REPLICA FORMULAS FOR THE INDEPENDENCE RATIO

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1. Formulas

For any fixed degree $d \geq 3$, the independence ratio of the random d-regular graph $\mathbb{G}(N,d)$ is known to converge to some constant α_d^* as $N \to \infty$. There exist r-step replica symmetry breaking formulas that are known to bound α_d^* from above [LO18].

The 1-RSB bound: for any $\lambda_0 > 1$ and any $q \in [0,1]$:

$$\alpha_d^* \log(\lambda_0) \leq \log\big(1 + (\lambda_0 - 1)(1-q)^d\big) - \frac{d}{2}\log\big(1 - (1 - 1/\lambda_0)q^2\big).$$

Choosing λ_0 and q optimally leads to an (implicit) formula for α_d^*. This 1-RSB bound is conjectured to be sharp for any $d \geq 20$ [BKZZ13] and known to be sharp for sufficiently large d [DSS16]. For $d \leq 19$ the 1-RSB bound is not tight and we want to find improved upper bounds via numerical optimization of the formulas below.

The 2-RSB bound: for any $\lambda_0 > 1$, $0 < m < 1$, and any $p_1, \ldots, p_n, q_1, \ldots, q_n \in [0,1]$ with $p_1 + \cdots + p_n = 1$ we have

$$\alpha_d^* \, m \log(\lambda_0) \leq \log \sum_{i_1=1}^{n} \cdots \sum_{i_d=1}^{n} \left(\prod_{\ell=1}^{d} p_{i_\ell} \right) \left(1 + (\lambda_0 - 1) \prod_{\ell=1}^{d} (1 - q_{i_\ell}) \right)^m - \frac{d}{2} \log \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} p_{i_1} p_{i_2} \Big(1 - (1 - 1/\lambda_0) q_{i_1} q_{i_2} \Big)^m .$$

The 3-RSB bound: for any $\lambda_0 > 1$, $0 < m_1, m_2 < 1$, $p_i \geq 0$ with $\sum p_i = 1$, $p_{i,j} \geq 0$ with $\sum_j p_{i,j} = 1$ for every fixed i, and $q_{i,j} \in [0,1]$ we have

$$\alpha_d^* \, m_1 m_2 \log(\lambda_0) \leq \log R^{\text{star}} - \frac{d}{2} \log R^{\text{edge}}, \text{ where}$$

$$R^{\text{star}} = \sum_{i_1} \cdots \sum_{i_d} \left(\prod_{\ell=1}^{d} p_{i_\ell} \right) \left(\sum_{j_1} \cdots \sum_{j_d} \left(\prod_{\ell=1}^{d} p_{i_\ell, j_\ell} \right) \left(1 + (\lambda_0 - 1) \prod_{\ell=1}^{d} (1 - q_{i_\ell, j_\ell}) \right)^{m_1} \right)^{m_2} ;$$

$$R^{\text{edge}} = \sum_{i_1} \sum_{i_2} p_{i_1} p_{i_2} \left(\sum_{j_1} \sum_{j_2} p_{i_1,j_1} p_{i_2,j_2} \Big(1 - (1 - 1/\lambda_0) q_{i_1,j_1} q_{i_2,j_2} \Big)^{m_1} \right)^{m_2} .$$

For general $r \geq 1$: we will index our parameters p_s, q_s with sequences $s = \big(s^{(1)}, \ldots, s^{(k)}\big)$ of length $|s| = k \leq r - 1$. We denote the empty sequence (of length 0) by $\emptyset$. Furthermore, we write $s' \succ s$ if s' is obtained by adding an element to the end of s, that is, $|s'| = |s| + 1$ and the first $|s|$ elements coincide.

Now let $\emptyset \in S$ be some set of sequences of length at most $r - 1$. We partition S into two parts $S_{\leq r-2} \cup S_{r-1}$ based on whether the length of the sequence is at most $r - 2$ or exactly $r - 1$, respectively.

The discrete version of the r-RSB bound has the following parameters:

- $\lambda_0 > 1$;
- $0 < m_1, \dots, m_{r-1} < 1$;
- $p_s \geq 0$, $s \in S$, satisfying
$$\sum_{s' \succ s} p_{s'} = 1 \text{ for each } s \in S_{\leq r-2};$$
- $q_s \in [0,1]$, $s \in S_{r-1}$.

For any d-tuple $s_1, \dots, s_d$ of sequences of length $r-2$, set

$$R^{\text{star}}_{s_1,\dots,s_d} := \sum_{s'_1 \succ s_1} \cdots \sum_{s'_d \succ s_d} p_{s'_1} \cdots p_{s'_d} \big(1 + (\lambda_0 - 1)(1 - q_{s'_1}) \cdots (1 - q_{s'_d})\big)^{m_1},$$

and then, recursively for $k = r-3, r-4, \dots, 0$, for any d-tuple $s_1, \dots, s_d$ of sequences of length k let

$$R^{\text{star}}_{s_1,\dots,s_d} := \sum_{s'_1 \succ s_1} \cdots \sum_{s'_d \succ s_d} p_{s'_1} \cdots p_{s'_d} \big(R^{\text{star}}_{s'_1,\dots,s'_d}\big)^{m_{r-1-k}}.$$

