Saturation problems about forbidden 0-1 submatrices

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joint work with Rado Fulek

Definition: pattern avoidance

A 0-1 matrix M *contains* a *pattern* P, which is a not-all-zeros 0-1 matrix, if M contains a submatrix P' that can be transformed into the matrix P by replacing some (potentially none) 1 entries with 0 entries.
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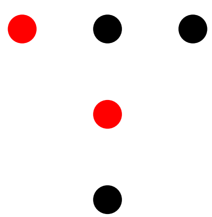
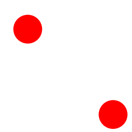
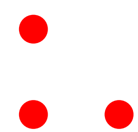
 contains  and avoids 

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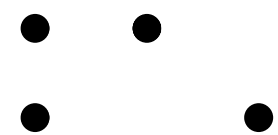
Thm (Füredi,Bienstock-Győri). $ex($  $, n) = \Theta(n \log n)$.

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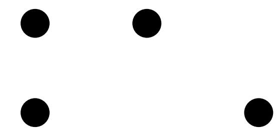
Thm (Füredi,Bienstock-Győri). $ex($  $, n) = \Theta(n \log n)$.

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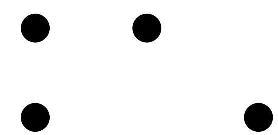
Thm (Füredi,Bienstock-Győri). $ex($  $, n) = \Theta(n \log n)$.

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Conjecture (Pach-Tardos). $ex(P, n)$ is quasi-linear for every P that is the incidence matrix of a tree.

Saturation functions

Saturation function. A matrix M is *saturating* for a pattern P if it avoids P as a submatrix and is maximal with this property, that is, if one changes any 0 entry to a 1 entry in the matrix M then the resulting matrix M' contains P. Let $sat(P, n)$ denote the minimal weight of a 0-1 matrix M of size $n \times n$ saturating for P. The matrix M is *saturating* for P.

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Observation. $ssat(P,n) \leq sat(P,n) \leq ex(P,n)$ and if $sat(P,n) = ex(P,n)$ then every maximal matrix avoiding P has the same weight.

Dichotomy for sat

Theorem (Fulek-K). For any $k \times l$ pattern P

$$sat(P,n) \le (k+l-2)n - (k-1)(l-1) \; and$$

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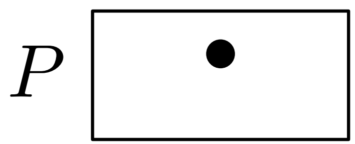

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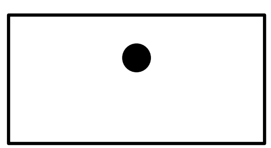
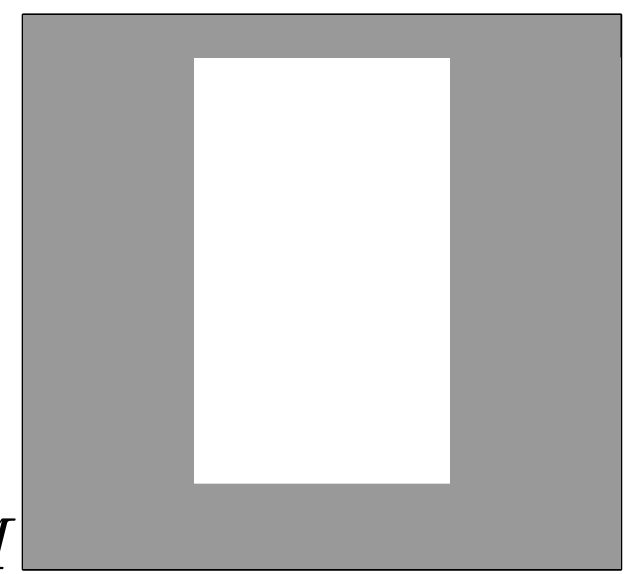


The following M is saturating for P: M has all 1 entries in its first $k'-1$ and last $k-k'$ rows, first $l'-1$ and last $l-l'$ columns.

The first part of the theorem is proved.

M


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Let $k' = \max(k, l)$. Assume that $sat(P, n_0) < \frac{n_0}{k'-1}$ for some $n_0 \geq k' - 1$, we want to show that in this case $sat(P, n) \leq sat(P, n_0) = O(1)$ for $n \geq n_0$.

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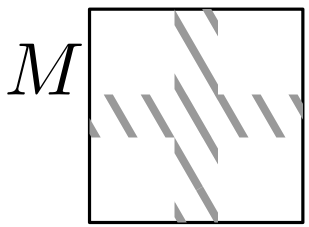
Then exists M $n_0 \times n_0$ saturating matrix with $k-1$ consecutive empty rows and $l-1$ consecutive empty columns.



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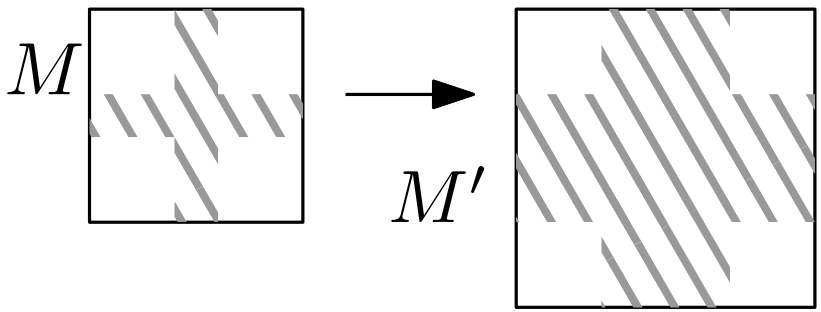

Multiplying these rows and columns we get an $n \times n$ matrix M' saturating for P.

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Theorem (Fulek-K). Given a pattern P, $ssat(P,n) = O(1)$ if and only if all the following properties hold for P:

1. The first and last row of P both contain a 1 entry that is the only 1 entry in its column,
2. The first and last column of P both contain a 1 entry that is the only 1 entry in its row,
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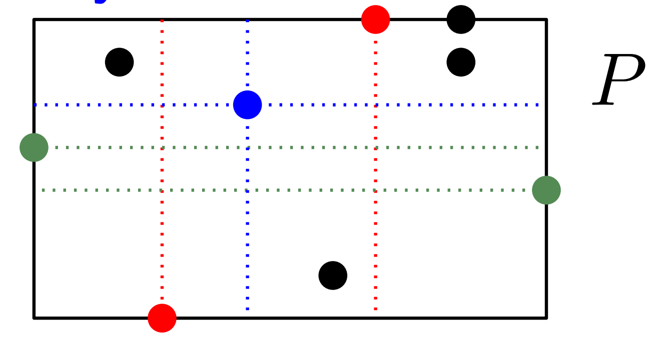

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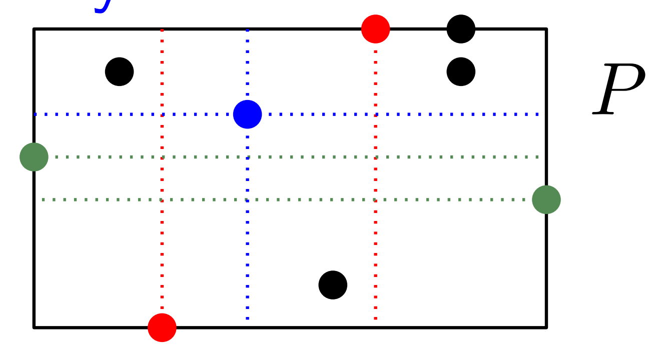
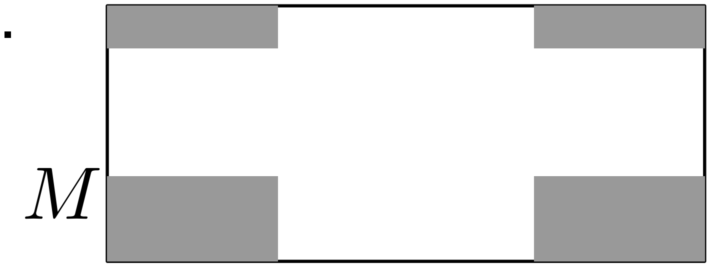


Proof. First suppose (1),(2),(3) all hold.

The $n \times n$ matrix M with 1 entries in $k-1 \times l-1$ size rectangles in all corners semisaturates P.



