INDEPENDENCE RATIO AND RANDOM EIGENVECTORS IN TRANSITIVE GRAPHS

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Abstract. A theorem of Hoffman gives an upper bound on the independence ratio of regular graphs in terms of the minimum $\lambda_{\min}$ of the spectrum of the adjacency matrix. To complement this result we use random eigenvectors to gain lower bounds in the vertex-transitive case. For example, we prove that the independence ratio of a 3-regular transitive graph is at least

$$q = \frac{1}{2} - \frac{3}{4\pi} \arccos\left(\frac{1-\lambda_{\min}}{4}\right).$$

The same bound holds for infinite transitive graphs: we construct factor of i.i.d. independent sets for which the probability that any given vertex is in the set is at least $q - o(1)$.

We also show that the set of the distributions of factor of i.i.d. processes is not closed provided that the spectrum of the graph is uncountable.

1. Introduction

1.1. **The independence ratio and the minimum eigenvalue.** An *independent set* is a set of vertices in a graph, no two of which are adjacent. The *independence ratio* of a graph G is the size of its largest independent set divided by the total number of vertices. If G is regular, then the independence ratio is at most $1/2$, and it is equal to $1/2$ if and only if G is bipartite.

The adjacency matrix of a d-regular graph has real eigenvalues between $-d$ and d. The least eigenvalue $\lambda_{\min}$ is at least $-d$, and it is equal to $-d$ if and only if the graph is bipartite.

So the distance of the independence ratio from $1/2$ and the distance of $\lambda_{\min}$ from $-d$ both measure how far a d-regular graph is from being bipartite. The following natural question arises: what kind of connection is there between these two graph parameters?

A theorem of Hoffman [5] gives a partial answer to this question. It says that the independence ratio of a d-regular graph is at most

$$\frac{-\lambda_{\min}}{d-\lambda_{\min}} = \frac{1}{2} - \frac{\frac{1}{2}(\lambda_{\min}+d)}{2d-(\lambda_{\min}+d)}. \tag{1}$$

(For a simple proof see [4, Theorem 11]. Also see [9, Section 4] for certain improvements.)

Hoffman's bound implies that $\lambda_{\min} \to -d$ as the independence ratio tends to $1/2$. The converse statement is not true in general: it is easy to construct d-regular graphs with $\lambda_{\min}$ arbitrarily close to $-d$ and the independence ratio separated from $1/2$. However, for transitive graphs the converse is also true. A graph G is said to be *vertex-transitive* (or *transitive* in short) if its automorphism group $\mathrm{Aut}(G)$ acts transitively on the vertex set $V(G)$.

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Theorem 1. *Let G be a finite, d-regular, vertex-transitive graph with least eigenvalue $\lambda_{\min}$. Then the independence ratio of G is at least*

$$\frac{1}{2} - \frac{1}{3}\sqrt{d(\lambda_{\min} + d)}.$$

In particular, if $\lambda_{\min} \to -d$, then the independence ratio converges to $1/2$.

The idea behind the proof is to consider random eigenvectors with eigenvalue $\lambda_{\min}$. Let λ be an arbitrary eigenvalue of the adjacency matrix of some transitive graph G and let E_λ denote the eigenspace corresponding to λ, that is, the space of eigenvectors with eigenvalue λ. (Note that E_λ is typically more than one dimensional, since G is transitive.) Furthermore, let S_λ be the unit sphere in E_λ. Now we pick a uniform random vector from S_λ. Note that S_λ is $\mathrm{Aut}(G)$-invariant, therefore the distribution of this random vector is $\mathrm{Aut}(G)$-invariant, too. Let us choose the vertices v with the property that the value of the eigenvector at v is larger than at each neighbor of v. (If λ is negative, then we expect many of the vertices with positive value to have this property.) Clearly, these vertices form an independent set. Since our random vector is invariant, the probability q that a given vertex is chosen is the same for all vertices. Therefore the expected size of this random independent set is $q|V(G)|$, and consequently, the independence ratio of G is at least q. An estimate of q yields Theorem 1 above. In many cases we obtain much sharper bounds.

When the graph has a lot of symmetry (for example, when any pair of neighbors of a fixed vertex can be mapped to any other pair by a suitable graph automorphism), then the probability q defined above is actually determined by λ. In this case it equals $q_d(\lambda)$, the relative volume of the $d-1$-dimensional regular spherical simplex defined by normal vectors with pairwise scalar product $\frac{d-2-\lambda}{2(d-1)}$ (see Definition 2.7). There is a simple formula for $q_3(\lambda)$, see Theorem 3.

We conjecture that $q \geq q_d(\lambda)$ for arbitrary transitive graphs (provided that λ is sufficiently small). In other words, the worst-case scenario is when the graph has a lot of symmetry. Of course, this would yield a lower bound $q_d(\lambda_{\min})$ for the independence ratio. We managed to prove this conjecture for 3-regular transitive graphs and 4-regular arc-transitive graphs. We also showed that a well-known conjecture in geometry would imply the d-regular, arc-transitive case. (A graph is said to be *arc-transitive* or *symmetric* if for any two pairs of adjacent vertices (u_1, v_1) and (u_2, v_2), there is an automorphism of the graph mapping u_1 to u_2 and v_1 to v_2.) The following theorems were obtained.

Theorem 2. *Suppose that G is a finite, d-regular, arc-transitive graph with least eigenvalue $\lambda_{\min}$. Then the independence ratio of G is at least*

$$\frac{1}{2} - \frac{1}{3}\sqrt{\lambda_{\min} + d}.$$

In fact, a well-known conjecture in geometry (see Conjecture 2.10) would imply that the independence ratio is at least $q_d(\lambda_{\min})$. This has been proven in the case $d = 4$: the independence ratio of a finite, 4-regular, arc-transitive graph is at least

$$q_4(\lambda_{\min}) \geq \frac{1}{2} - \frac{1}{4}\sqrt{\lambda_{\min} + 4}. \tag{2}$$

Theorem 3. *Suppose that G is a finite, 3-regular, vertex-transitive graph with minimum eigenvalue $\lambda_{\min}$. Then the independence ratio of G is at least*

$$q_3(\lambda_{\min}) = \frac{1}{8} + \frac{3}{4\pi}\arcsin\left(\frac{1-\lambda_{\min}}{4}\right) = \frac{1}{2} - \frac{3}{4\pi}\arccos\left(\frac{1-\lambda_{\min}}{4}\right).$$

In fact, the following stronger statement holds: G contains two disjoint independent sets I_1, I_2 with total size $|I_1 \cup I_2| \geq 2q_3(\lambda_{\min})|V(G)|$. This means that the induced subgraph $G[I_1 \cup I_2]$ is bipartite and has at least $2q_3(\lambda_{\min})|V(G)|$ vertices.

See Figure 1 to compare the lower bound given in Theorem 3 to Hoffman's upper bound (1). Note that $-3 \leq \lambda_{\min} \leq -2$ for any 3-regular transitive graph with the only exception of the complete graph K_4 for which $\lambda_{\min} = -1$. (See Proposition 4.3 in the Appendix.)

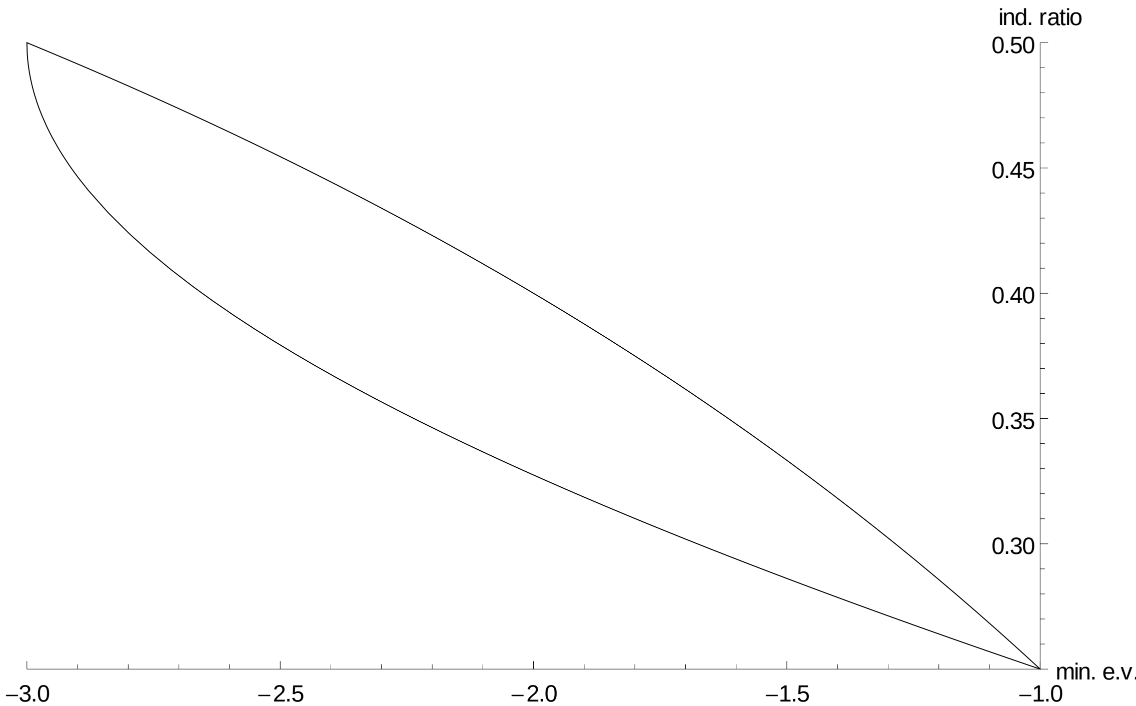

FIGURE 1. Hoffman's upper bound (1) and the lower bound of Theorem 3 for $\lambda_{\min} \in [-3, -1]$

1.2. **Random wave functions on infinite transitive graphs.** In order to generalize the above theorems we define random wave functions on infinite transitive graphs G. A *wave function* with eigenvalue λ on G is a function $f\colon V(G) \to \mathbb{R}$ such that

$$\sum_{u \in N(v)} f(u) = \lambda f(v) \text{ for each vertex } v \in V(G),$$

where $N(v)$ denotes the set of neighbors of v in G. So a wave function is basically an eigenvector of the adjacency operator of G, except that it does not need to be in $\ell_2(V(G))$.

