Real Functions and Measures, BSM, Fall 2014
Assignment 6

1. The *Cantor function* $f\colon [0,1] \to [0,1]$ is defined as follows. Let $f(1) = 1$. For any $0 \le x < 1$ we express x in base 3: $x = 0.a_1a_2a_3\ldots$, where we exclude representations for which $a_n = a_{n+1} = a_{n+2} = \ldots = 2$ for some n. First we take the first digit a_k that is equal to 1 (if exists) and replace all subsequent digits by 0. (If x contains no digit equal to 1, then we leave x as it is.) Then we replace each digit 2 by 1, consider the obtained 0-1 sequence in base 2, and we define $f(x)$ to be the real number corresponding to this binary representation.
a) Show that f is constant on each of the "middle third intervals" in the complement of the Cantor set.
b) Write f as the limit of piecewise linear functions.
c) Show that f is continuous.
d) Determine the integral $\int_{[0,1]} f\, d\lambda$.

2. At the beginning of this term we defined the Lebesgue outer measure as

$$\lambda^*(E) := \inf\left\{\sum_{n=1}^{\infty} |b_n - a_n| : E \subset \bigcup_{n=1}^{\infty} [a_n, b_n)\right\}$$

for any $E \subset \mathbb{R}$.

Now we have another definition of the Lebesgue measure, via the Riesz Representation Theorem. In a nuthsell: we defined the positive linear functional $\Lambda\colon C_c(\mathbb{R}) \to \mathbb{C}$ as the limit of the functionals $\Lambda_n f = 2^{-n}\sum_{x\in P_n} f(x)$, where P_n denotes the lattice $\{a2^{-n} : a \in \mathbb{Z}\}$. In the proof of the Riesz Representation Theorem for any such linear functional we defined an outer measure μ as follows:

$$\text{for open } G\text{: } \mu(G) := \sup\{\Lambda f : f \prec G\};$$
$$\text{for arbitrary } E\text{: } \mu(E) := \inf\{\mu(G) : E \subseteq G \text{ open}\}.$$

We showed that μ is an outer measure and that there exists a complete σ-algebra (containing the Borel sets) on which μ is a measure.
Prove that our original definition λ^* coincides with the outer measure μ obtained in the Riesz Representation Theroem.
(You can use the following facts: (1) if I is an interval, then $\mu(I)$ is equal to the length of the interval; (2) any open set in $\mathbb{R}$ can be written as the **disjoint** union of at most countably many open intervals; (3) μ is a measure on Borel sets.)

3. For any fixed $x \in \mathbb{R}$ let L_x denote the vertical line $\{(x,y) : y \in \mathbb{R}\}$ and let $\pi\colon \mathbb{R}^2 \to \mathbb{R}$ be the projection map $(x,y) \mapsto y$. Set

$$\tau := \left\{G \subseteq \mathbb{R}^2 : \pi(G \cap L_x) \text{ is open for all } x \in \mathbb{R}\right\}.$$

a) Show that $(\mathbb{R}^2, \tau)$ is a locally compact topological space.
b) Prove that $K \subseteq \mathbb{R}^2$ is compact if and only if $\pi(K \cap L_x) \subseteq \mathbb{R}$ is compact for all

x and $K \cap L_x = \emptyset$ for all but finitely many x.
c) For any compactly supported continuous function f on this space let

$$\Lambda f := \sum_{x\in\mathbb{R}} \int_{\mathbb{R}} f(x,y)\,dy.$$

Show that Λ is a positive linear functional and find the representing measure μ.

4. A measurable set A of a measure space $(X, \mathcal{M}, \mu)$ is said to be an *atom* (w.r.t. μ) if $\mu(A) > 0$ and for every $E \subset A$ either $\mu(E) = 0$ or $\mu(A \setminus E) = 0$.
a) Find all atoms in the space $(\mathbb{R}, P(\mathbb{R}), \texttt{counting measure})$.
b) Let μ be a finite Borel measure on $\mathbb{R}$. Show that every atom A contains a point $x \in A$ such that $\mu(A) = \mu(\{x\})$.
c) Let A be an atom in a measure space $(X, \mathcal{M}, \mu)$. Prove that any measurable function $f\colon X \to \mathbb{R}$ is constant almost everywhere on A.

5.* (extra problem, no points) Let $(X, \mathcal{M}, \mu)$ be an atom-free measure space with $\mu(X) = 1$. Prove that μ takes every value in $[0, 1]$.