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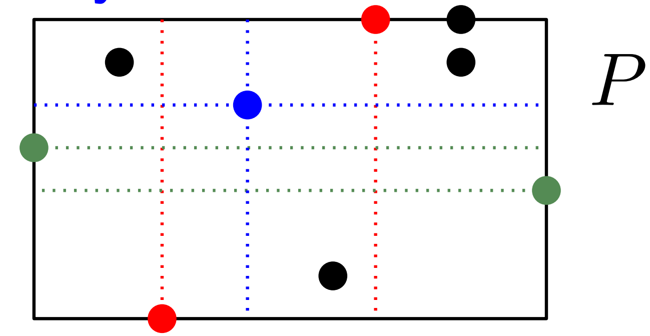

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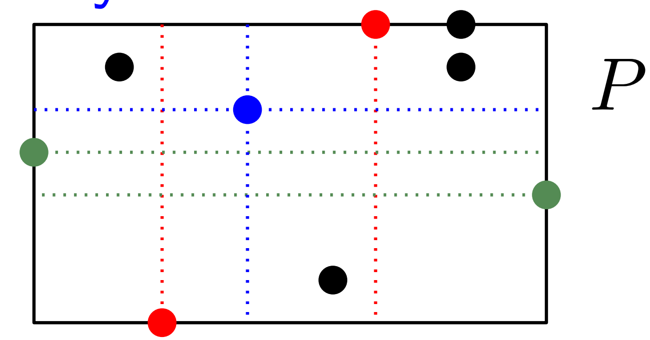

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Then if M is semisaturating for P then it has no empty column. Indeed, otherwise we can put a 1 entry in the first/last position of the column without introducing a new copy of P.

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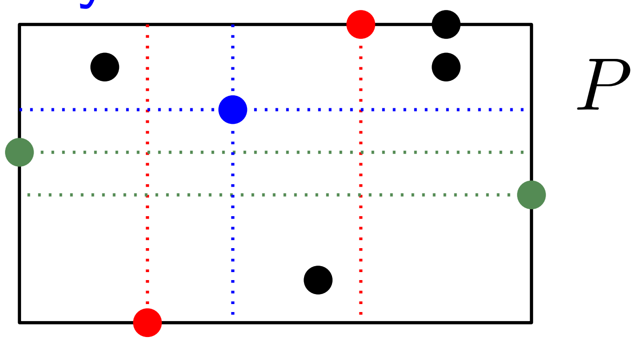

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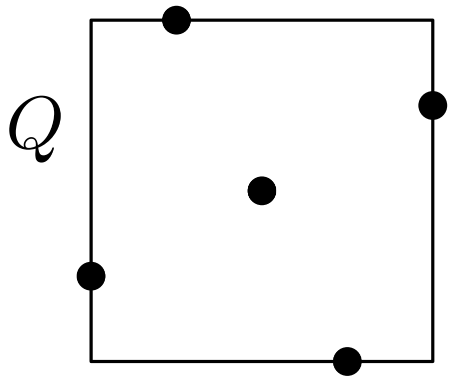
Similarly if (2) then there is no empty row and if (3) then there is no empty row **and** empty column in M.

When is the saturation function bounded?

Theorem (F-K). $sat(Q, n) < 400$.



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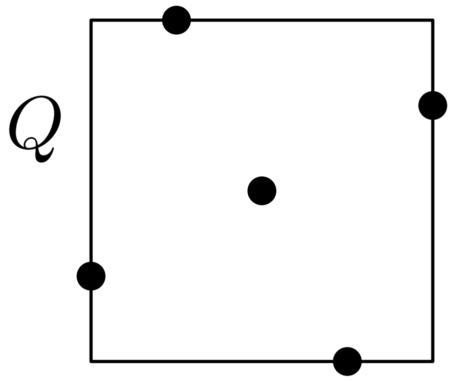

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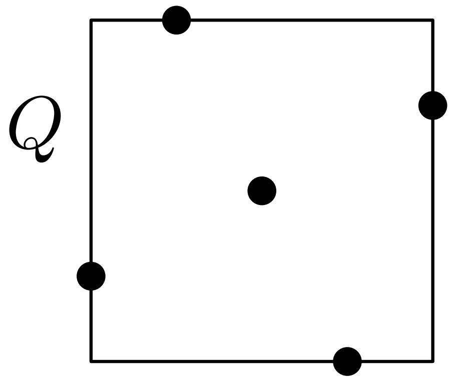

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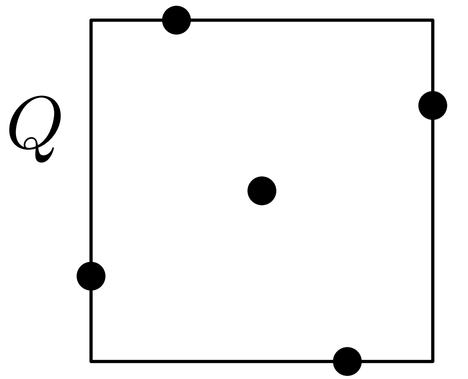

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Theorem (F-K). If $P = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix}$, for some 0-1 not all-0 submatrices A, B, then

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Pattern Q with bounded saturation function

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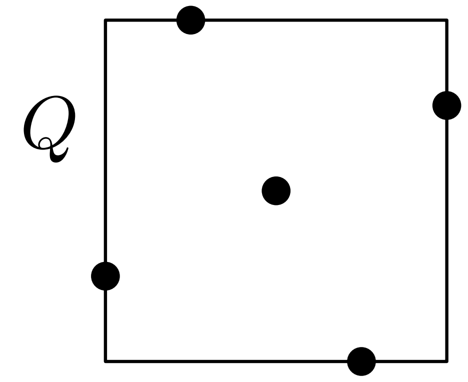

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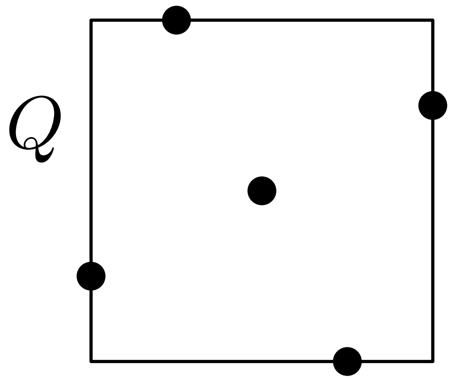

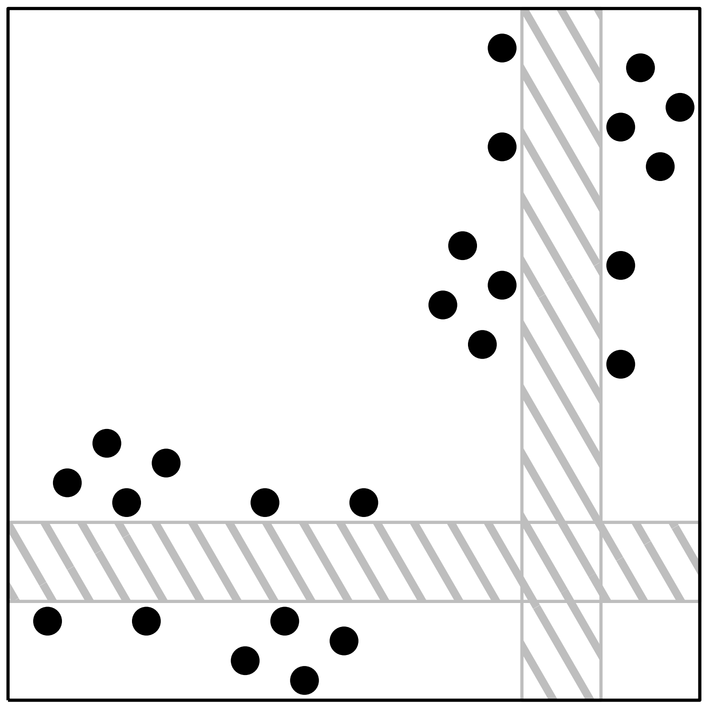

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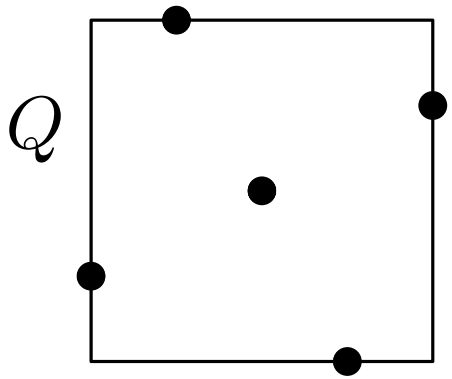