These random wave functions will also let us answer an open question concerning factor of i.i.d. processes. Suppose that we have independent standard normal random variables Z_u assigned to each vertex u of an infinite transitive graph G. By a *factor of i.i.d. process* on G we mean random variables X_v, $v \in V(G)$ that are all obtained as measurable functions of the random variables Z_u, $u \in V(G)$ and that are $\mathrm{Aut}(G)$-equivariant (i.e., they commute with the natural action of $\mathrm{Aut}(G)$). It is easy to see that for any factor of i.i.d. process the correlation of X_v and $X_{v'}$ converges to 0 as the distance of v and v' goes to infinity. So a random process that is 0 everywhere with probability 1/2 and 1 everywhere with probability 1/2 cannot be a factor of i.i.d. However, it can be seen easily that this process can be approximated by factor of i.i.d. processes provided that G is amenable. So the space of factor of i.i.d. processes is not closed, that is, the distributions of these processes do not form a closed set w.r.t. the weak topology. It has been an open question whether the same is true on non-amenable graphs, for example, on the d-regular tree, see [1, Section 4, Question 4]. We will show that the space of factor of i.i.d. processes is not closed provided that the spectrum of G is uncountable.

We say that a factor of i.i.d. process X_v, $v \in V(G)$ is a *linear factor of i.i.d.* if each X_v is obtained as a (possibly infinite) linear combination of Z_u, $u \in V(G)$. Note that linear factors have the following properties.

Definition 1.1. We call a collection of random variables X_v, $v \in V(G)$ a *Gaussian process* on G if they are jointly Gaussian and each X_v is centered (i.e., has mean 0). (Random variables are jointly Gaussian if any finite linear combination of them is Gaussian.) We say that a Gaussian process X_v is $\mathrm{Aut}(G)$-invariant (or simply invariant) if for any $\Phi \in \mathrm{Aut}(G)$ the joint distribution of the Gaussian process $X_{\Phi(v)}$ is the same as that of the original process.

We will prove that the adjacency operator A_G has approximate eigenvectors (satisfying a certain invariance property) for any λ in the spectrum $\lambda \in \sigma(A_G)$. Then we will use these approximate eigenvectors as coefficients to define linear factor of i.i.d. processes converging in distribution to an invariant Gaussian process X_v that satisfies the eigenvector equation at each vertex.

Theorem 4. *Let G be an infinite vertex-transitive graph with adjacency operator A_G. Then for each point λ of the spectrum $\sigma(A_G)$ there exists a nontrivial invariant Gaussian process X_v, $v \in V(G)$ such that*

$$\sum_{u \in N(v)} X_u = \lambda X_v \text{ for each vertex } v \in V(G), \tag{3}$$

where $N(v)$ denotes the set of neighbors of v in G. Furthermore, the process X_v can be approximated (in distribution) by linear factor of i.i.d. processes. Clearly, we can assume that these approximating linear factors have only finitely many nonzero coefficients.

An invariant Gaussian process satisfying (3) will be called a *Gaussian wave function* with eigenvalue λ. If the spectrum of G is not countable, then we can conclude that some of these Gaussian wave functions cannot be obtained as factor of i.i.d. processes.

Theorem 5. *Let G be an infinite transitive graph such that the spectrum of the adjacency oparator A_G is not countable. Then there exist (linear) factor of i.i.d. processes on G with the property that the weak limit of their distributions cannot be obtained as the distribution of a factor of i.i.d. process.*

We can say more for Cayley graphs.

Theorem 6. *Suppose that G is the Cayley graph of a finitely generated infinite group. Then a Gaussian wave function with eigenvalue $\lambda_{\max} \stackrel{def}{=} \sup \sigma(A_G)$ can never be obtained as the distribution of a factor of i.i.d. process.*

In view of Theorems 4 and 6 there exists a Gaussian wave function with eigenvalue $\lambda_{\max}$ that can be approximated by factor of i.i.d. processes but cannot be obtained as one. An independent and different proof of this result was given by Russell Lyons in the special case when G is a regular tree (personal communication).

1.3. **Factor of i.i.d. independent sets.** Let X_v, $v \in V(G)$ be a random process on our infinite transitive graph G. As in the finite setting, $I_+ \stackrel{\text{def}}{=} \{v : X_v > X_u, \forall u \in N(v)\}$ is a random independent set. If our process is invariant, then the probability that $v \in I_+$ is the same for each vertex v, and thus this probability can be used to measure the size of I_+. If our process is a factor of some i.i.d. process Z_v, then the resulting independent set is also a factor of Z_v.

In the infinite setting let $\lambda_{\min}$ denote the minimum of the spectrum $\sigma(A_G)$ and let X_v be a linear factor of Z_v approximating the Gaussian eigenvector with eigenvalue $\lambda_{\min}$ (see Theorem 4). As the process X_v converges in distribution to the Gaussian eigenvector, the probability $P(v \in I_+)$ approaches the corresponding probability for the Gaussian eigenvector process, which, as we will see, can be computed the exact same way as in the finite case.

Theorem 7. *Theorems 1, 2 and 3 give lower bounds q (in terms of $\lambda_{\min}$) for the independence ratio of finite transitive graphs with least eigenvalue $\lambda_{\min}$. These bounds remain true in the following framework. Let $\lambda_{\min}$ denote the minimum of the spectrum of an infinite transitive graph G. Then for any $\varepsilon > 0$ there exists a factor of i.i.d. independent set on G such that the probability that any given vertex is in the set is at least $q - \varepsilon$.*

A special case of this infinite setting was investigated in [3]. When G is the d-regular tree T_d, then any factor of i.i.d. independent set on G automatically gives a lower bound for the independence ratio of d-regular finite graphs with sufficiently large girth. In particular, for the 3-regular tree T_3 one has $\lambda_{\min} = -2\sqrt{2}$. Therefore the infinite version of Theorem 3 tells us that there exists factor of i.i.d. independent set in T_3 with density

$$\frac{1}{2} - \frac{3}{4\pi} \arccos\left(\frac{1+2\sqrt{2}}{4}\right) \approx 0.4298.$$

In [3] the somewhat better bound 0.4361 was obtained, which is the current best. In fact, [3] was the starting point for the work in the present paper. For previous results on the independence ratio of large-girth graphs see [2, 10, 11, 12, 8, 6].

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2. Finite vertex-transitive graphs

Throughout this section G will denote a vertex-transitive, finite graph with degree d for some positive integer $d \geq 3$. The least eigenvalue of its adjacency matrix A_G will be denoted by $\lambda_{\min}$. For now let λ be an arbitrary eigenvalue of A_G. Eventually, we will choose λ as the minimum eigenvalue. First we define what we mean by a random eigenvector.

Definition 2.1. Let E_λ be the eigenspace corresponding to λ, that is,

$$E_\lambda \stackrel{\text{def}}{=} \{x \in \ell_2(V(G)) \ : \ A_G x = \lambda x\}.$$

We fix some orthonormal basis $e_1, \ldots, e_l$ in E_λ, and take independent standard normal random variables $\gamma_1, \ldots, \gamma_l$. We call $\sum_{i=1}^{l} \gamma_i e_i$ the *random eigenvector with eigenvalue* λ.

Remark 2.2. The (distribution of the) random eigenvector is clearly independent of the choice of the basis $e_1, \ldots, e_l$, so it is well defined. It also follows that the distribution of the random eigenvector is $\mathrm{Aut}(G)$-invariant. (Note that in the introduction we defined the random eigenvector differently: a uniform random vector on the unit sphere of E_λ, which is just the normalized version of the random eigenvector of Definition 2.1.)

We will think of this random eigenvector as a collection of real-valued random variables X_v, $v \in V(G)$ with the property that they are jointly Gaussian and $\mathrm{Aut}(G)$-invariant, each X_v is centered, and

$$\sum_{u \in N(v)} X_u = \lambda X_v \text{ for each vertex } v,$$

where $N(v)$ denotes the set of neighbors of v in G. Since G is transitive, each X_v has the same variance. After multiplying these random variables with a suitable positive constant we might assume that $\mathrm{var}(X_v) = 1$ for each vertex v. Next we define random independent sets by means of these random eigenvectors.

Definition 2.3. Let

$$I_+ = I_+^\lambda \stackrel{\text{def}}{=} \{v \in V(G) \ : \ X_v > X_u \text{ for each } u \in N(v)\}, \text{ and}$$
$$I_- = I_-^\lambda \stackrel{\text{def}}{=} \{v \in V(G) \ : \ X_v < X_u \text{ for each } u \in N(v)\}.$$

Clearly, I_+ and I_- are disjoint (random) independent sets in G.

The $\mathrm{Aut}(G)$-invariance implies that the probability of the event $v \in I_+$ is the same for all vertices v. So from now on, we will focus on a fixed vertex v (that we will call the root) and its neighbors $u_1, \ldots, u_d$. For X_v and X_{u_i} we will simply write X and Y_i, respectively. Therefore we have

$$\sum_{i=1}^{d} Y_i = \lambda X. \tag{4}$$

Let us denote the covariance $\mathrm{cov}(Y_i, Y_j)$ by $c_{i,j}$. It follows from (4) that

$$\lambda^2 = \mathrm{cov}(\lambda X, \lambda X) = \sum_{i,j} c_{i,j} = d + 2 \sum_{i<j} c_{i,j}, \text{ thus } \sum_{i<j} c_{i,j} = \frac{\lambda^2 - d}{2}. \tag{5}$$

Setting $U_i \stackrel{\text{def}}{=} X - Y_i$ we have

$$P(v \in I_+) = P(U_i > 0, 1 \le i \le d).$$

As we will see, this probability can be expressed as the volume of a certain spherical simplex.

Definition 2.4. Let S^{d-1} denote the unit sphere in $\mathbb{R}^d$. A half-space is said to be *homogeneous* if the defining hyperplane (i.e., the boundary of the half-space) passes through the origin. A vector n orthogonal to the defining hyperplane and "pointing outward" is called an *outer normal vector*. Then the given (open) half-space consists of those $x \in \mathbb{R}^d$ for which the inner product $n \cdot x$ is negative.

