean space in such a way that*

$$|P_i - P_j| \leq |P'_i - P'_j| \leq (1+\delta)|P_i - P_j| \quad (1 \leq i, j \leq m).$$

Theorem 3.4. *There exist absolute constants $c, C > 0$ such that for any positive integer r and positive real $\varepsilon < 1$, 2^r points can be given in the $\lceil Cr^3/\varepsilon^2 \rceil$-dimensional Euclidean space with the property that for any angle α determined by three given points the following holds:*

$$c\frac{\varepsilon}{r} < \left|\alpha - \frac{\pi}{3}\right| < \varepsilon.$$

Moreover, for any four distinct points A, B, C, D of these points we have

$$\left|\angle(A-B, C-D) - \frac{\pi}{2}\right| < \varepsilon.$$

Proof. Let $\lambda > 1$ be a real number. We use Lemma 3.2 with $l_i = \lambda^{r-i}$ $(i = 1, \ldots, r)$. The lemma gives us 2^r points which have configuration $\mathcal{P}_r(\lambda^{r-1}, \ldots, \lambda, 1)$. Let us denote the set of these points by S, and take three distinct points in S. By construction, the triangle determined by these points has two sides with the same length λ^s and a third side with a smaller length λ^t for some integers $0 \leq t < s \leq r-1$. Let this third side be A_1A_2 and let B denote the remaining vertex. (That is, $|A_1 - A_2| = \lambda^t < \lambda^s = |A_1 - B| = |A_2 - B|$.)

Now we apply the Johnson-Lindenstrauss lemma for the points in S with some $0 < \delta < 1$; by S' we will denote the set of the points obtained. We consider the points $A'_1, A'_2, B' \in S'$ corresponding to the points A_1, A_2, B. Using the fact that $(1+\delta)^2 < 1 + 3\delta$ we get that

$$\lambda^{2t} \leq |A'_1 - A'_2|^2 < (1+3\delta)\lambda^{2t}; \quad \lambda^{2s} \leq |A'_i - B'|^2 < (1+3\delta)\lambda^{2s} \quad (i = 1, 2).$$

By the cosine law we have

$$\cos\left(\angle A'_1 A'_2 B'\right) = \frac{|A'_1 - A'_2|^2 + |A'_2 - B'|^2 - |A'_1 - B'|^2}{2|A'_1 - A'_2||A'_2 - B'|} <$$

$$\frac{(1+3\delta)\left(\lambda^{2s} + \lambda^{2t}\right) - \lambda^{2s}}{2\lambda^s\lambda^t} = \frac{1}{2\lambda^{s-t}} + 3\delta\frac{\lambda^{2s} + \lambda^{2t}}{2\lambda^s\lambda^t} \leq \frac{1}{2\lambda} + 3\delta\frac{\lambda^r + 1}{2}.$$

Set $\lambda = 1 + \frac{c\varepsilon}{r}$ and $\delta = \frac{c\varepsilon}{36r}$ with a sufficiently small constant c. Then

$$\lambda^r = \left(1 + \frac{c\varepsilon}{r}\right)^r < \exp(c\varepsilon) < 1 + 2c\varepsilon < 2.$$

Thus

$$\cos\left(\angle A_1'A_2'B'\right) < \frac{1}{2\lambda} + 3\delta\frac{\lambda^r+1}{2} < \left(\frac{1}{2} - \frac{\lambda-1}{2\lambda}\right) + \frac{9}{2}\delta < \frac{1}{2} - \frac{c\varepsilon}{4r} + \frac{c\varepsilon}{8r} = \frac{1}{2} - \frac{c\varepsilon}{8r}.$$

Since cos is a Lipschitz function with Lipschitz constant 1, it follows that $\angle A_1'A_2'B' > \pi/3 + c\varepsilon/8r$. The same holds for the angle $\angle A_2'A_1'B'$. Therefore for the third angle in the triangle we get $\angle A_1'B'A_2' < \pi/3 - c\varepsilon/4r$.

On the other hand, the distance of any two points in S' is at least 1 and at most $(1+\delta)\lambda^{r-1} < \lambda^r < 1+2c\varepsilon$. Now let us take four distinct points A, B, C, D in S'. As we have seen in the proof of Lemma 2.3, $|\angle(ABC) - \pi/3| < \varepsilon$ and $|\angle(A-B, C-D) - \pi/2| < \varepsilon$ provided that c is sufficiently small.

Finally, by the Johnson-Lindenstrauss lemma the set S' is contained in a Euclidean space of dimension at most $\lceil C\log(2^r)/\delta^2\rceil = \lceil Cr^3/\varepsilon^2\rceil$. □

This discrete set of points can be blown up (using the self-similar construction described in Section 2) to a large dimensional set that does not contain the angles $\pi/3$ and $2\pi/3$.