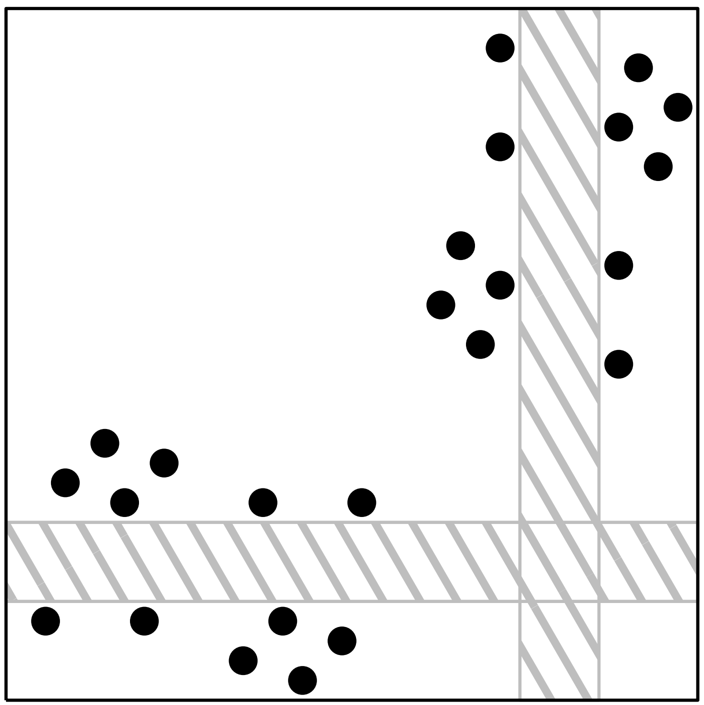

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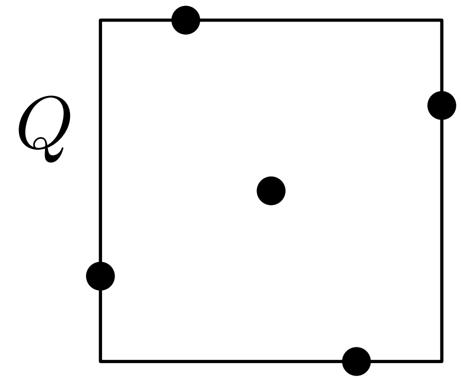

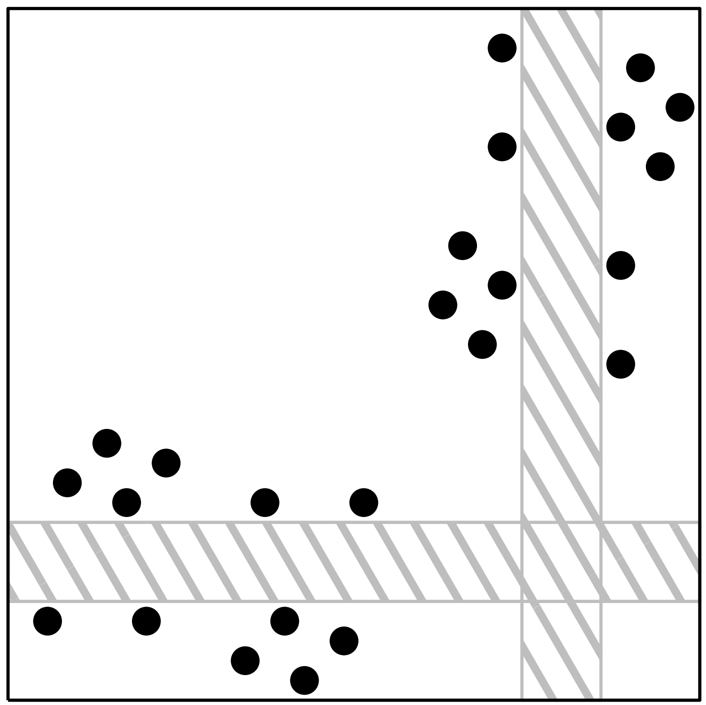

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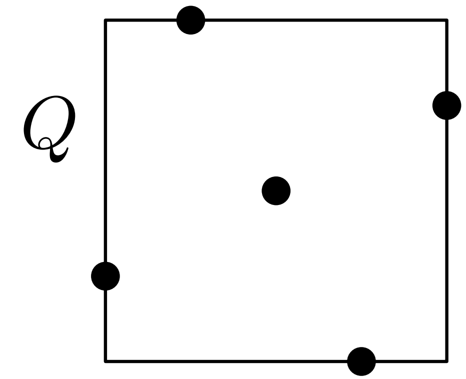

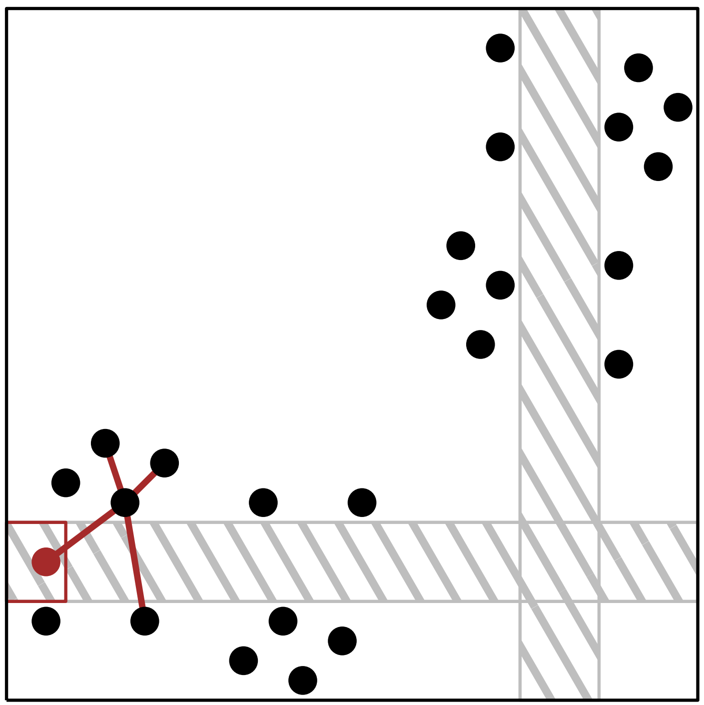

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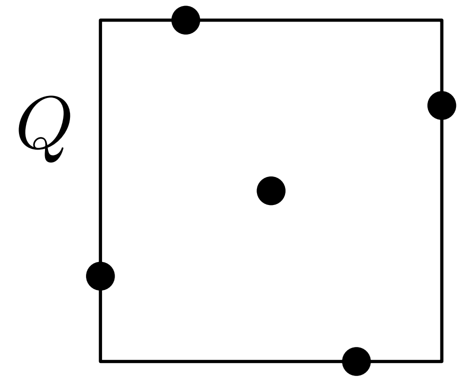

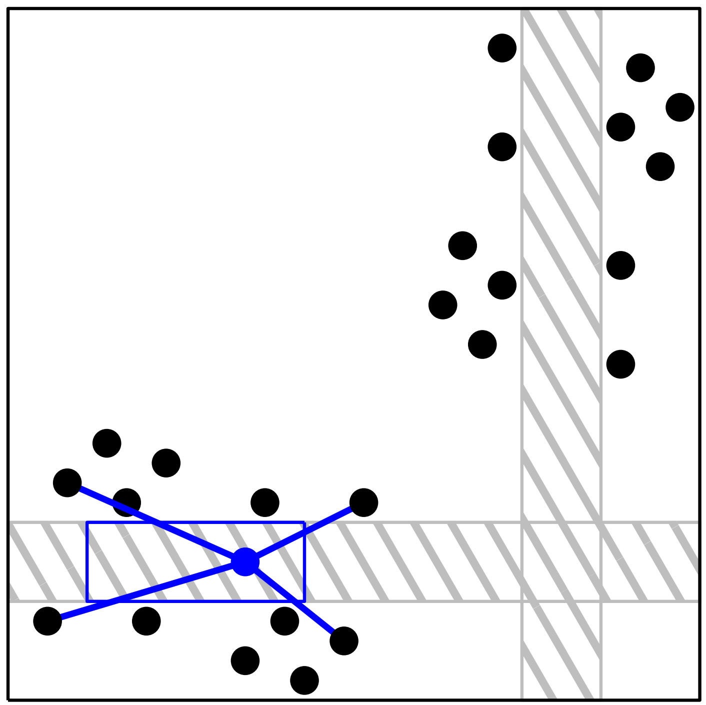

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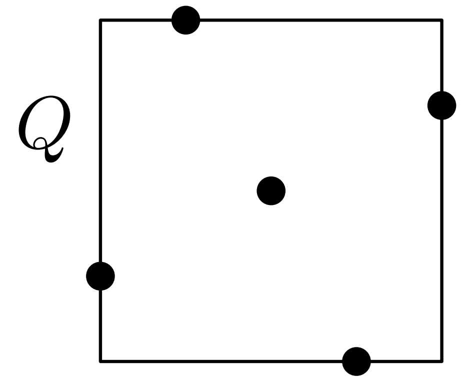

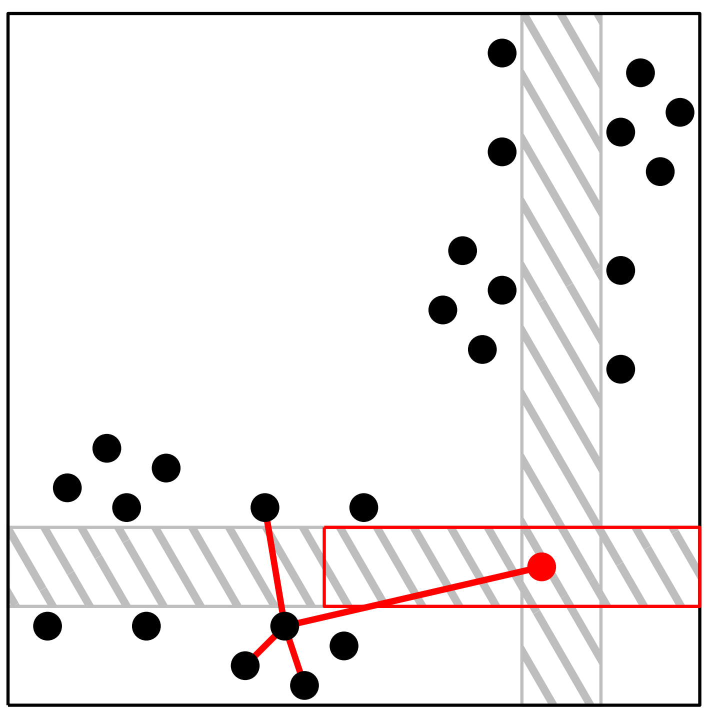

We get a saturating M from this by adding 1's to the white areas until the matrix is saturated for Q.