A $d-1$-dimensional *spherical simplex* is the intersection of S^{d-1} and d homogeneous half-spaces in $\mathbb{R}^d$. Up to congruence, a spherical simplex is determined by the $\binom{d}{2}$ pairwise angles enclosed by the outer normal vectors of the d half-spaces. If these $\binom{d}{2}$ angles are all equal, then we say that the spherical simplex is *regular*.

Since $Y_1, \ldots, Y_d$ are centered and jointly Gaussian, they can be written as the linear combinations of independent standard normal variables: there exist independent standard

Gaussians $Z_1, \dots, Z_d$ and (deterministic) vectors $y_1, \dots, y_d \in \mathbb{R}^d$ such that Y_i is the inner product of y_i and $Z = (Z_1, \dots, Z_d)$. Setting $x = (y_1 + \dots + y_d)/\lambda$ and $u_i = x - y_i$ we have

$$Y_i = y_i \cdot Z; X = x \cdot Z; U_i = u_i \cdot Z.$$

It is easy to see that for any deterministic vectors $a, b \in \mathbb{R}^d$ the covariance $\operatorname{cov}(a \cdot Z, b \cdot Z)$ is equal to the inner product $a \cdot b$. In particular,

$$x \cdot x = \operatorname{var}(X) = 1;\ y_i \cdot y_j = \operatorname{cov}(Y_i, Y_j) = c_{i,j};\ u_i \cdot u_j = \operatorname{cov}(U_i, U_j). \tag{6}$$

In this formulation the event $U_i > 0$ is that the random point Z lies in the homogeneous open half-space with outer normal vector $-u_i$. So the probability in question is equal to the measure of the intersection of the homogeneous half-spaces with outer normal vectors $-u_i$ with respect to the standard multivariate Gaussian measure on $\mathbb{R}^d$. This is simply the volume of the corresponding $d-1$-dimensional spherical simplex divided by the volume $\operatorname{vol}(S^{d-1})$ of the unit sphere S^{d-1}, which is determined by the pairwise angles

$$\varphi_{i,j} \stackrel{\text{def}}{=} \angle(u_i, u_j) = \arccos\left(\frac{u_i \cdot u_j}{\|u_i\| \|u_j\|}\right), \tag{7}$$

which, in turn, can be expressed using the inner products $y_i \cdot y_j = c_{i,j}$.

The probability $P(v \in I_+)$ seems to be the smallest when G has a lot of symmetry. To make this more precise, we first define what we mean by a "lot of symmetry".

Definition 2.5. We say that G is *cherry-transitive* if any cherry (path of length 2) in G can be mapped to any other cherry using a suitable graph automorphism of G.

Proposition 2.6. *If G is cherry-transitive, then*

$$c_{i,j} = \frac{\lambda^2 - d}{d(d-1)} \text{ for all } i \neq j,$$

and, consequently, the pairwise angles $\varphi_{i,j}$ are all equal to

$$\arccos\left(\frac{d-2-\lambda}{2(d-1)}\right). \tag{8}$$

Proof of Proposition 2.6. If G is cherry-transitive, then for any $i_1 \neq j_1$ and $i_2 \neq j_2$ there exists an automorphism $\Phi \in \operatorname{Aut}(G)$ such that Φ fixes the root v and takes the unordered pair u_{i_1}, u_{j_1} to u_{i_2}, u_{j_2}, that is,

$$\Phi v = v, \Phi u_{i_1} = \Phi u_{i_2}, \Phi u_{j_1} = \Phi u_{j_2} \text{ or } \Phi v = v, \Phi u_{i_1} = \Phi u_{j_2}, \Phi u_{j_1} = \Phi u_{i_2}.$$

Together with the $\operatorname{Aut}(G)$-invariance of the random eigenvector this implies that $c_{i_1,j_1} = c_{i_2,j_2}$. Since this holds for any two pairs of indices, it follows that all $c_{i,j}$, $i \neq j$ are the same. Using (5) we conclude that for $i \neq j$

$$c_{i,j} = \frac{\lambda^2 - d}{d(d-1)}.$$

Then easy calculation shows (using notations introduced earlier) that

$$u_i \cdot u_j = \frac{2(d-\lambda)}{d} \text{ and } \|u_i\|^2 = \|u_j\|^2 = \frac{(d-\lambda)(d-2-\lambda)}{d(d-1)}.$$

Plugging this into (7) gives

$$\varphi_{i,j} = \arccos\left(\frac{d-2-\lambda}{2(d-1)}\right).$$

□

We are now in a position to define the functions $q_d(\lambda)$.

Definition 2.7. For $-d \le \lambda \le d$ let $q_d(\lambda)$ denote the volume of the $d-1$-dimensional regular spherical simplex corresponding to the angle (8) divided by $\mathrm{vol}(S^{d-1})$. Then $P(v \in I_+) = q_d(\lambda)$ for any cherry-transitive G. In particular, the independence ratio of any cherry-transitive graph G is at least $q_d(\lambda_{\min})$.

So $P(v \in I_+) = q_d(\lambda)$ provided that G has enough symmetry. The following conjecture says that in the general (i.e., vertex-transitive) case the probability should be larger than that.

Conjecture 2.8. *For any transitive graph G it holds that*

$$P(v \in I_+) \ge q_d(\lambda)$$

for any λ, or at least for sufficiently small λ: $\lambda \le \lambda_0$ for some λ_0.

This would, of course, imply that the independence ratio of G is at least $q_d(\lambda_{\min})$ provided that $\lambda_{\min} \le \lambda_0$.

We will prove this conjecture for $d = 3$ and $\lambda_0 = -2$ in Section 2.1. The conjecture might be true for arbitrary λ, but proving for $\lambda \le \lambda_0 = -2$ will be sufficient for our purposes, because $\lambda_{\min} \le -2$ for any 3-regular transitive graph except K_4.

A few properties of the functions $q_d(\lambda)$ are collected in the next proposition.

Proposition 2.9. *For any $d \ge 3$ q_d is a monotone decreasing continuous function on $[-d, -1]$ with*

$$q_d(-d) = \frac{1}{2} \text{ and } q_d(-1) = \frac{1}{d+1}.$$

As for the behavior of q_d around $-d$ we have

$$q_d(\lambda) \ge \frac{1}{2} - \frac{\pi \, \mathrm{vol}(S^{d-2})}{4 \, \mathrm{vol}(S^{d-1})} \sqrt{\frac{\lambda + d}{d}} \ge \frac{1}{2} - \frac{1}{3}\sqrt{\lambda + d}.$$

Proof. Monotonicity and continuity follow readily from the definition of q_d.

For $\lambda = -d$ the angles $\varphi_{i,j}$ are 0, so the corresponding (degenerate) spherical simplex is a hemisphere, thus $q_d(-d) = 1/2$ as claimed.

For $\lambda = -1$ the angles $\varphi_{i,j}$ are $\pi/3$. It is not hard to see that the vertices of our spherical simplex in that case will be the d vertices of a face of a regular (Euclidean) simplex in $\mathbb{R}^d$. Then each of the $d+1$ spherical simplices belonging to the $d+1$ faces has volume $\mathrm{vol}(S^{d-1})/(d+1)$. (We could also argue that for $G = K_{d+1}$ and $\lambda = -1$ we have $P(v \in I_+) = 1/(d+1)$, and since K_{d+1} is cherry-transitive, $P(v \in I_+) = q_d(-1)$.)

See Section 2.3 for a proof of the claimed behavior around $-d$. □

2.1. **The 3-regular, vertex-transitive case.** Now we turn to the proof of Theorem 3 that gives a lower bound for the independence ratio of 3-regular transitive graphs. We will basically show that Conjecture 2.8 is true when $d = 3$ and $\lambda_0 = -2$.

For $d = 3$ the surface of the unit sphere $S^{d-1} = S^2$ is 4π and the area of a spherical triangle is $\alpha + \beta + \gamma - \pi$, where α, β, γ are the angles enclosed by the sides of the spherical triangle. As we have seen, the probability $P(v \in I_+)$ equals the area of a certain spherical triangle divided by 4π. The angles of the spherical triangle in question are $\pi - \varphi_{1,2}$, $\pi - \varphi_{1,3}$

and $\pi - \varphi_{2,3}$. Therefore

$$P(v \in I_+) = \frac{1}{4\pi}\left(\sum_{1\le i<j\le 3} (\pi - \varphi_{i,j}) - \pi\right) = \frac{1}{4\pi}\left(\frac{\pi}{2} + \sum_{1\le i<j\le 3} (\frac{\pi}{2} - \varphi_{i,j})\right) = \frac{1}{4\pi}\left(\frac{\pi}{2} + \sum_{1\le i<j\le 3} \arcsin\left(\frac{u_i\cdot u_j}{\|u_i\|\|u_j\|}\right)\right). \tag{9}$$

By Proposition 2.6 we have $c_{i,j} = (\lambda^2-3)/6$ and $\varphi_{i,j} = \arccos((1-\lambda)/4)$ in the cherry-transitive case, thus

$$q_3(\lambda) = \frac{1}{8} + \frac{3}{4\pi}\arcsin\left(\frac{1-\lambda}{4}\right) = \frac{1}{2} - \frac{3}{4\pi}\arccos\left(\frac{1-\lambda}{4}\right). \tag{10}$$

Proof of Theorem 3. The statement of the theorem is true for the complete graph K_4 as the independence ratio is $1/4$ and the minimum eigenvalue is -1 in that case. For any other 3-regular transitive graph G we have $\lambda_{\min} \le -2$. (See Proposition 4.3 in the Appendix.) Therefore it suffices to prove that $P(v \in I_+) \ge q_3(\lambda)$, whenever $\lambda \le -2$.

Recall that Y_1, Y_2, Y_3 are standard Gaussians with pairwise covariances $c_{i,j}$. Therefore the matrix

$$\begin{pmatrix} 1 & c_{1,2} & c_{1,3} \\ c_{1,2} & 1 & c_{2,3} \\ c_{1,3} & c_{2,3} & 1 \end{pmatrix}$$

is positive semidefinite. In particular, its determinant is nonnegative:

$$1 + 2c_{1,2}c_{1,3}c_{2,3} - c_{1,2}^2 - c_{1,3}^2 - c_{2,3}^2 \ge 0.$$

Furthermore, according to (5) we have $c_{1,2}+c_{1,3}+c_{2,3} = (\lambda^2-3)/2 \ge 1/2$, because $\lambda \le -2$. It follows that each $c_{i,j}$ must be between $-1/2$ and 1.