Theorem 3.5. *There exist absolute constants $c, C > 0$ such that for any $0 < \delta < \varepsilon < 1$ with $\varepsilon/\delta > C$ there exists a self-similar set of dimension*

$$s \geq \frac{c\varepsilon/\delta}{\log(1/\delta)}$$

in a Euclidean space of dimension

$$n \leq \frac{C\varepsilon}{\delta^3}$$

such that any angle determined by three points of the set is inside the ε-neighbourhood of $\{0, \pi/3, \pi/2, 2\pi/3, \pi\}$ but outside the δ-neighbourhood of $\{\pi/3, 2\pi/3\}$.

Proof. Set $r = [c\varepsilon/\delta]$. The previous theorem claims that for

$$n = \lceil Cr^3/\varepsilon^2\rceil \leq C\varepsilon/\delta^3; \quad m = 2^r,$$

there exist m points $P_1, \ldots, P_m \in \mathbb{R}^n$ such that for any three distinct points P_i, P_j, P_k

$$2\delta < \left|\angle P_iP_jP_k - \frac{\pi}{3}\right| < \frac{\varepsilon}{2},$$

and for any four different points P_i, P_j, P_k, P_l

$$\left|\angle(P_i - P_j, P_k - P_l) - \frac{\pi}{2}\right| < \frac{\varepsilon}{2}.$$

Now we take the self-similar set of Section 2 with $q_i = q = c\delta$. The set obtained has dimension

$$\frac{\log m}{\log(1/(c\delta))} \geq \frac{cr}{\log(1/\delta)} \geq \frac{c\varepsilon/\delta}{\log(1/\delta)}.$$

Moreover, Corollary 2.2 implies that all the angles occuring in this set are inside the ε-neighbourhood of $\{0, \pi/3, \pi/2, 2\pi/3, \pi\}$ but outside the δ-neighbourhood of $\{\pi/3, 2\pi/3\}$. □

By fixing a small ε and setting $\delta = c/\sqrt[3]{n}$ in the above theorem, we obtain the following corollaries.

Corollary 3.6. *A self-similar set $K \subset \mathbb{R}^n$ can be given such that the dimension of K is at least*

$$s \geq \frac{c\sqrt[3]{n}}{\log n},$$

and K does not contain the angle $\pi/3$ and $2\pi/3$ (moreover, K does not contain any angle in the $c/\sqrt[3]{n}$-neighbourhood of $\pi/3$ and $2\pi/3$).

So there exists a compact set in $\mathbb{R}^n$ of dimension at least $\frac{c\sqrt[3]{n}}{\log n}$ that avoids a small neighbourhood of the angles $\pi/3$ and $2\pi/3$. Probably, this result is quite far from being sharp. However, the following corollary is surprisingly sharp.

Corollary 3.7. *For any $0 < \delta < 1$ there exists a self-similar set K of dimension at least $c\delta^{-1}/\log(\delta^{-1})$ in some Euclidean space such that K does not contain any angle in $(\pi/3 - \delta, \pi/3 + \delta) \cup (2\pi/3 - \delta, 2\pi/3 + \delta)$.*

Remark 3.8. The previous result is essentially sharp. It was proved in [3] that if $\dim K > C\delta^{-1}\log(\delta^{-1})$, then K contains an angle $\alpha \in (\pi/3 - \delta, \pi/3 + \delta)$ and also an angle $\alpha' \in (2\pi/3 - \delta, 2\pi/3 + \delta)$.

Finally, we mention that the self-similar sets K constructed in this paper avoid α even in the sense that for any $A, B, C, D \in K$ with $A \neq B$ and $C \neq D$ we have $\angle(A-B, C-D) \neq \alpha$ (see the proof of Corollary 2.2).

References

[1] P. Erdős, Z. Füredi, The greatest angle among n points in the d-dimensional Euclidean space, *Annals of Discrete Mathematics* **17** (1983), 275–283.

[2] K.J. Falconer, On a problem of Erdős on fractal combinatorial geometry, *J. Combin. Theory Ser. A* **59** (1992), 142–148.

[3] V. Harangi, T. Keleti, G. Kiss, P. Maga, A. Máthé, P. Mattila, B. Strenner, How large dimension guarantees a given angle?, arXiv:1101.1426.

[4] W.B. Johnson, J. Lindenstrauss, Extensions of Lipschitz mappings into a Hilbert space, *Contemporary Mathematics* **26** (1984), 189–206.

[5] T. Keleti, Construction of 1-dimensional subsets of the reals not containing similar copies of given patterns, *Anal. PDE* **1** (2008), 29–33.

[6] P. Maga, Full dimensional sets without given patterns, *Real Anal. Exchange*, **36** (2010) no. 1, 79–90.

[7] G. Salmon, *A Treatise on the Analytic Geometry of Three Dimensions*, Hodges, Smith, and co., 2nd edition, 1865, p. 33.