We are left to prove that adding a 1 intro the empty grey area introduces a copy of Q.

Identity patterns and company

Theorem (Brualdi-Cao 2020).

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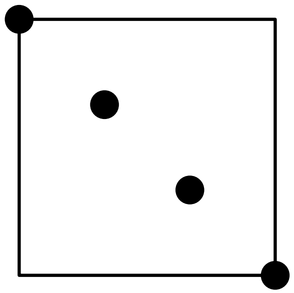

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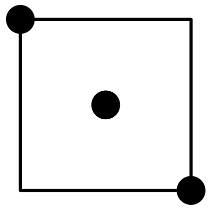

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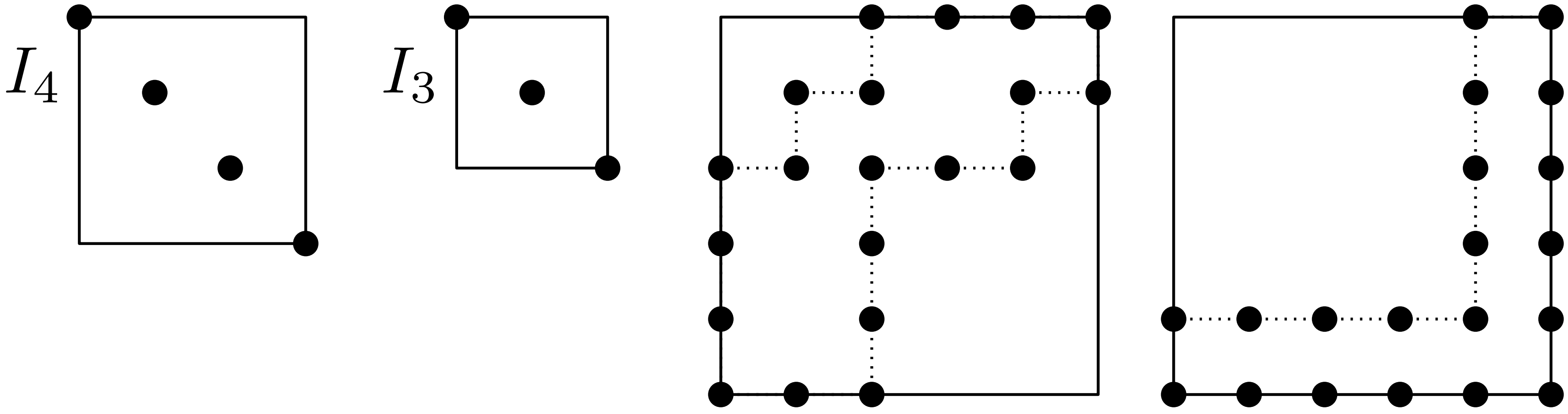

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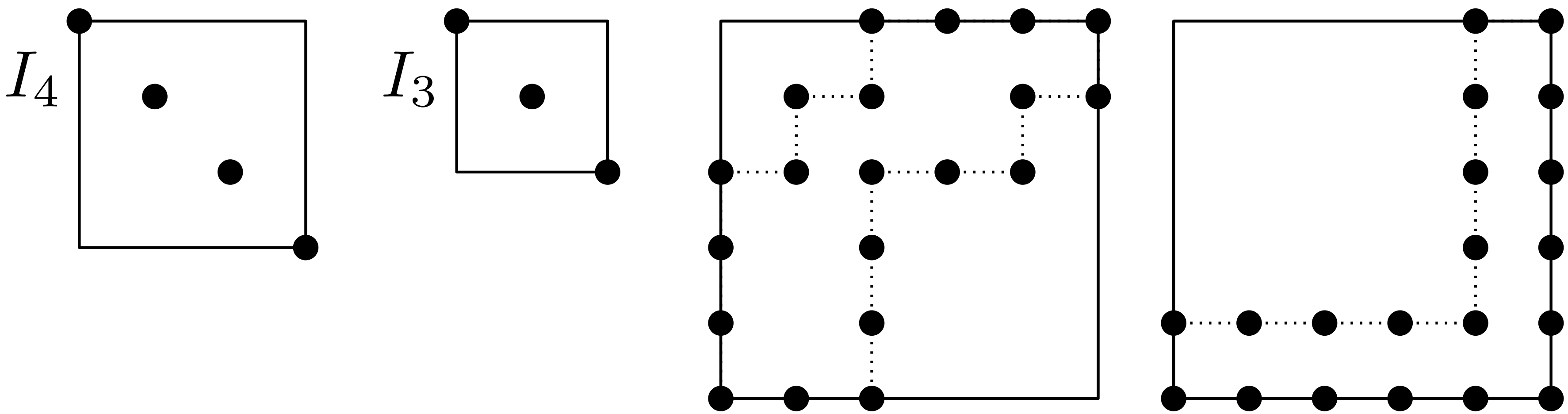

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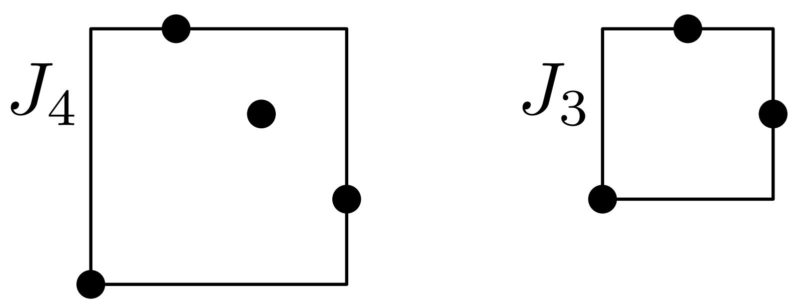

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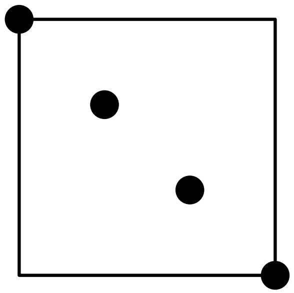
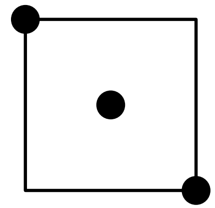
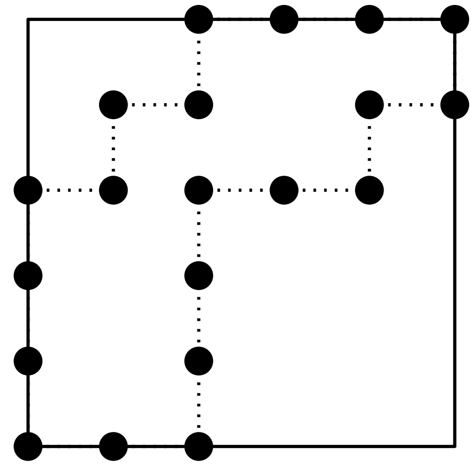
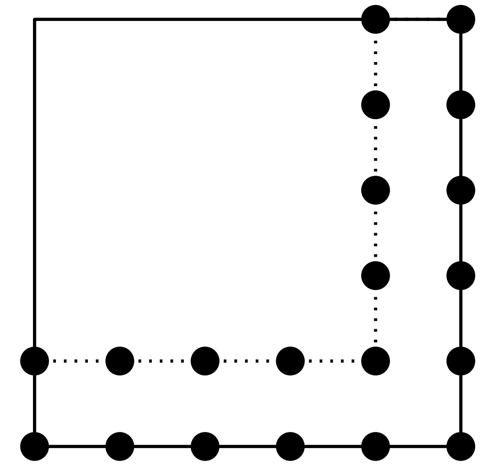
I_4  I_3   

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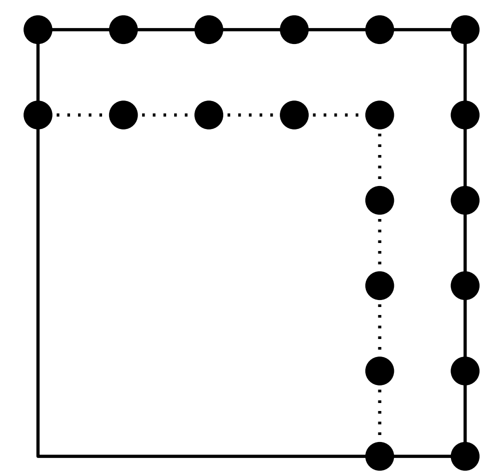
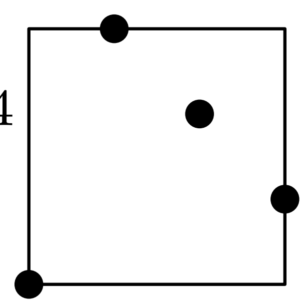
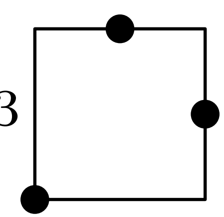


