Real Functions and Measures, BSM, Fall 2014

Assignment 3

1. Suppose that $f_n\colon X \to [0,\infty)$ are measurable functions on a measure space $(X, \mathcal{M}, \mu)$ with the property that $f_1 \geq f_2 \geq f_3 \geq \dots$ and $f_1 \in L^1(\mu)$.
a) Prove that

$$\int_X \left(\lim_{n\to\infty} f_n(x)\right) d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu.$$

b) Is the statement true without the assumption $f_1 \in L^1(\mu)$?

2. Suppose $\mu(X) < \infty$ for a measure space $(X, \mathcal{M}, \mu)$. Prove that if the functions $f_n\colon X \to \mathbb{C}$ are uniformly bounded (that is, there exists a $K > 0$ such that $|f_n(x)| \leq K$ for all $n \in \mathbb{N}$ and all $x \in X$) and $f(x) = \lim_{n\to\infty} f_n(x)$ exists for all $x \in X$, then

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

3. Let $(X, \mathcal{M}, \mu)$ be a measure space and $\alpha \in (0,\infty)$. Suppose that $\int_X f \, d\mu = c \in (0,\infty)$ for a measurable function $f\colon X \to [0,\infty]$. Prove that

$$\lim_{n\to\infty} \int_X n \log\left(1 + (f/n)^\alpha\right) d\mu = \begin{cases} \infty & \text{if } 0 < \alpha < 1, \\ c & \text{if } \alpha = 1, \\ 0 & \text{if } 1 < \alpha < \infty. \end{cases}$$

(Hint: if $\alpha \geq 1$, the integrands are dominated by αf; if $\alpha < 1$, Fatou's lemma can be applied.)

4. Find a measure space $(X, \mathcal{M}, \mu)$ and measurable functions $f_n\colon X \to [0,\infty)$ for which the inequality in Fatou's lemma is strict, that is,

$$\int_X \left(\liminf_{n\to\infty} f_n\right) d\mu < \liminf_{n\to\infty} \int_X f_n \, d\mu.$$

5. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f \in L^1(\mu)$.
a) Show that for any $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that

$$\int_{\{x : |f(x)| \geq n_0\}} |f| \, d\mu < \varepsilon.$$

b) Prove that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $\mu(E) < \delta$ for some measurable set E we have $\int_E |f| \, d\mu < \varepsilon$.