Indeed, let x, y, z be real numbers between -1 and 1 with $x+y+z \ge 1/2$ and $1+2xyz-x^2-y^2-z^2 \ge 0$. Assume that $z < -1/2$. Then

$$0 \le 1 + 2xyz - x^2 - y^2 - z^2 = 1 + 2(z+1)xy - (x+y)^2 - z^2 \le$$
$$1+2(z+1)\left(\frac{x+y}{2}\right)^2 - (x+y)^2 - z^2 = 1 + \frac{z-1}{2}(x+y)^2 - z^2 \le 1 + \frac{z-1}{2}\left(\frac{1}{2} - z\right)^2 - z^2 < 0,$$

contradiction. Therefore $z \ge -1/2$. Similarly, $x, y \ge -1/2$, too.

Next we bound $u_i \cdot u_j/(\|u_i\|\|u_j\|)$ from below. Using (6), $x = (y_1+y_2+y_3)/\lambda$ and $c_{1,2}+c_{1,3}+c_{2,3} = (\lambda^2-3)/2$:

$$x \cdot y_1 = \frac{1}{\lambda}(1 + c_{1,2} + c_{1,3}) = \frac{1}{\lambda}\left(1 + \frac{\lambda^2-3}{2} - c_{2,3}\right) = \frac{\lambda}{2} - \frac{1}{2\lambda} - \frac{1}{\lambda}c_{2,3},$$
$$\|u_1\|^2 = \|x - y_1\|^2 = 2 - 2x\cdot y_1 = 2 - \lambda + \frac{1}{\lambda} + \frac{2}{\lambda}c_{2,3}.$$

Similar formulas hold for $x \cdot y_i$ and $\|u_i\|$, $i = 2, 3$. By the inequality of arithmetic and geometric means it follows that

$$\|u_1\|\|u_2\| \le \frac{\|u_1\|^2 + \|u_2\|^2}{2} = 2 - \lambda + \frac{1}{\lambda} + \frac{1}{\lambda}(c_{1,3} + c_{2,3}) = \frac{-1}{\lambda}\left(\frac{1}{2} - 2\lambda + \frac{\lambda^2}{2} + c_{1,2}\right).$$

Note that this holds with equality when all $c_{i,j}$ are equal. Furthermore,

$$u_1 \cdot u_2 = (x-y_1)\cdot(x-y_2) = 1 + c_{1,2} - x\cdot(y_1+y_2) = 1 + c_{1,2} + x\cdot(y_3 - \lambda x) =$$
$$1 + c_{1,2} + \left(\frac{\lambda}{2} - \frac{1}{2\lambda} - \frac{1}{\lambda}c_{1,2}\right) - \lambda = \frac{-1}{\lambda}\left(\frac{1}{2} - \lambda + \frac{\lambda^2}{2} + (1-\lambda)c_{1,2}\right).$$

It follows that

$$\frac{u_1\cdot u_2}{\|u_1\|\|u_2\|} \geq \frac{\frac{1}{2} - \lambda + \frac{\lambda^2}{2} + (1-\lambda)c_{1,2}}{\frac{1}{2} - 2\lambda + \frac{\lambda^2}{2} + c_{1,2}},$$

because the numerator is positive (note that $-3 \leq \lambda \leq -2$ and $c_{1,2} \geq -1/2$). The analogous inequality holds for any other pair of indices i,j. Since arcsin is a monotone increasing function, (9) yields that

$$P(v\in I_+) \geq \frac{1}{8} + \frac{1}{4\pi}\sum_{1\leq i<j\leq 3} \arcsin\left(\frac{\frac{1}{2} - \lambda + \frac{\lambda^2}{2} + (1-\lambda)c_{i,j}}{\frac{1}{2} - 2\lambda + \frac{\lambda^2}{2} + c_{i,j}}\right).$$

Setting

$$f(t) \overset{\text{def}}{=} \arcsin\left(\frac{\frac{1}{2} - \lambda + \frac{\lambda^2}{2} + (1-\lambda)t}{\frac{1}{2} - 2\lambda + \frac{\lambda^2}{2} + t}\right),$$

we have

$$P(v\in I_+) \geq \frac{1}{8} + \frac{1}{4\pi}\sum_{1\leq i<j\leq 3} f(c_{i,j}). \tag{11}$$

On the other hand,

$$q_3(\lambda) = \frac{1}{8} + \frac{3}{4\pi} f\left(\frac{\lambda^2-3}{6}\right), \tag{12}$$

which follows from (10) and the definition of f. (It also follows from the fact that when each $c_{i,j}$ is equal to $(\lambda^2-3)/6$, then (11) should hold with equality.) In view of (11) and (12) we need to show that

$$\frac{1}{3}\sum_{1\leq i<j\leq 3} f(c_{i,j}) \geq f\left(\frac{\lambda^2-3}{6}\right), \tag{13}$$

where each $c_{i,j}$ is between $-1/2$ and 1 and their average is $(\lambda^2-3)/6$. This, of course, would follow from the convexity of f. Unfortunately, f is not convex on the entire interval $[-1/2,1]$. We claim, however, that the tangent line to f at $t_0 = (\lambda^2-3)/6$ is below f on the entire interval $[-1/2,1]$, which still implies (13). The rather technical proof of this claim can be found in the Appendix (Lemma 4.6).

Now let $\lambda = \lambda_{\min} \leq -2$, then $P(v\in I_+) \geq q_3(\lambda_{\min})$. So the expected size of the random independent set I_+ is at least $q_3(\lambda_{\min})|V(G)|$, thus the independence ratio of G is at least $q_3(\lambda_{\min})$.

To prove the second part of the statement we notice that the random independent set I_- (see Definition 2.3) has the same expected size. Indeed, if we replace X_v, $v\in V(G)$ with $X'_v = -X_v$, then X'_v, $v\in V(G)$ have the same joint distribution and the roles of I_+ and I_- interchange. Since I_+ and I_- are always disjoint, the expected size of their union $I_+\cup I_-$ is at least $2q_3(\lambda_{\min})|V(G)|$. Consequently, there must exist disjoint independent sets I_1, I_2 in G with $|I_1\cup I_2| \geq 2q_3(\lambda_{\min})|V(G)|$. ∎

For graphs with very large odd-girth Theorem 4.1 of the Appendix gives a slightly better bound. The proof is based on the same random eigenvector but uses a different method to find large independent sets.

2.2. **The arc-transitive case.** The following innocent-looking, and very plausible, conjecture is open in dimension $n \geq 4$.

Conjecture 2.10. *Let S be a sphere in the n-dimensional Euclidean space $\mathbb{R}^n$. We have $n+1$ spherical caps with the same given radius on S. We want to find the configuration for which the volume of the union of the caps is maximal. It is conjectured that this optimal configuration is always the one where the $n+1$ centers are the vertices of a regular simplex in $\mathbb{R}^n$.*

The statement of the conjecture is trivial for $n = 2$, while the $n = 3$ case follows from the so-called Moment Theorem of L. Fejes Tóth [14, Theorem 2].

In what follows we will explain how the case $n = d-1$ of the above conjecture implies that $P(v \in I_+) \geq q_d(\lambda)$ holds for every d-regular arc-transitive graph G, and consequenlty the independence ratio of G is at least $q_d(\lambda_{\min})$. In particular, the $d = 4$ case follows from the $n = 3$ case of the conjecture which is known to be true, see Theorem 2. Using our previous notations, $P(v \in I_+)$ is the volume of the spherical simplex T determined by the half-spaces with outer normal vectors $-u_i$, $i = 1, \ldots, d$, while $q_d(\lambda)$ is the volume of the same simplex in the case when all the angles $\varphi_{i,j} = \angle(u_i, u_j)$, $i \neq j$ are the same. In other words, we need to show that the volume of the spherical simplex T is minimal when the angles $\angle(u_i, u_j)$ are the same.

If G is arc-transitive, then the covariances $\operatorname{cov}(X, Y_i) = x \cdot y_i$ are all equal. Since

$$x \cdot y_1 + \cdots + x \cdot y_d = x \cdot (y_1 + \cdots + y_d) = x \cdot (\lambda x) = \lambda,$$

we get that $x \cdot y_i = \lambda/d$ for each i. It follows that the angle enclosed by x and u_i

$$\angle(x, u_i) = \delta \stackrel{\text{def}}{=} \frac{\pi - \arccos(\lambda/d)}{2} = \frac{\arccos(-\lambda/d)}{2} \text{ for each } i. \tag{14}$$

Now let S_l be the set of points on S^{d-1} that has some fixed distance l from x, thus S_l is a $d-2$-dimensional sphere for any l. The intersection of S_l and the half-space with outer normal vector u_i is a spherical cap of radius depending only on l and λ. So the intersection of S_l and our spherical simplex T can be obtained by removing d spherical caps of the same given radius from S_l. If Conjecture 2.10 is true for $n = d-1$, then the total volume of the removed area is maximal for the "regular configuration" when each $\angle(u_i, u_j)$ is the same. Therefore the $d-2$-dimensional volume of $T \cap S_l$ is minimal for the regular configuration for any l. It follows that the $d-1$-dimensional volume of T is also minimal for the regular configuration, and this is what we wanted to prove.

2.3. **Bounds near $-d$.** Even if Conjecture 2.10 is not assumed to be true, the above observations yield a lower bound for the independence ratio of d-regular arc-transitive graphs in the case when the least eigenvalue is close to $-d$. As we have seen in (14), $\angle(x, u_i) = \delta$ for each i, which means that each point of S^{d-1} at (spherical) distance less than $\pi/2 - \delta$ from x is contained in our spherical simplex T. These points form a spherical cap with center x and radius $\pi/2 - \delta$. (In fact, this spherical cap is the "inscribed ball" of T.) Using (14) and that $\arccos(t) \leq \pi/2\sqrt{1-t}$ for any $t \in [0, 1]$, we get

$$\delta = \frac{\arccos(-\lambda/d)}{2} \leq \frac{\pi}{4}\sqrt{1 + \lambda/d}$$

provided that $\lambda \le 0$.