J_4  J_3 

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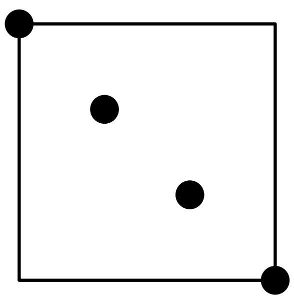

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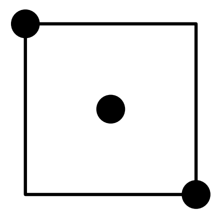

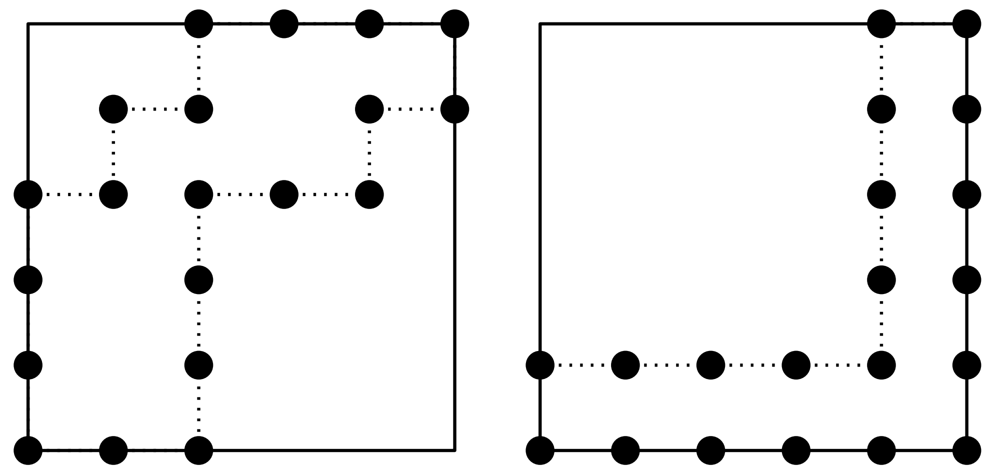

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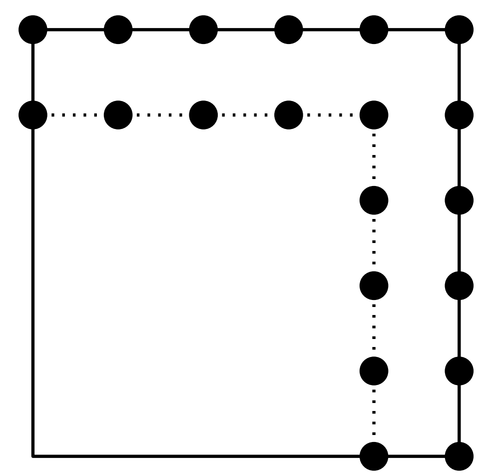

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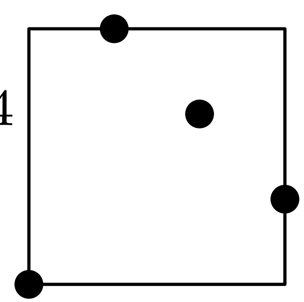

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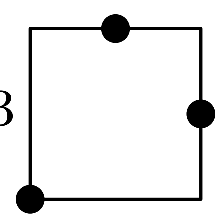

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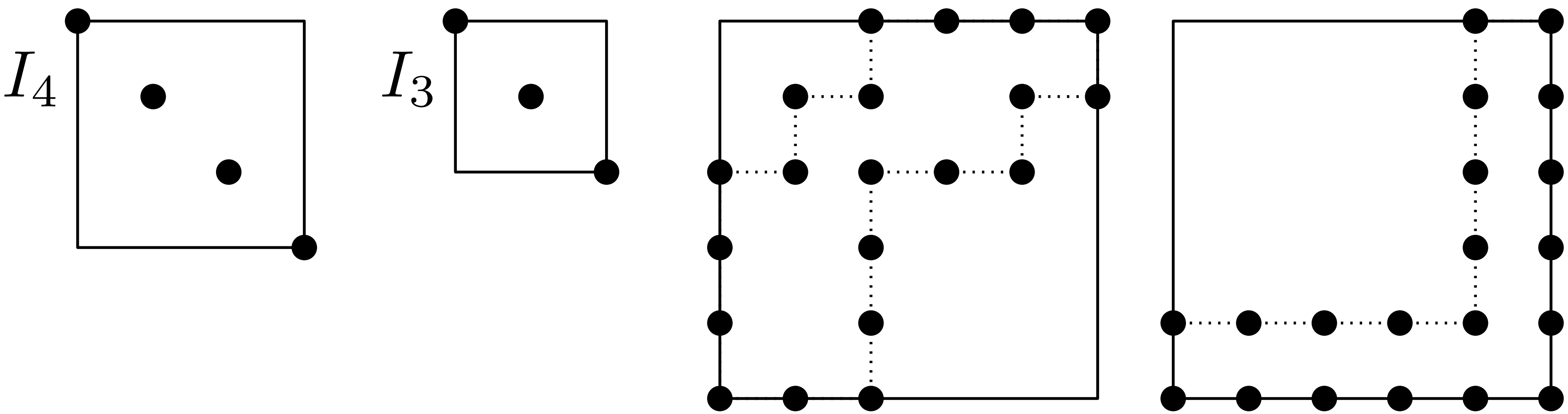

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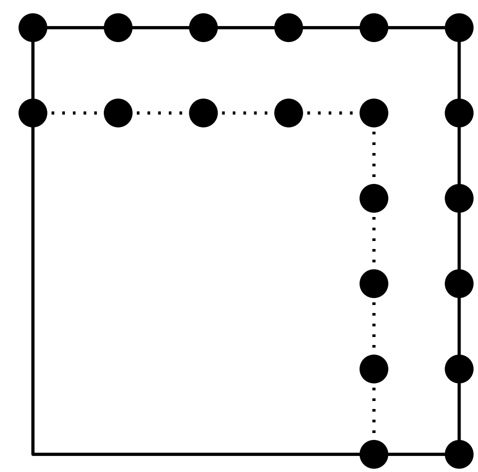

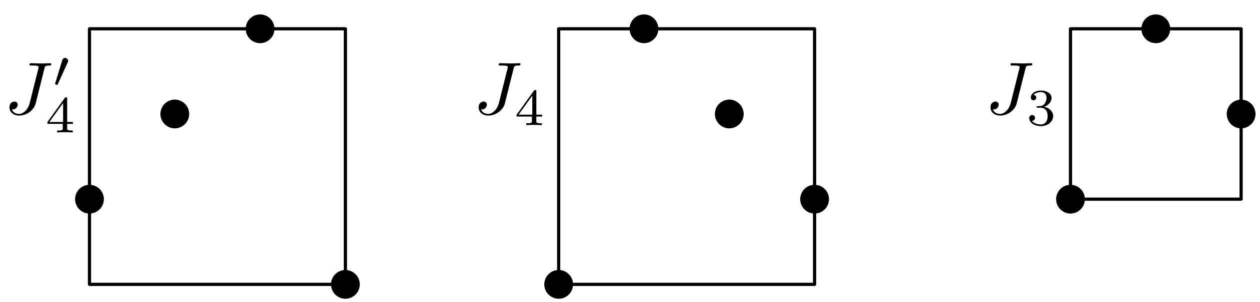

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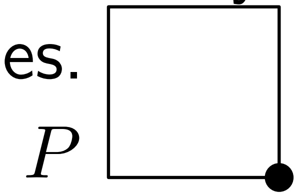
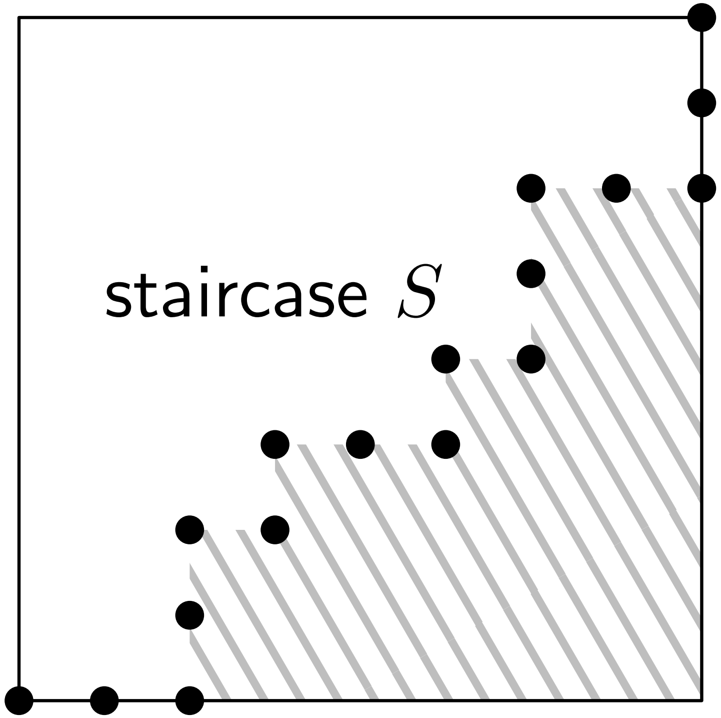
Patterns with a corner 1 entry

Lemma. Given a non-(1×1) pattern P in which the last row and column both contain exactly one 1 entry, which is in their intersection. Then in any matrix M saturating for P there is a staircase S in M such that all positions in S contain a 1 entry and all positions that are below S contain only 0 entries.





