Real Functions and Measures, BSM, Fall 2014

Assignment 5

1. Let X be an infinite set and consider the discrete topology on X, that is $\tau = P(X)$.
a) Describe $C_c(X)$, the space of compactly-supported continuous $X \to \mathbb{C}$ functions.
b) Describe all positive linear functionals $\Lambda \colon C_c(X) \to \mathbb{C}$.
c) For any positive linear functional Λ on $C_c(X)$ determine the corresponding measure (see *Riesz representation theorem*).
d) Which functional corresponds to the counting measure?

2. Let K denote the Cantor set (that is, we start with the segment $[0,1]$, and at each step we remove the open "middle third" of each segment).
a) Show that K is compact.
b) Determine the Lebesgue measure of K.
c) Modify the construction of K to obtain a compact set in $[0,1]$ that has positive Lebesgue measure and that does not contain any intervals.
d) For any $0 < \varepsilon < 1$ construct an open set $G \subset [0,1]$ such that G is dense in $[0,1]$ (that is, the closure of G is $[0,1]$) and $\lambda(G) = \varepsilon$.
e) Construct a Borel set $E \subset \mathbb{R}$ such that $0 < \lambda(E \cap I) < \lambda(I)$ holds for every interval I.

3. Determine the integral of the functions defined in HW2/4 and HW2/5 (over $[0,1)$ and with respect to the Lebesgue measure).

4. Prove that

$$\lim_{n\to\infty} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x} dx = 1.$$

(Hint: use that Riemann-integral and Lebesgue-integral coincide for continuous functions.)

5.* (extra problem, no points) Does there exist a measurable $f \colon \mathbb{R} \to [0,\infty)$ function that has infinite integral w.r.t. the Lebesgue measure over every interval?