This spherical cap can be obtained by taking the hemisphere (around x) and removing a strip of "width" δ (in spherical distance). The volume of this strip is clearly at most $\delta \operatorname{vol}(S^{d-2})$, therefore the volume of the spherical cap is at least $\operatorname{vol}(S^{d-1})/2 - \delta \operatorname{vol}(S^{d-2})$, whence

$$P(v \in I_+) \ge \frac{\operatorname{vol}(S^{d-1})/2 - \delta \operatorname{vol}(S^{d-2})}{\operatorname{vol}(S^{d-1})} = \frac{1}{2} - \frac{\pi \operatorname{vol}(S^{d-2})}{4 \operatorname{vol}(S^{d-1})} \sqrt{\frac{\lambda + d}{d}}.$$

For $d = 4$ we have $\operatorname{vol}(S^2)/\operatorname{vol}(S^3) = (4\pi)/(2\pi^2) = 2/\pi$, so the bound is

$$\frac{1}{2} - \frac{1}{4}\sqrt{\lambda + 4}.$$

For general d, we use the estimate $\operatorname{vol}(S^{d-2})/\operatorname{vol}(S^{d-1}) \le \sqrt{d}/\sqrt{2\pi}$ (see Lemma 4.5 of the Appendix) to obtain the following bound

$$\frac{1}{2} - \frac{\sqrt{\pi}}{4\sqrt{2}}\sqrt{\lambda + d} > \frac{1}{2} - \frac{1}{3}\sqrt{\lambda + d}.$$

These are lower bounds for the probability $P(v \in I_+)$, in particular, for $q_d(\lambda)$. Thus the first part of Theorem 2 follows, as well as the estimate (2) for $q_4(\lambda)$ and the last statement of Proposition 2.9.

We can even say something in the general (vertex-transitive) case. Using $x \cdot y_1 + \dots + x \cdot y_d = \lambda$ and $x \cdot y_j \ge -1$:

$$x \cdot y_i \le \lambda + d - 1 \text{ for each } 1 \le i \le d.$$

Therefore the angle $\angle(x, y_i)$ is at least $\arccos(\lambda + d - 1)$. Using that $\arccos(t) \le \pi/2\sqrt{1-t}$ for any $t \in [0,1]$, it follows that

$$\angle(x, u_i) \le \delta' \overset{\text{def}}{=} \frac{\pi - \arccos(\lambda + d - 1)}{2} = \frac{\arccos(1 - \lambda - d)}{2} \le \frac{\pi}{4}\sqrt{\lambda + d}$$

provided that $\lambda \le -d + 1$. This means that our spherical simplex T contains the spherical cap with center x and radius $\pi/2 - \delta'$. Therefore

$$P(v \in I_+) \ge \frac{\operatorname{vol}(S^{d-1})/2 - \delta' \operatorname{vol}(S^{d-2})}{\operatorname{vol}(S^{d-1})} = \frac{1}{2} - \frac{\pi \operatorname{vol}(S^{d-2})}{4 \operatorname{vol}(S^{d-1})}\sqrt{\lambda + d} \ge \frac{1}{2} - \frac{\sqrt{\pi}}{4\sqrt{2}}\sqrt{d(\lambda + d)}.$$

Since $\sqrt{\pi}/(4\sqrt{2}) < 1/3$, Theorem 1 follows.

3. Infinite transitive graphs

3.1. **Random wave functions.** Our goal now is to generalize the random eigenvectors we introduced in Section 2 for infinite transitive graphs G. For an infinite graph G the adjacency operator $A_G : \ell_2(V(G)) \to \ell_2(V(G))$ might not have any eigenvectors (i.e., the point spectrum might be empty). So the approach we used in the finite setting will not work here. Instead, we will define random wave functions as the limit of linear factor of i.i.d. processes. The coefficients of these linear factors will be approximate eigenvectors of A_G that are invariant under automorphisms fixing some root $x \in V(G)$. We start with proving that such approximate eigenvectors exist for any λ in the spectrum $\sigma(A_G)$. Let $\operatorname{Stab}_x(G)$ denote the *stabilizer subgroup*, that is, the group of automorphisms fixing x.

Theorem 3.1. *Let G be an infinite vertex-transitive graph with adjacency operator A_G and with some fixed root x. Then for any $\varepsilon > 0$ and any λ in the spectrum $\sigma(A_G)$ there exists a $\mathrm{Stab}_x(G)$-invariant vector $\alpha \in \ell_2(V(G))$ such that*

$$\|\alpha\| = 1 \text{ and } \|A_G\alpha - \lambda\alpha\| \leq \varepsilon.$$

Proof. Consider the projection-valued measure P_λ corresponding to the self-adjoint operator A_G. This "measure" assigns an orthogonal projection P_S to each Borel set $S \subseteq \mathbb{R}$. According to spectral theory, one can integrate with respect to this measure. For instance, the following formula holds:

$$A_G = \int_{\mathbb{R}} \lambda \, \mathrm{d}P_\lambda.$$

Furthermore, the projections P_S have the property that if an operator T commutes with A_G, then it also commutes with each projection P_S. There is a unitary operator U_Φ corresponding to each $\Phi \in \mathrm{Aut}(G)$ (the one that permutes the coordinates of $\ell_2(V(G))$ according to Φ). Since U_Φ commutes with A_G, it also commutes with the projections P_S.

Now let λ_0 be an arbitrary element of the spectrum $\sigma(A_G)$ and set $S = [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$. We define α as the image of the indicator function $\mathbb{1}_x$ under the projection P_S:

$$\alpha \stackrel{\mathrm{def}}{=} P_{[\lambda_0-\varepsilon,\lambda_0+\varepsilon]}\mathbb{1}_x.$$

Note that $\mathbb{1}_x$ is a fixed point of U_Φ for any $\Phi \in \mathrm{Stab}_x(G)$, therefore

$$U_\Phi\alpha = U_\Phi P_{[\lambda_0-\varepsilon,\lambda_0+\varepsilon]}\mathbb{1}_x = P_{[\lambda_0-\varepsilon,\lambda_0+\varepsilon]}U_\Phi\mathbb{1}_x = P_{[\lambda_0-\varepsilon,\lambda_0+\varepsilon]}\mathbb{1}_x = \alpha,$$

thus α is $\mathrm{Stab}_x(G)$-invariant. On the other hand, since $P_S P_{\mathbb{R}\setminus S} = 0$, we have

$$A_G\alpha - \lambda_0\alpha = \left(\int_{\mathbb{R}} \lambda - \lambda_0 \, \mathrm{d}P_\lambda\right)\alpha = \left(\int_{[\lambda_0-\varepsilon,\lambda_0+\varepsilon]} \lambda - \lambda_0 \, \mathrm{d}P_\lambda\right)\alpha,$$

which clearly implies that

$$\|A_G\alpha - \lambda_0\alpha\| \leq \varepsilon\|\alpha\|.$$

It remains to show that $\alpha = P_{[\lambda_0-\varepsilon,\lambda_0+\varepsilon]}\mathbb{1}_x \neq 0$. Assume that $P_{[\lambda_0-\varepsilon,\lambda_0+\varepsilon]}\mathbb{1}_x = 0$. It follows that $P_{[\lambda_0-\varepsilon,\lambda_0+\varepsilon]}\mathbb{1}_v = 0$ for every vertex $v \in V(G)$. Indeed, let $\Phi \in \mathrm{Aut}(G)$ such that $\Phi x = v$. Then $U_\Phi\mathbb{1}_x = \mathbb{1}_v$ and

$$P_{[\lambda_0-\varepsilon,\lambda_0+\varepsilon]}\mathbb{1}_v = P_{[\lambda_0-\varepsilon,\lambda_0+\varepsilon]}U_\Phi\mathbb{1}_x = U_\Phi P_{[\lambda_0-\varepsilon,\lambda_0+\varepsilon]}\mathbb{1}_x = 0.$$

This holds for each vertex v, which clearly implies that $P_{[\lambda_0-\varepsilon,\lambda_0+\varepsilon]} = 0$. Then the operator

$$B = \int_{\mathbb{R}\setminus[\lambda_0-\varepsilon,\lambda_0+\varepsilon]} \frac{1}{\lambda - \lambda_0} \, \mathrm{d}P_\lambda$$

would be the inverse of $A_G - \lambda_0 I$ contradicting our assumption that $\lambda_0 \in \sigma(A_G)$. □

Remark 3.2. There is a general theorem for Hilbert spaces saying that every point of the spectrum of a self-adjoint operator is an approximate eigenvalue [13, Corollary 4.1.3]. So the real content of the above theorem is that one can find approximate eigenvectors that are $\mathrm{Stab}_x(G)$-invariant. This invariance will be crucial for us later on, when we will use these approximate eigenvectors as coefficients to define linear factor of i.i.d. processes.

Suppose now that we have an i.i.d. process on G: independent standard normal random variables Z_u assigned to each vertex u. We will consider processes X_v, $v \in V(G)$, where each X_v is a (possibly infinite) linear combination of Z_u, $u \in V(G)$. We collected some obvious properties of such processes in the next proposition.

Proposition 3.3. *Let $\beta_{v,u}$, $v,u \in V(G)$ be real numbers and let*

$$X_v = \sum_{u \in V(G)} \beta_{v,u} Z_u. \tag{15}$$

The infinite sum in (15) *converges almost surely if and only if*

$$\sum_{u \in V(G)} \beta_{v,u}^2 < \infty. \tag{16}$$

If (16) *is satisfied, then X_v is a centered Gaussian with variance* $\mathrm{var}(X_v) = \sum_{u \in V(G)} \beta_{v,u}^2$.

The process X_v, $v \in V(G)$ is $\mathrm{Aut}(G)$*-invariant if and only if*

$$\beta_{v,u} = \beta_{\Phi v, \Phi u} \text{ for all } \Phi \in \mathrm{Aut}(G). \tag{17}$$

Now we are in a position to formally define linear factor of i.i.d. processes.

Definition 3.4. We say that a process X_v, $v \in V(G)$ is a *linear factor* of the i.i.d process Z_u if it can be written as in (15) for some real numbers $\beta_{v,u}$, $v,u \in V(G)$ satisfying (16) and (17).