Similarly, for any pair s_1, s_2 of sequences of length $r-2$, set

$$R^{\text{edge}}_{s_1,s_2} := \sum_{s'_1 \succ s_1} \sum_{s'_2 \succ s_2} p_{s'_1} p_{s'_2} \big(1 - (1 - 1/\lambda_0) q_{s'_1} q_{s'_2}\big)^{m_1},$$

and then, recursively for $k = r-3, r-4, \dots, 0$. for any pair s_1, s_2 of sequences of length k, let

$$R^{\text{edge}}_{s_1,s_2} := \sum_{s'_1 \succ s_1} \sum_{s'_2 \succ s_2} p_{s'_1} p_{s'_2} \big(R^{\text{edge}}_{s'_1,s'_2}\big)^{m_{r-1-k}}.$$

Then the bound is

$$\alpha^*_d \, m_1 \dots m_{r-1} \log(\lambda_0) \leq \log R^{\text{star}}_{\emptyset,\dots,\emptyset} - \frac{d}{2} \log R^{\text{edge}}_{\emptyset,\emptyset}.$$

Remark 1.1. Normally we fix integers $n_1, \dots, n_{r-1} \geq 2$ and assume that the k-th elements of our sequences come from the set $\{1, \dots, n_k\}$. This way the number of free parameters (after taking the sum restrictions on the parameters p_s into account) is

$$(r-1) + 2 n_1 n_2 \cdots n_{r-1}.$$

In the tables we will refer to such a parameter space as $[n_{r-1}, \cdots, n_1]$.

2. Program codes

GitHub repository with our program codes: `https://github.com/harangi/rsb`

Website with examples: `https://www.renyi.hu/~harangi/rsb.htm`

Our codes use the following variables:

$$\begin{aligned} \text{deg} &\longleftrightarrow d \\ \text{depth} &\longleftrightarrow r-1 \\ \text{nrs} &\longleftrightarrow [n_{r-1},\dots,n_1] \\ \text{la} &\longleftrightarrow \lambda_0 \\ \text{ms} &\longleftrightarrow [m_{r-1},\dots,m_1] \\ \text{prs}[k] &\longleftrightarrow \text{numpy array consisting of all } p_s \text{ with } |s|=k+1 \quad (k=0,\dots,r-2) \\ \text{prs}[r-1] &\longleftrightarrow \text{numpy array consisting of all } 1-q_s \text{ with } |s|=r-1 \end{aligned}$$

Appendix

Below we list our best r-RSB bounds of α_d^* for each degree $3 \leq d \leq 19$ in the following format: r $[n_{r-1}, \ldots, n_1]$ bound (see Remark 1.1 for the definition of n_k).

For comparison, we included $r = 1$, that is, the 1-RSB bound from [LO18] that we improve on.

degree: 3

1	[]	0.450859654
2	[32]	0.450789936
3	[8, 4]	0.450786018
4	[8, 2, 2]	0.450785346
5	[4, 2, 2, 2]	0.450785210

degree: 4

1	[]	0.411194564
2	[18]	0.411100755
3	[6, 4]	0.411095101
4	[4, 3, 2]	0.411094131

degree: 5

1	[]	0.379268170
2	[8]	0.379176250
3	[3, 3]	0.379170372
4	[2, 2, 3]	0.379170310

degree: 6

1	[]	0.352984549
2	[7]	0.352905514
3	[4, 2]	0.352900232
4	[3, 2, 2]	0.352899485

degree: 7

1	[]	0.330884354
2	[5]	0.330821477
3	[5, 2]	0.330817014

degree: 8

1	[]	0.311972567
2	[6]	0.311925387
3	[3, 2]	0.311922227

degree: 9

1	[]	0.295553902
2	[5]	0.295520273
3	[2, 2]	0.295519497

degree: 10

1	[]	0.281128003
2	[5]	0.281105186
3	[2, 2]	0.281104953

degree: 11

1	[]	0.268324856
2	[7]	0.268310124

degree: 12

1	[]	0.256864221
2	[5]	0.256855205

degree: 13

1	[]	0.246529415
2	[6]	0.246524236

degree: 14

1	[]	0.237149865
2	[4]	0.237147193

degree: 15

1	[]	0.228589175
2	[4]	0.228587914

degree: 16

1	[]	0.220736776
2	[4]	0.220736278

degree: 17

1	[]	0.213501935208
1+	[]	0.213501905193

degree: 18

1	[]	0.20680939479
2	[2]	0.20680939005

degree: 19

1	[]	0.2005961242697
1+	[]	0.2005961242567

References

[BKZZ13] Jean Barbier, Florent Krzakala, Lenka Zdeborová, and Pan Zhang. The hard-core model on random graphs revisited. *Journal of Physics: Conference Series*, 473:012021, dec 2013.

[DSS16] Jian Ding, Allan Sly, and Nike Sun. Maximum independent sets on random regular graphs. *Acta Math.*, 217(2):263–340, 2016.

[LO18] Marc Lelarge and Mendes Oulamara. Replica bounds by combinatorial interpolation for diluted spin systems. *Journal of Statistical Physics*, 173(3):917–940, Nov 2018.

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