Real Functions and Measures, BSM, Fall 2014

Assignment 7

1. Let (X, τ) be a topological space and $\mathcal{B}$ the Borel σ-algebra (that is, the σ-algebra generated by τ). Let $Y \subset X$ be an arbitrary subset.
a) Show that $(Y, \tau|_Y)$ is a topological space, where

$$\tau|_Y \overset{\text{def}}{=} \{G \cap Y : G \in \tau\}.$$

b) Let $\mathcal{B}_Y$ be the Borel σ-algebra of $(Y, \tau|_Y)$, and

$$\mathcal{B}|_Y \overset{\text{def}}{=} \{B \cap Y : B \in \mathcal{B}\}.$$

Show that $\mathcal{B}|_Y$ is a σ-algebra that contains $\mathcal{B}_Y$.
c) Prove that $\mathcal{B}|_Y = \mathcal{B}_Y$.
(Hint: to show that $B \cap Y \in \mathcal{B}_Y$ for any $B \in \mathcal{B}$, prove that $\mathcal{M} = \{E : E \cap Y \in \mathcal{B}_Y\}$ is a σ-algebra that contains τ.)

2. We identify $\mathbb{R}$ with the x-axis $\{(x, 0) : x \in \mathbb{R}\}$ of $\mathbb{R}^2$. Are the following statements true?
a) If B is Borel set in $\mathbb{R}^2$, then $B \cap \mathbb{R}$ is a Borel set in $\mathbb{R}$.
b) If $E \subset \mathbb{R}^2$ is Lebesque measurable in $\mathbb{R}^2$, then $E \cap \mathbb{R}$ is Lebesque measurable in $\mathbb{R}$.

3. Prove the following statements.
a) If $A, B \subset \mathbb{R}$ are open, then so is $A \times B \subset \mathbb{R}^2$.
b) If $A, B \subset \mathbb{R}$ are closed, then so is $A \times B \subset \mathbb{R}^2$.
c) If $A, B \subset \mathbb{R}$ are G^δ, then so is $A \times B \subset \mathbb{R}^2$.
d) If $A, B \subset \mathbb{R}$ are F^σ, then so is $A \times B \subset \mathbb{R}^2$.
e) If $A, B \subset \mathbb{R}$ are Borel, then so is $A \times B \subset \mathbb{R}^2$.
(Hint: $A \times B = (A \times \mathbb{R}) \cap (\mathbb{R} \times B)$.)
f) If $A, B \subset \mathbb{R}$ are Lebesgue measurable, then so is $A \times B \subset \mathbb{R}^2$.
(Hint: use the fact that $E \subseteq \mathbb{R}^k$ is Lebesque measurable if and only if there exist an F_σ-set E_1 and a G_δ-set E_2 such that $E_1 \subseteq E \subseteq E_2$ and $\lambda(E_2 \setminus E_1) = 0$. In one dimension the latter means that $E_2 \setminus E_1$ can be covered by countably many intervals with arbitrary small total length.)

4. For an $\mathbb{R} \to \mathbb{R}$ function f let

$$A_f \overset{\text{def}}{=} \{(x, y) : y < f(x)\} \subset \mathbb{R}^2.$$

a) Express A_f as the union of countably many sets of the form $E \times (-\infty, t)$.
b) Prove that f is Borel measurable if and only if $A_f \subset \mathbb{R}^2$ is a Borel set.

5. Let $(X, \mathcal{M})$ be a measurable space. A *finite signed measure* on $(X, \mathcal{M})$ is a mapping $\mu \colon \mathcal{M} \to \mathbb{R}$ such that $\mu(\emptyset) = 0$ and μ is σ-additive, that is,

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$$

for any sequence $E_1, E_2, \ldots$ of pairwise disjoint measurable sets.

Given a finite signed measure μ we define $\mu^+, \mu^- \colon \mathcal{M} \to [0, \infty]$ as follows:

$$\mu^+(E) \stackrel{\text{def}}{=} \sup\left\{\mu(A) : A \subseteq E; A \in \mathcal{M}\right\};$$
$$\mu^-(E) \stackrel{\text{def}}{=} -\inf\left\{\mu(A) : A \subseteq E; A \in \mathcal{M}\right\}.$$

a) Show that $\mu^- = (-\mu)^+$.
b) Prove that μ^+ and μ^- are measures.
c) Show that $\mu^+(X)$ and $\mu^-(X)$ are finite.
(μ^+ and μ^- are called the *positive part/positive variation* and the *negative part/negative variation* of μ, respectively. It can be shown that $\mu = \mu^+ - \mu^-$.)

6.* (extra problem, no points) Let f and A_f be as in Question 4. Prove that f is Lebesque measurable if and only if $A_f \subset \mathbb{R}^2$ is Lebesgue measurable.