Remark 3.5. Let us fix a root $x \in V(G)$. For a linear factor the coefficients $\alpha_u \stackrel{\text{def}}{=} \beta_{x,u}$ clearly determine each $\beta_{v,u}$. Here $\alpha = (\alpha_u)_{u \in V(G)}$ can be any $\mathrm{Stab}_x(G)$-invariant vector in $\ell_2(V(G))$. So there is a one-to-one correspondance between linear factor of i.i.d. processes on G and $\mathrm{Stab}_x(G)$-invariant vectors $\alpha \in \ell_2(V(G))$. Also, by Proposition 3.3 we have $\mathrm{var}(X_v) = \|\alpha\|^2$.

Recall Definition 1.1 of invariant Gaussian processes.

Definition 3.6. We call an invariant Gaussian process X_v, $v \in V(G)$ a *Gaussian wave function* with eigenvalue λ if

$$\sum_{u \in N(v)} X_u = \lambda X_v \text{ for each vertex } v \in V(G),$$

where $N(v)$ denotes the set of neighbors of v in G.

Example 3.7. It was shown in [3] that for the d-regular tree T_d there exists an essentially unique Gaussian wave function for each $\lambda \in [-d, d]$. Furthermore, this Gaussian wave function can be approximated by factor of i.i.d. processes provided that λ is in the spectrum $\sigma(T_d) = [-2\sqrt{d-1}, 2\sqrt{d-1}]$.

In general, it is not clear for which λ such Gaussian wave functions exist and whether they are unique.

Definition 3.8. For a transitive graph G we call the closed set

$$\widetilde{\sigma}(G) \stackrel{\text{def}}{=} \{\lambda : \text{there exists a Gaussian wave function on } G \text{ with eigenvalue } \lambda\}$$

the *Gaussian spectrum* of G.

Theorem 4 claims that for any $\lambda \in \sigma(A_G)$ there exists a Gaussian wave function on G, which can be approximated by linear factor of i.i.d. processes. Therefore $\widetilde{\sigma}(G) \supseteq \sigma(A_G)$.

Proof of Theorem 4. We use the $\mathrm{Stab}_x(G)$-invariant approximate eigenvectors of Theorem 3.1 to define linear factor of i.i.d. processes. So let α be a $\mathrm{Stab}_x(G)$-invariant vector with

$\|\alpha^\varepsilon\| = 1$ and $\|A_G\alpha^\varepsilon - \lambda\alpha^\varepsilon\| \leq \varepsilon$. By Remark 3.5 for each α^ε there is a corresponding linear factor X_v^ε, $v \in V(G)$. Note that the process

$$Y_v^\varepsilon \stackrel{\text{def}}{=} \sum_{u \in N(v)} X_u^\varepsilon - \lambda X_v^\varepsilon$$

is also a linear factor, the corresponding coefficient vector is $\delta^\varepsilon \stackrel{\text{def}}{=} A_G\alpha^\varepsilon - \lambda\alpha^\varepsilon$. Therefore X_v^ε is an invariant Gaussian process with $\operatorname{var}(X_v^\varepsilon) = \|\alpha^\varepsilon\|^2 = 1$ and

$$\operatorname{var}\left(\sum_{u \in N(v)} X_u^\varepsilon - \lambda X_v^\varepsilon\right) = \operatorname{var}\left(Y_v^\varepsilon\right) = \|\delta^\varepsilon\|^2 = \|A_G\alpha^\varepsilon - \lambda\alpha^\varepsilon\|^2 \leq \varepsilon^2.$$

Since the space of invariant Gaussian processes with variance 1 is compact, it follows that there exists a sequence ε_n converging to 0 such that the processes $X_v^{\varepsilon_n}$ converge in distribution. The limit process will be a nontrivial invariant Gaussian process X_v that satisfies the eigenvector equation (3) at each vertex. □

3.2. **Factor of i.i.d. processes.** For a graph G we defined an i.i.d. process on G as independent standard normal random variables Z_v, $v \in V(G)$. In other words, $Z = (Z_v)_{v \in V(G)}$ is a random point in the measure space (Ω, μ), where Ω is $\mathbb{R}^{V(G)}$ with the product topology and μ is the product of standard Gaussian measures (one on each copy of $\mathbb{R}$). The natural action of $\operatorname{Aut}(G)$ on $V(G)$ gives rise to an action of $\operatorname{Aut}(G)$ on Ω: for $\Phi \in \operatorname{Aut}(G)$ and $\omega = (\omega_v)_{v \in V(G)} \in \Omega$ let

$$(\Phi \cdot \omega)_v \stackrel{\text{def}}{=} \omega_{\Phi^{-1}v}.$$

Let G be an infinite transitive graph and suppose that F is a measurable $\Omega \to \Omega$ function that is $\operatorname{Aut}(G)$-equivariant (i.e., commutes with the $\operatorname{Aut}(G)$-action). Then $X = F(Z)$ is an invariant process on G. Such a process $X = (X_v)_{v \in V(G)}$ is called a *factor* of the i.i.d. process Z.

An $\operatorname{Aut}(G)$-equivariant $F\colon \Omega \to \Omega$ function is determined by $f = \pi_x \circ F$, where $\pi_x\colon \Omega \to \mathbb{R}$ is the projection corresponding to the coordinate of some fixed root x. Here f can be any $\operatorname{Stab}_x(G)$-invariant $\Omega \to \mathbb{R}$ function. So factor of i.i.d. processes can be identified with measurable, $\operatorname{Stab}_x(G)$-invariant functions $f\colon \Omega \to \mathbb{R}$.

Next we will prove Theorem 5 and Theorem 6 by showing that certain Gaussian wave functions X_v, $v \in V(G)$ cannot be obtained as factor of i.i.d. processes. Since X_v has finite variance in that case, we can restrict ourselves to functions $f \in L_2(\Omega, \mu)$. Let $H_{\text{inv}} \subset L_2(\Omega, \mu)$ be the subspace containing those $f \in L_2(\Omega, \mu)$ that are $\operatorname{Stab}_x(G)$-invariant. There is a natural way to define an adjacency operator $\mathcal{A}$ on the Hilbert space H_{inv}. Let

$$(\mathcal{A}f)(\omega) \stackrel{\text{def}}{=} \sum_{y \in N(x)} f\left(\Phi_{y \to x} \cdot \omega\right),$$

where $\Phi_{y \to x}$ is an (arbitrary) automorphism of G taking y to x. Since f is $\operatorname{Stab}_x(G)$-invariant, $\mathcal{A}$ is well defined.

Suppose now that we have a Gaussian wave function with eigenvalue λ that can be obtained as a factor of i.i.d. process. Then the corresponding f satisfies the eigenvector equation $\mathcal{A}f = \lambda f$. In particular, λ needs to be in the point spectrum of $\mathcal{A}$. (Note that an eigenvector f of $\mathcal{A}$ does not necessarily give us a Gaussian wave function: although the corresponding factor of i.i.d. process will satisfy the eigenvector equation at each vertex, $f(Z)$ might not have a Gaussian distribution.)

Proof of Theorem 5. Since $L_2(\Omega, \mu)$ is a separable Hilbert space, so is H_{inv}, and consequently the point spectrum of $\mathcal{A} : H_{\text{inv}} \to H_{\text{inv}}$ is countable.

Therefore only for countably many λ's can we have a Gaussian wave function on G that can be obtained as a factor of i.i.d. process. However, if $\sigma(A_G)$ is uncountable, then by Theorem 4 G has Gaussian wave functions for uncountably many different eigenvalues λ; moreover, they can all be approximated by linear factor of i.i.d. processes. □

Proof of Theorem 6. We will use two basic facts about the point spectra of the adjacency operators A_G and $\mathcal{A}$. First, $\lambda_{\max}$ is never in the point spectrum $\sigma_p(A_G)$ (we will give a short proof for this in the Appendix, see Lemma 4.4). Second, $\sigma_p(\mathcal{A}) \subseteq \sigma_p(A_G) \cup \{d\}$ for Cayley graphs (this will be explained after the proof). Therefore $\lambda_{\max}$ is not in the point spectrum of $\mathcal{A}$ provided that $\lambda_{\max} < d$, and consequently, a Gaussian wave function with eigenvalue $\lambda_{\max}$ cannot be obtained as a factor of i.i.d. process.

In the case $\lambda_{\max} = d$ the Gaussian wave function has to be constant, that is, $X_u = X_v$ for any two vertices u, v. However, for a factor of i.i.d. process the correlation between X_u and X_v should tend to 0 as the distance of u and v goes to infinity. □

Next we will explain the relation between the adjacency operators A_G and $\mathcal{A}$. This can be found in [7, Section 3] in a more general setting; see also [9, Theorem 2.1 and Corollary 2.2]. Let ν denote the standard Gaussian measure. Since $L_2(\mathbb{R}, \nu)$ is a separable Hilbert space, it has a countable orthonormal basis: $g_0, g_1, g_2, \ldots$, where g_0 will be assumed to be the constant 1 function. Let $\mathcal{I}$ denote the set of finitely supported $V(G) \to \{0, 1, 2, \ldots\}$ functions. For each $q \in \mathcal{I}$ we define an $\Omega \to \mathbb{R}$ function:

$$W_q(\omega) \stackrel{\text{def}}{=} \prod_{v \in V(G)} g_{q(v)}(\omega_v).$$

Note that this is actually a finite product, since all but finitely many terms are equal to $g_0 \equiv 1$. According to [7, Lemma 3.1] the functions W_q, $q \in \mathcal{I}$ form an orthonormal basis of $L_2(\Omega, \mu)$. It follows that $L_2(\Omega, \mu)$ is separable, which fact was used in the proof of Theorem 5.

We defined the operator $\mathcal{A}$ on the space $H_{\text{inv}} \subset L_2(\Omega, \mu)$ containing $\text{Stab}_x(G)$-invariant functions. When G is a Cayley graph, there is a natural way to extend $\mathcal{A}$ to an adjacency operator over the whole space $L_2(\Omega, \mu)$. Suppose that Γ is a finitely generated infinite group. Let S be a finite, symmetric set of generators and let G be the corresponding Cayley graph, that is, $V(G) = \Gamma$ and the vertex $v \in \Gamma$ is adjacent to the vertices γv, $\gamma \in S$. The natural action of Γ on itself gives rise to the following Γ-action on Ω:

$$(\gamma \cdot \omega)_v \stackrel{\text{def}}{=} (\omega)_{\gamma^{-1} v}.$$

(This is often called the *generalized Bernoulli shift.*) Then for $f \in L_2(\Omega, \mu)$ let

$$(\mathcal{A} f)(\omega) \stackrel{\text{def}}{=} \sum_{\gamma \in S} f(\gamma \cdot \omega).$$

This clearly extends our earlier definition of $\mathcal{A}$.

