Real Functions and Measures, BSM, Fall 2014

Exercise sheet: Jordan measure

Notation: m_* and m^* denote the Jordan inner and outer measure, respectively, while the Lebesgue outer measure is denoted by λ^*.

1. Determine all pairs (a, b) of real numbers for which there exists a bounded set $E \subset \mathbb{R}$ such that $m_*(E) = a$ and $m^*(E) = b$.

2. Find the Jordan outer measure of the following subset of $\mathbb{R}^2$:

$$\{(1/n, y) : y \in [0,1]; n = 1, 2, 3, \ldots\}.$$

3. Let $C \subset [0,1]$ denote the Cantor set. Find $m_*(C)$ and $m^*(C)$. Is C Jordan measurable?

4. Suppose that $E_1, E_2, \ldots$ are Jordan measurable sets. Show that the countable union $\bigcup_{n=1}^{\infty} E_n$ and the countable intersection $\bigcap_{n=1}^{\infty} E_n$ need not be Jordan measurable, even when bounded.

5. Is it possibile that for some bounded sets $E_1 \supset E_2 \supset E_3 \supset \cdots$ we have

$$m^*(E_1) = m^*(E_2) = \ldots = 1$$

and

$$m^*\left(\bigcap_{n=1}^{\infty} E_n\right) = 0?$$

6. Give an example of a sequence of uniformly bounded, Riemann integrable functions $f_n \colon [0,1] \to \mathbb{R}$ for $n = 1, 2, \ldots$ that converge pointwise to a bounded function $f \colon [0,1] \to \mathbb{R}$ that is not Riemann integrable. What happens if we replace pointwise convergence with uniform convergence?

7. Is it true that every bounded closed set is Jordan measurable?

8. Find a bounded set $E \subset \mathbb{R}$ such that $m_*(E) < \lambda^*(E) < m^*(E)$.

9. Suppose that $E_1, \ldots, E_n$ are Jordan measurable subsets of the unit cube. Prove that if the sum of their Jordan measures is more than k, then there exists a point that is contained by at least $k+1$ sets.

10. a) Show that there exists a set $A \subset [0,1]$ such that $[0,1]$ contains infinitely many disjoint translated copies of A, yet, $\mathbb{R}$ can be covered by countably many translated copies of A.

b) Let $P(\mathbb{R})$ denote the power set of $\mathbb{R}$, that is, the set of all subsets of $\mathbb{R}$. Prove that there does not exist a σ-additive, translation-invariant function $\mu \colon P(\mathbb{R}) \to [0, \infty]$ with $\mu([0,1]) = 1$ (that is, it is impossible to „assign measure” to **all** subsets of $\mathbb{R}$).