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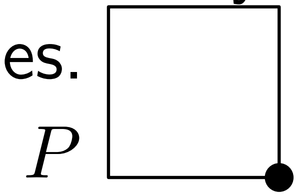

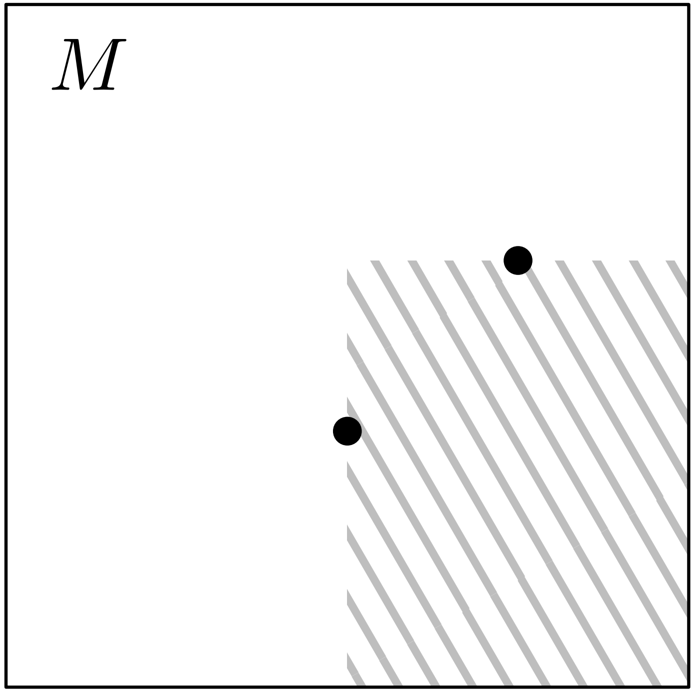

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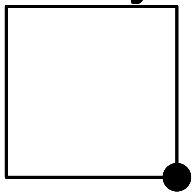

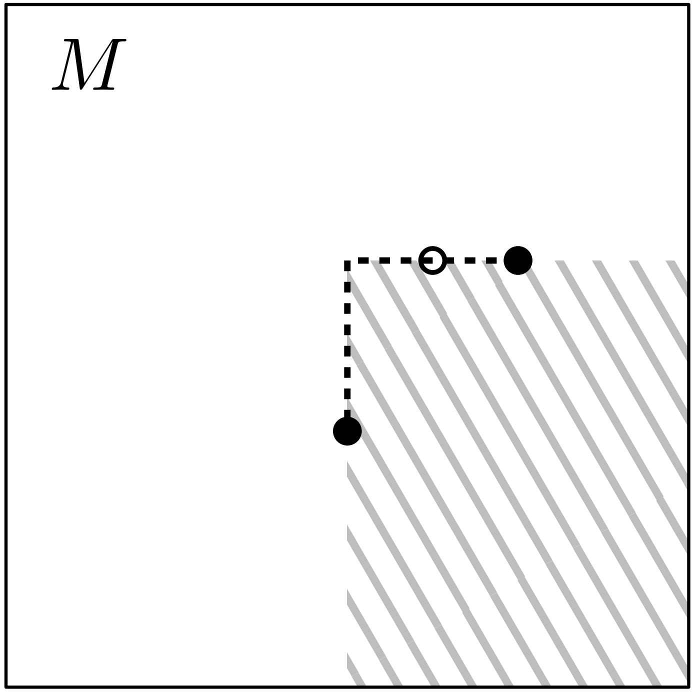

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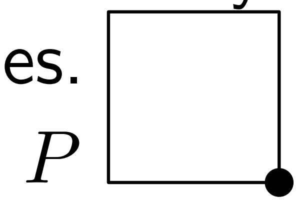

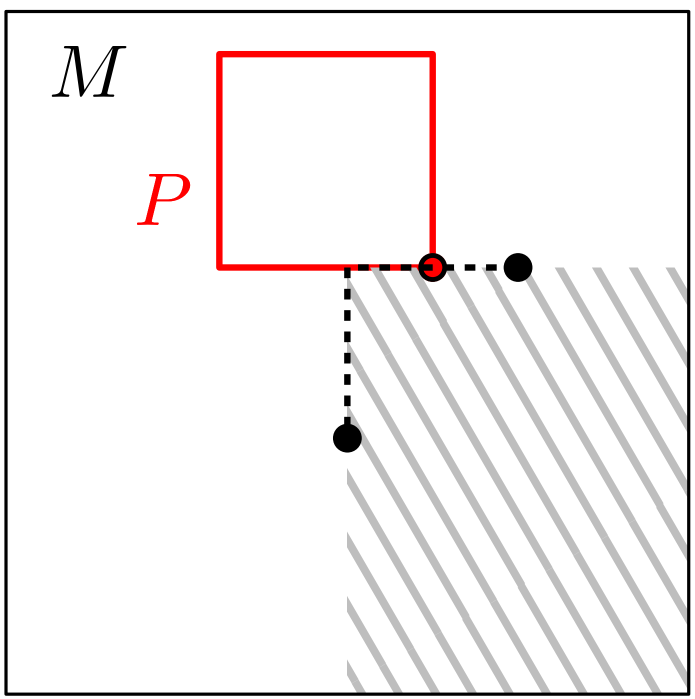

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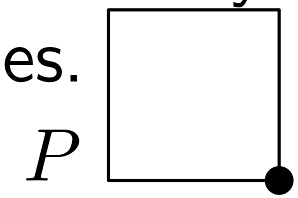

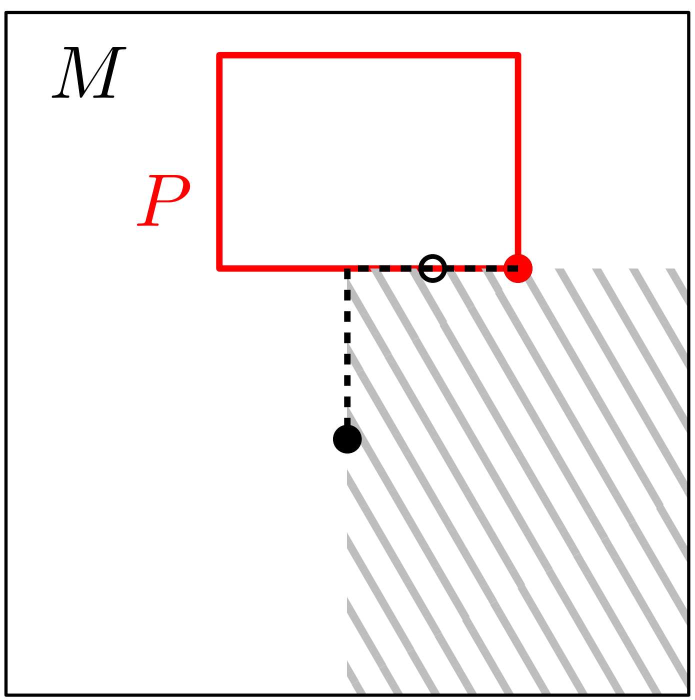

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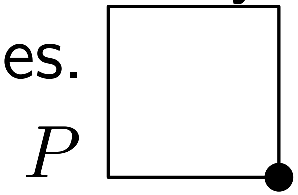

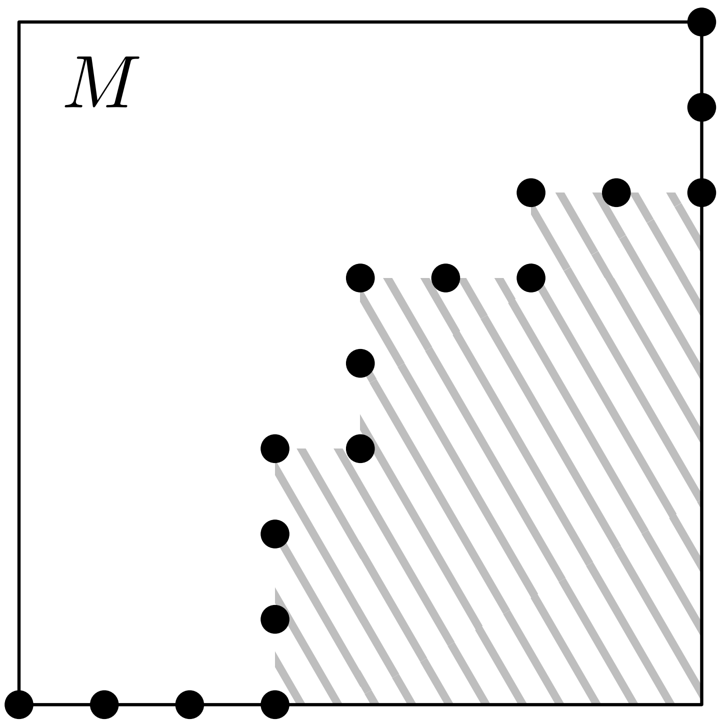

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Repeating this we get the staircase S.