There is a natural Γ-action on $\mathcal{I}$ as well: for $q \in \mathcal{I}$

$$(\gamma \cdot q)(v) \stackrel{\text{def}}{=} q\left(\gamma^{-1} v\right).$$

It is compatible with the Γ-action on Ω in the following sense:

$$W_{\gamma \cdot q}(\omega) = W_q\left(\gamma^{-1} \cdot \omega\right).$$

It means that

$$\mathcal{A}W_q = \sum_{\gamma\in S} W_{\gamma\cdot q}. \tag{18}$$

We now consider the the orbit $\{\gamma \cdot p \,:\, \gamma \in \Gamma\}$ of a given element $p \in \mathcal{I}$ and the closure of the space spanned by the corresponding functions $W_{\gamma\cdot p}$:

$$H_p \stackrel{\text{def}}{=} \mathrm{cl}\left(\mathrm{span}\left\{W_{\gamma\cdot p} \,:\, \gamma \in \Gamma\right\}\right) \subset L_2(\Omega, \mu).$$

It is clear from (18) that H_p is $\mathcal{A}$-invariant. If $p \equiv 0$, then H_p consists of the contant functions on Ω and both the point spectrum and the spectrum of $\mathcal{A}\big|_{H_p}$ is $\{d\}$. Otherwise the stabilizer Γ_p of p is a finite subgroup of Γ , and $\mathcal{A}\big|_{H_p}$ is closely related to the original adjacency operator A_G. Indeed, let $T_p\colon H_p \to \ell_2(V(G)) \cong \ell_2(\Gamma)$ be the operator defined by

$$T_p\colon W_q \mapsto \mathbb{1}_{\{\gamma\in\Gamma\,:\,\gamma\cdot p=q\}},$$

where q is in the orbit of p. It is easy to see that T_p is a bounded operator for which $T_p\mathcal{A}\big|_{H_p} = A_G T_p$. Since T_p is also bounded below, it follows that

$$\sigma\left(\mathcal{A}\big|_{H_p}\right) \subseteq \sigma(A_G) \text{ and } \sigma_p\left(\mathcal{A}\big|_{H_p}\right) \subseteq \sigma_p(A_G)$$

with equality when the stabilizer Γ_p is trivial.

Therefore for Cayley graphs the operators $A_G\colon \ell_2(V(G)) \to \ell_2(V(G))$ and $\mathcal{A}\colon L_2(\Omega,\mu) \to L_2(\Omega,\mu)$ have the same spectra and point spectra with the possible exception of the point d:

$$\sigma(\mathcal{A}) = \sigma(A_G) \cup \{d\} \text{ and } \sigma_p(\mathcal{A}) = \sigma_p(A_G) \cup \{d\}.$$

Consequently,

$$\sigma_p\left(\mathcal{A}\Big|_{H_{\text{inv}}}\right) \subseteq \sigma_p(\mathcal{A}) = \sigma_p(A_G) \cup \{d\},$$

which we used in the proof of Theorem 6.

3.3. **Independent sets.** Let G be an infinite transitive graph and $\lambda_{\min}$ be the minimum of its spectrum $\sigma(A_G)$. Consider linear factor of i.i.d. processes X_v^n converging in distribution to a Gaussian wave function X_v with eigenvalue $\lambda_{\min}$ as $n \to \infty$ as in Theorem 4. We define the following independent sets on G:

$$I_+ \stackrel{\text{def}}{=} \{v \,:\, X_v > X_u, \forall u \in N(v)\} \text{ and } I_+^n \stackrel{\text{def}}{=} \{v \,:\, X_v^n > X_u^n, \forall u \in N(v)\}.$$

Then for each n the independent set I_+^n is a factor of the i.i.d. process Z_v (i.e., it is obtained as a measurable function of Z_v, $v \in V(G)$ that commutes with the natural action of $\mathrm{Aut}(G)$.) Furthermore, since the event $v \in I_+$ corresponds to an open set, we have

$$\liminf_{n\to\infty} P(v \in I_+^n) \geq P(v \in I_+).$$

Therefore whenever we have a lower bound q for $P(v \in I_+)$, it yields that for any $\varepsilon > 0$ there exists a factor of i.i.d. independent set with "size" greater than $q - \varepsilon$.

Bounding $P(v \in I_+)$, however, leads us to the same optimization problem as in the finite case. We need to estimate the volume of the same spherical simplex with the exact same constraints. (Of course, there might be a difference between the finite and infinite setting in terms of what covariances $c_{i,j}$ can actually come up, but our proofs used only the trivial constraints that they form a positive semidefinite matrix and their sum is $(\lambda_{\min}^2 - d)/2$, which are true in the infinite case, too.) Thus we obtain the exact same bounds and Theorem 7 follows.

Actually, in Theorem 3 we proved the bound only for graphs with $\lambda_{\min} \leq -2$ and argued that the only finite, 3-regular, transitive graph for which this does not hold is the complete graph K_4. For infinite transitive graphs $\lambda_{\min} \leq -2$ holds with no exception. This follows from the fact that they contain arbitrarily long paths as induced subgraphs.

4. Appendix

Theorem 4.1. *Suppose that G is a finite, 3-regular, vertex-transitive graph with minimum eigenvalue $\lambda_{\min}$ and odd-girth g. Then the independence ratio of G is at least*

$$\frac{5g-3}{16g} + \frac{g+1}{2g}\frac{3}{4\pi}\arcsin\left(\frac{\lambda_{\min}^2 - 3}{6}\right) \geq \frac{5}{16} + \frac{3}{8\pi}\arcsin\left(\frac{\lambda_{\min}^2 - 3}{6}\right) - \frac{3}{16g}.$$

In fact, there exist two disjoint independent sets in G such that their average size divided by $|V(G)|$ is not less than the above bound.

Proof of Theorem 4.1. It is easy to check the statement for K_4. According to Proposition 4.3 $\lambda_{\min} \leq -2$ holds for any other finite, 3-regular, transitive graph G. Let X_v, $v \in V(G)$ be the random eigenvector corresponding to $\lambda_{\min}$. Let V_+ denote the set of "positive vertices", that is,

$$V_+ \stackrel{\text{def}}{=} \{v \in V(G) \ : \ X_v > 0\}.$$

The expected size of V_+ is $|V(G)|/2$.

Since $\lambda_{\min}$ is negative, a vertex and its three neighbors cannot all be positive. Therefore each vertex has degree at most two in the induced subgraph $G[V_+]$. Thus each connected component of this subgraph is a path or a cycle. We want to choose an independent set from each component. We can choose at least half the vertices from paths and even cycles. From an odd cycle of length $l \geq g$ we can choose $(l-1)/2$ vertices, which is at least a $(g-1)/(2g)$ proportion of all vertices in that component. (Recall that g denotes the odd-girth of G, that is, the length of the shortest odd cycle in G.)

We need one more observation, namely, that many of the components actually contain only one vertex. Using our earlier notations, let v be an arbitrary vertex with neighbors u_1, u_2, u_3, the corresponding random variables are X and Y_1, Y_2, Y_3. Note that $Y_1 < 0$, $Y_2 < 0$ and $Y_3 < 0$ imply that $X > 0$. Therefore the probability p that v is an isolated vertex in $G[V_+]$ is

$$\begin{aligned} p &\stackrel{\text{def}}{=} P\left(X > 0; Y_1 < 0; Y_2 < 0; Y_3 < 0\right) = P\left(Y_1 < 0; Y_2 < 0; Y_3 < 0\right) = \\ &P\left(y_i \cdot Z < 0; i = 1,2,3\right) = \frac{1}{2} - \frac{1}{4\pi}\sum_{1 \leq i,j \leq 3} \arccos(c_{i,j}) = \frac{1}{8} + \frac{1}{4\pi}\sum_{1 \leq i,j \leq 3} \arcsin(c_{i,j}). \end{aligned} \tag{19}$$

Note that arcsin it is a monotone increasing odd function on $[-1,1]$, which is convex on $[0,1]$. Furthermore, the average of $c_{i,j}$ is $(\lambda_{\min}^2 - 3)/6 \geq (2^2-3)/6 > 0$. It is easy to see that these imply that the right hand side of (19) decreases (not increases) if we replace each $c_{i,j}$ with their average $(\lambda_{\min}^2 - 3)/6$. Thus

$$p \geq \frac{1}{8} + \frac{3}{4\pi}\arcsin\left(\frac{\lambda_{\min}^2 - 3}{6}\right). \tag{20}$$

Our independent set will contain all isolated vertices and at least a $(g-1)/(2g)$ proportion of all the other vertices in V_+. This yields the following lower bound for the independence ratio of G:

$$p + \frac{g-1}{2g}\left(\frac{1}{2} - p\right) = \frac{g-1}{4g} + \frac{g+1}{2g}p.$$

Combining this with (20) yields the desired bound.

We can choose an independent set with the same expected size from the "negative vertices":

$$V_- \stackrel{\text{def}}{=} \{v \in V(G) \ : \ X_v < 0\}.$$

This implies the second part of the theorem.

We mention that the proof also works in the infinite setting, so there is an analogous theorem for infinite transitive graphs (as in Theorem 7). □

Remark 4.2. Any non-trivial lower bound for the density of components of size $3, 5, \dots$ in $G[V_+]$ would immediately yield an improvement in the above theorem. In [3] such non-trivial bounds were obtained for the 3-regular tree T_3.

Proposition 4.3. *Suppose that G is a finite, connected, 3-regular, vertex-transitive graph. Then either G is isomorphic to the complete graph K_4, or the least eigenvalue $\lambda_{\min}$ of its adjacency matrix is at most -2.*

The proof below is due to Péter Csikvári.

