Real Functions and Measures, BSM, Fall 2014

Assignment 1

1. a) Prove that any open set in $\mathbb{R}$ can be expressed as the union of **countably** many open intervals.
b) Prove that any open set in $\mathbb{R}^2$ can be expressed as the union of **countably** many open boxes.

2. Let X be an uncountable set and $\mathcal{M}$ be the σ-algebra generated by the finite subsets of X. Describe $\mathcal{M}$ and the $\mathcal{M}$-measurable $X \to \mathbb{R}$ functions. Explain your answer!

3. Let $(X, \mathcal{M})$ be a measurable space and let $f\colon X \to \mathbb{R}$ be a real-valued function. Prove that if $\{x : f(x) \geq r\} \in \mathcal{M}$ for every rational number r, then f is measurable.

4. Let $(X, \mathcal{M})$ be a measurable space. Suppose that $f\colon X \to [-\infty, \infty]$ and $g\colon X \to [-\infty, \infty]$ are measurable functions. Prove that

$$\{x : f(x) < g(x)\} \in \mathcal{M} \text{ and } \{x : f(x) = g(x)\} \in \mathcal{M}.$$

5. Prove that the set of points at which a sequence of measurable real-valued functions converges (to a finite limit) is measurable.