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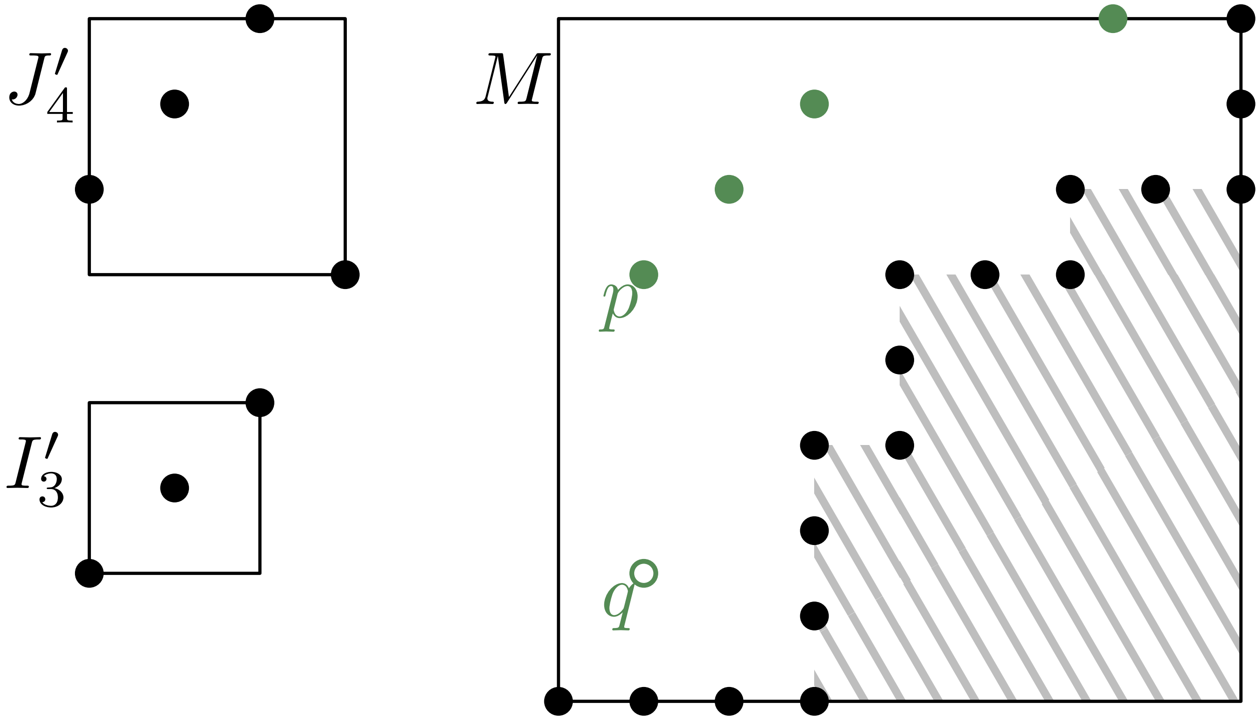

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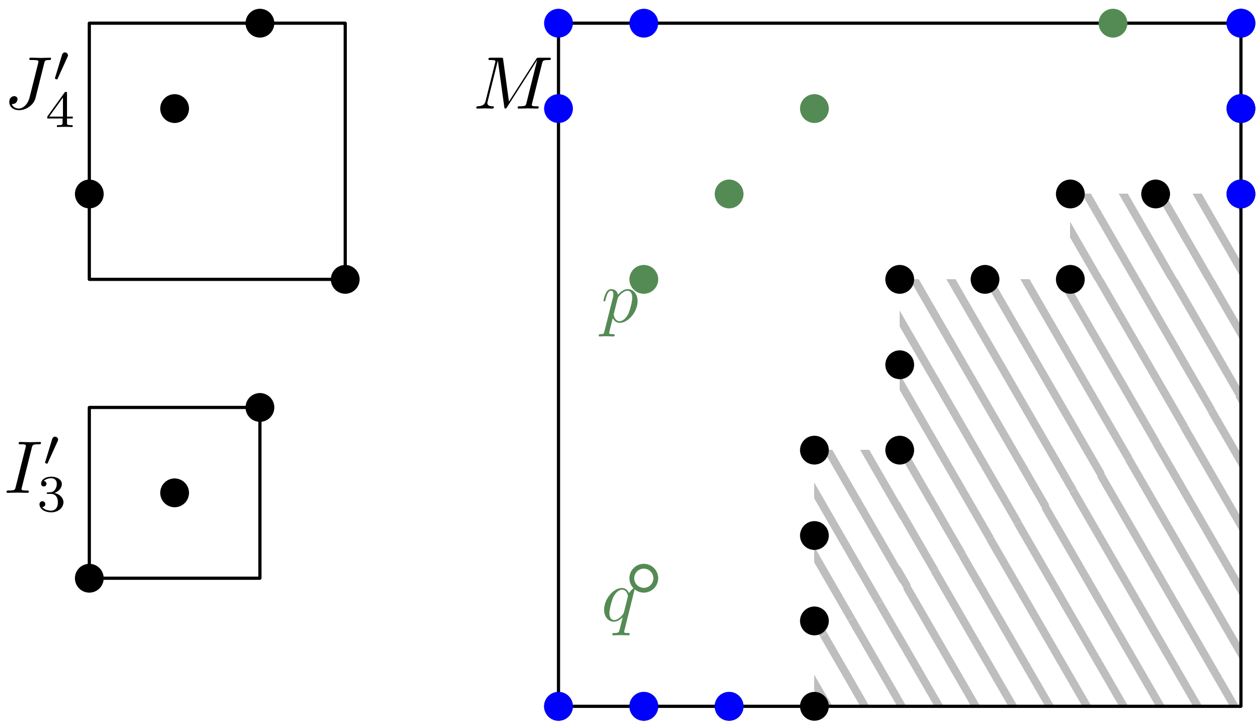

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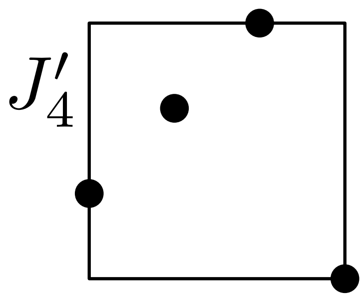