Proof. Let G be a connected, 3-regular, vertex-transitive graph with $\lambda_{\min}(G) > -2$. We need to show that G must be the complete graph K_4.

Cauchy's interlacing theorem implies that $\lambda_{\min}(G) \leq \lambda_{\min}(H)$ whenever H is an induced subgraph of G. Therefore $\lambda_{\min}(H) > -2$ must hold for any induced subgraph. Let T denote the tree shown in Figure 2. It is easy to see that the smallest eigenvalue of T is -2. We also have $\lambda_{\min}(C_{2k}) = -2$ for the cycle of length $2k$ for any $k \geq 2$. Therefore G can contain neither T, nor C_{2k} as an induced subgraph.

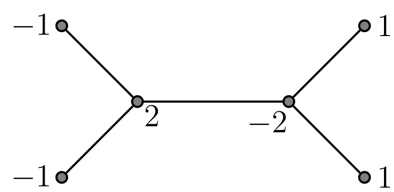


FIGURE 2. The graph T and the eigenvector corresponding to its least eigenvalue -2

We will distinguish three cases.

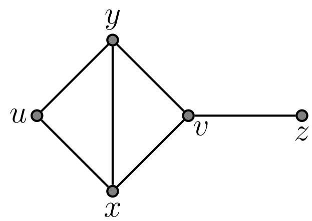
Case 1. *G does not contain triangles.*
Let u, v be two neighboring vertices, and let u_1, u_2 and v_1, v_2 denote the remaining two neighbors of u and v, respectively. Since G contains no triangles, u_1,u_2,v_1,v_2 are pairwise distinct vertices. The induced subgraph on the set $\{u, u_1, u_2, v, v_1, v_2\}$ must be isomorphic to T (the graph shown in Figure 2), otherwise G would contain a triangle or an induced C_4. Since G cannot contain T as an induced subgraph, this is a contradiction.

Case 2. *G contains triangles but no two share a common edge.*
Since G is vertex-transitive, there must be at least one triangle through every vertex. We claim that any two triangles must be disjoint. If they had two common vertices, then they would share an edge, and if they had exactly one common vertex, then that vertex would have degree at least 4.

So we have disjoint triangles in G, exactly one through every vertex. We claim that there can be at most one edge between two triangles (with one endpoint in one triangle and one in the other). Indeed, otherwise we would either have an induced C_4 or a vertex with degree at least 4.

Let us consider the following graph G^*. To each triangle in G corresponds a vertex in G^*, and we join two such vertices with an edge if there is an edge between the corresponding triangles. It is easy to see that G^* will be 3-regular as well. Take a cycle in G^* with minimum length $g \geq 3$. There is a corresponding cycle of length $2g$ in the original graph G. It is easy to see that this must be an induced cycle, contradiction.

Case 3. *G contains two triangles sharing an edge.*
Let xy be an edge shared by triangles xyu and xyv (see the figure below).



Then x and y already have degree 3, while u and v still need an edge. We claim that uv must be an edge. Otherwise v would have a neighbour z different from x, y, u. Since z cannot be adjacent to x and y, there is only one triangle through v, while there are two triangles through x, contradticting the transitivity of G. So uv is an edge, therefore each of x, y, u, v has degree 3. Since G is connected, G cannot have any other vertices and thus isomorphic to K_4. □

The following lemma is probably known, but we did not find an explicit reference, so we give a short proof.

Lemma 4.4. *If G is an infinite transitive graph, then the maximum $\lambda_{\max}$ of the spectrum of A_G is never in the point spectrum of A_G.*

Proof. In the case $\lambda_{\max} = d$, the equation $A_G f = df$ means that the vector f is harmonic. However, the maximum principle implies that there are no ℓ^2 harmonic functions. Thus there is no eigenvector for $\lambda_{\max}$, which is equivalent to saying that $\lambda_{\max}$ is not in the point spectrum of A_G.

For the non-amenable case (i.e. $\lambda_{\max} < d$), Theorem II.7.8 in [15] implies that for any vertex v

$$\sum_{n=0}^{\infty} \lambda_{max}^{-2n} \langle 1_v, A_G^{2n} 1_v \rangle < \infty,$$

where the left hand side can be written in terms of the spectral measure μ_G as

$$\sum_{n=0}^{\infty} \lambda_{\max}^{-2n} \int x^{2n} d\mu_G(x) \geq \sum_{n=0}^{\infty} \lambda_{\max}^{-2n} \lambda_{\max}^{2n} \mu_G(\{\lambda_{\max}\}).$$

This forces $\mu_G(\{\lambda_{\max}\}) = 0$, which means that $\lambda_{\max}$ is not in the point spectrum of A_G. □

Lemma 4.5.

$$\frac{\operatorname{vol}(S^{d-2})}{\operatorname{vol}(S^{d-1})} < \frac{\sqrt{d}}{\sqrt{2\pi}}.$$

Proof. Using the formula

$$\mathrm{vol}(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

we need to show that

$$\frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d-2}{2}\right)} < \sqrt{\frac{d}{2}}.$$

Since Γ is log-convex, the increments of its logarithm over intervals of length, say, $1/2$ are increasing. Thus

$$\frac{\Gamma(\frac{d-1}{2})}{\Gamma(\frac{d-2}{2})} \le \frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d-1}{2})}$$

and multiplying both sides by the left hand side, we get

$$\left(\frac{\Gamma(\frac{d-1}{2})}{\Gamma(\frac{d-2}{2})}\right)^2 \le \frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d-2}{2})} = \frac{d-2}{2} < \frac{d}{2}$$

as required. □

Lemma 4.6. *Let* $\lambda \in [-3,-2]$ *and*

$$f(t) \overset{def}{=} \arcsin\left(\frac{\frac{1}{2} - \lambda + \frac{\lambda^2}{2} + (1-\lambda)t}{\frac{1}{2} - 2\lambda + \frac{\lambda^2}{2} + t}\right).$$

Then the tangent line to f *at* $t_0 = (\lambda^2 - 3)/6$ *is below* f *on the entire interval* $[-0.5, 1]$.

Proof. We need to prove that

$$f(t) - f'(t_0)t$$

takes its minimum value at t_0 on the interval $[-0.5, 1]$. This will follow from the fact that $f'(t) < f'(t_0)$ for $-0.5 \le t < t_0$ and $f'(t) > f'(t_0)$ for $t_0 < t < 1$.

In order to make calculations easier we will use the following notations:

$$a = \frac{1}{2} - \lambda + \frac{\lambda^2}{2} \ge 4.5; \quad b = 1 - \lambda \ge 3; \quad c = a + b - 1 \ge 6.5;$$

then

$$f(t) = \arcsin\left(\frac{a+bt}{c+t}\right).$$

It is easy to see that $0 < a + bt < c + t$ for $t \in [-0.5, 1)$. Therefore we have

$$f'(t) = \left(1 - \left(\frac{a+bt}{c+t}\right)^2\right)^{-\frac{1}{2}} \frac{b(c+t) - (a+bt)}{(c+t)^2} = \frac{bc - a}{(c+t)\sqrt{(c+t)^2 - (a+bt)^2}}.$$

Since $bc - a > 0$ it follows that f' is positive on $[-0.5, 1)$ and thus f is monotone increasing. Next we study the intervals of monotonicity of f'. First we note that

$$(c+t)^2 - (a+bt)^2 = (c + a + (1+b)t)\,(c - a + (1-b)t)\,.$$

Using $c - a = b - 1$ we get that

$$(c+t)^2 - (a+bt)^2 = (b^2 - 1)(t + d)(1 - t),$$

where

$$d = \frac{c+a}{b+1} = \frac{1 - 3\lambda + \lambda^2}{2 - \lambda} = 1 - \lambda - \frac{1}{2-\lambda} \ge \frac{11}{4}.$$

It follows that

$$\frac{1}{(f'(t))^2} = \frac{b^2-1}{(bc-a)^2}(t+c)^2(t+d)(1-t).$$

If we restrict ourselves to the interval $[-0.5, 1)$ (where f' is positive), then it suffices to examine the function

$$g(t) = (t+c)^2(t+d)(1-t).$$

Wherever g is monotone increasing, f' is monotone decreasing, and vice versa.

So we have a fourth-degree polynomial g with leading coefficient -1, whose roots are $-c$ (with multiplicity 2), $-d$, and 1. Consequently, the derivative g' is a third-degree polynomial with negative leading coefficient and with roots $-c$, u, v, where $-c < u < -d < v < 1$. We distinguish the following two cases.

Case 1: $v \leq -0.5$. Then g is monotone decreasing on $[-0.5, \infty)$, therefore f' is monotone increasing on $[-0.5, 1)$, and thus f is convex on the whole interval, which clearly implies the statement of the lemma.

Case 2: $v > -0.5$. Since the other two roots of g' are less than $-d < -0.5$, we know that g is monotone increasing on $[-0.5, v]$ and monotone decreasing on $[v, 1)$. We claim that

$$g\left(-\frac{1}{2}\right) > g\left(\frac{1}{6}\right). \tag{21}$$

This would yield that $v < 1/6$. Since $1/6 \leq t_0 = (\lambda^2 - 3)/6$, we have $g(-1/2) > g(1/6) > g(t_0)$. This means that $g(t) > g(t_0)$ for $-0.5 \leq t < t_0$ and $g(t) < g(t_0)$ for $t_0 < t < 1$. As for f', $f'(t) < f'(t_0)$ for $-0.5 \leq t < t_0$ and $f'(t) > f'(t_0)$ for $t_0 < t < 1$, and the statement of the lemma clearly follows.

It remains to show (21). Let $-1/2 = t_2 < t_1 = 1/6$. Then $t_1 - t_2 = 2/3$; $t_2 + c \geq 6$ and $t_2 + d \geq 9/4$, and consequently

$$\frac{g(t_1)}{g(t_2)} = \frac{1-t_1}{1-t_2}\left(1+\frac{t_1-t_2}{t_2+c}\right)^2\left(1+\frac{t_1-t_2}{t_2+d}\right) \leq$$
$$\frac{5/6}{3/2}\left(1+\frac{2/3}{6}\right)^2\left(1+\frac{2/3}{9/4}\right) = \frac{5}{9}\left(\frac{10}{9}\right)^2\frac{35}{27} = \frac{17500}{19683} < 1.$$

□

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