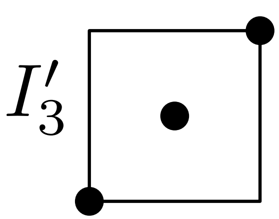

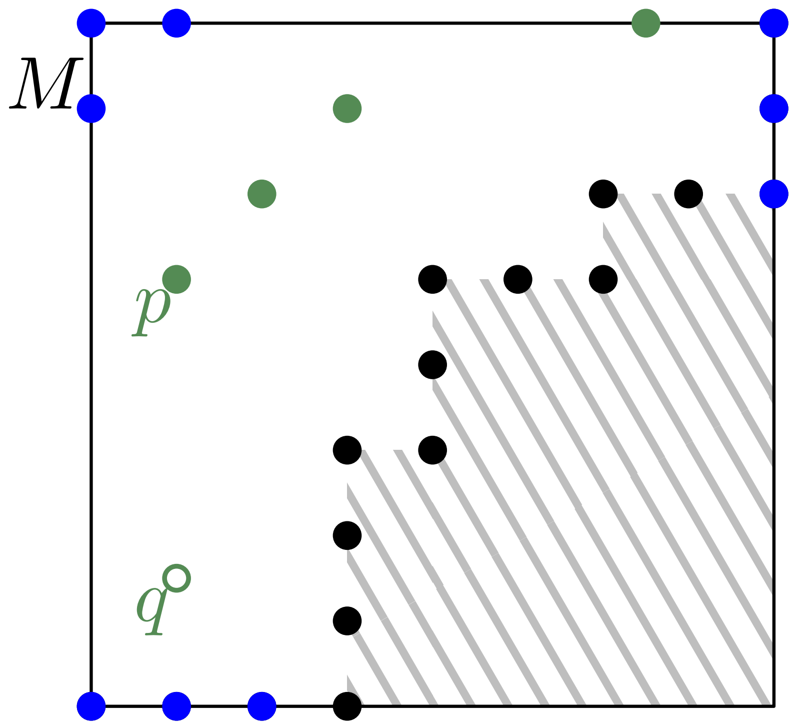

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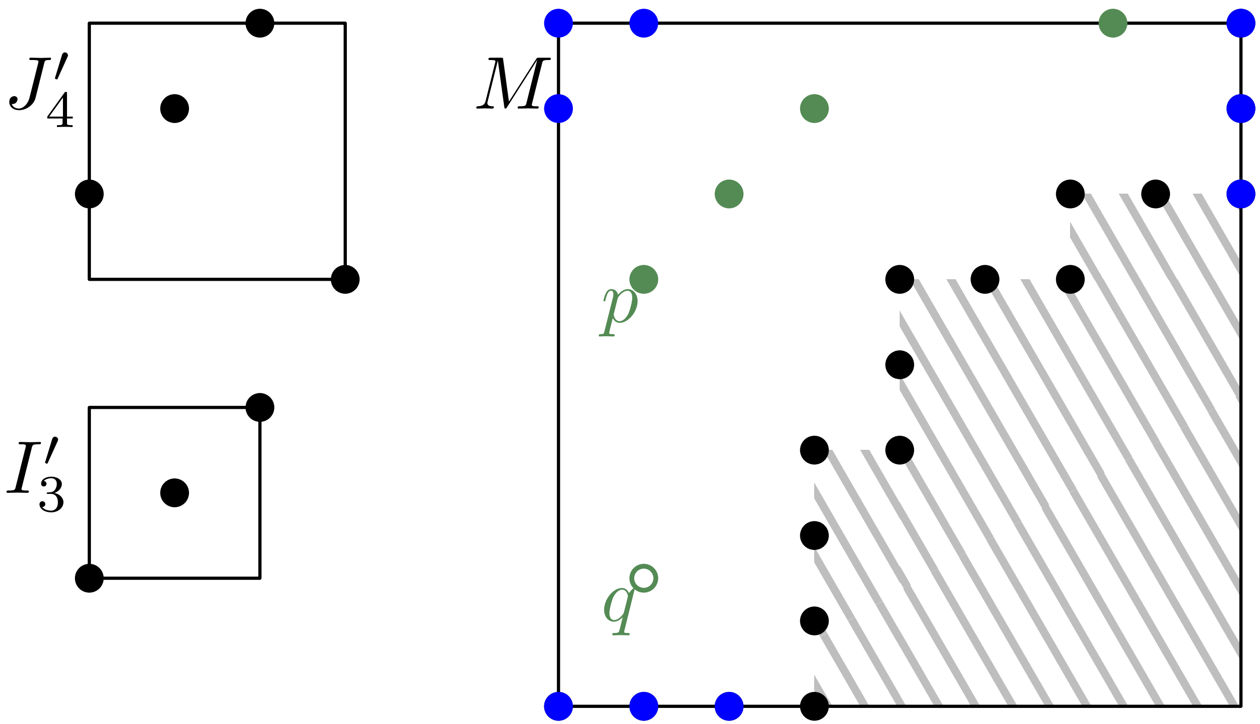

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The right-most position in each row before S has level 1.

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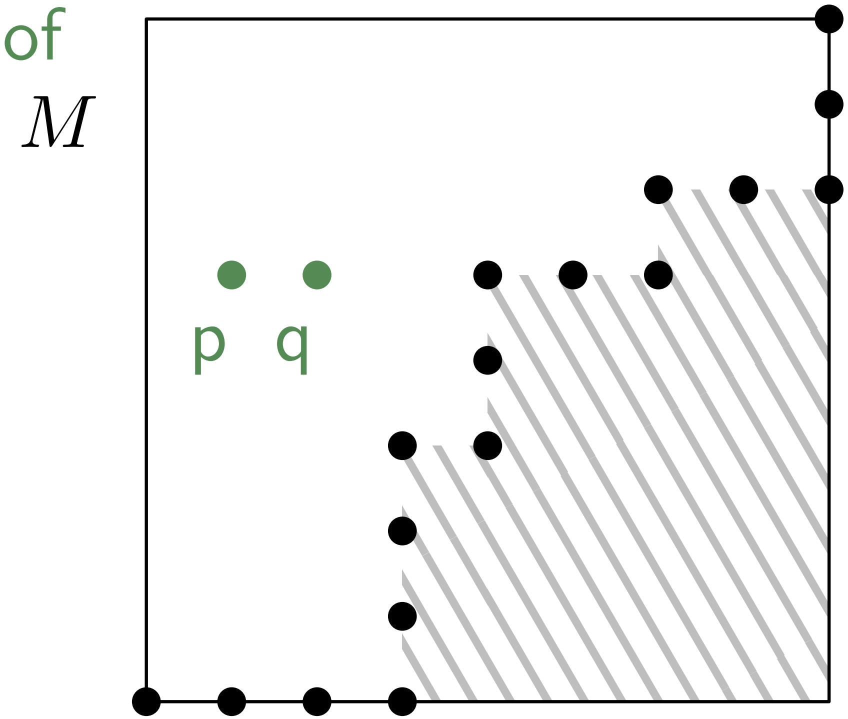

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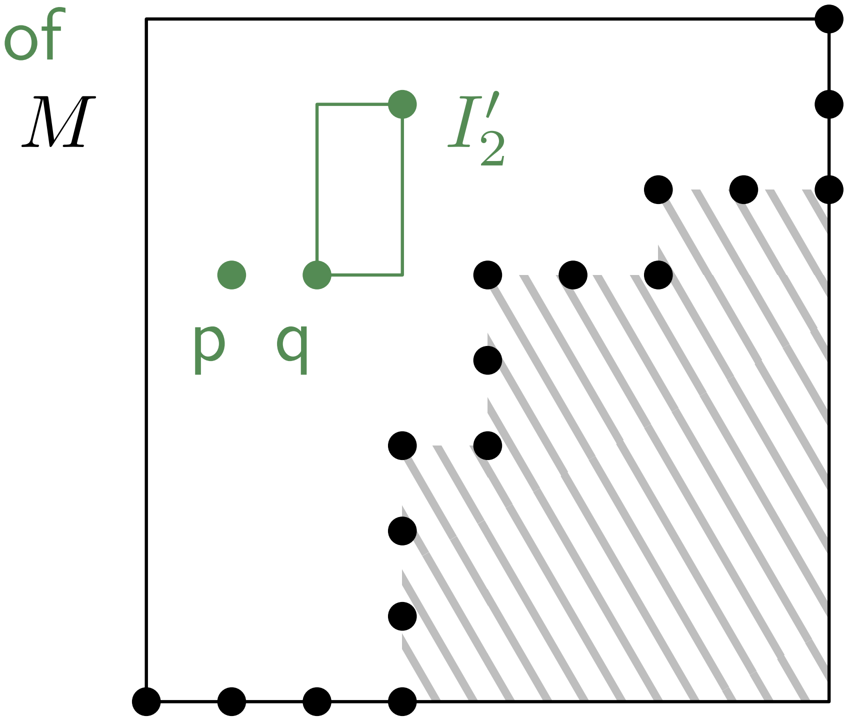

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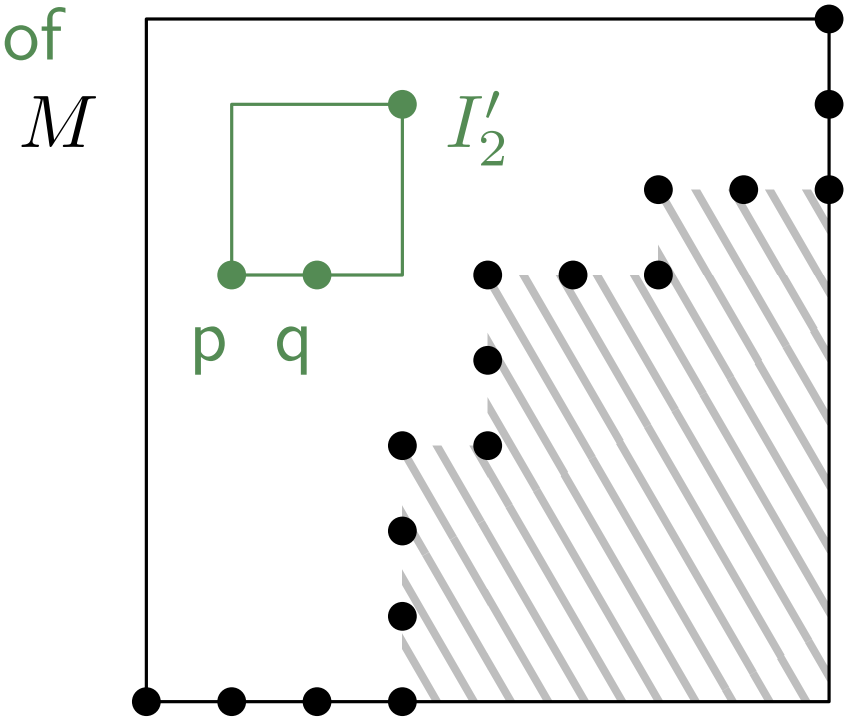

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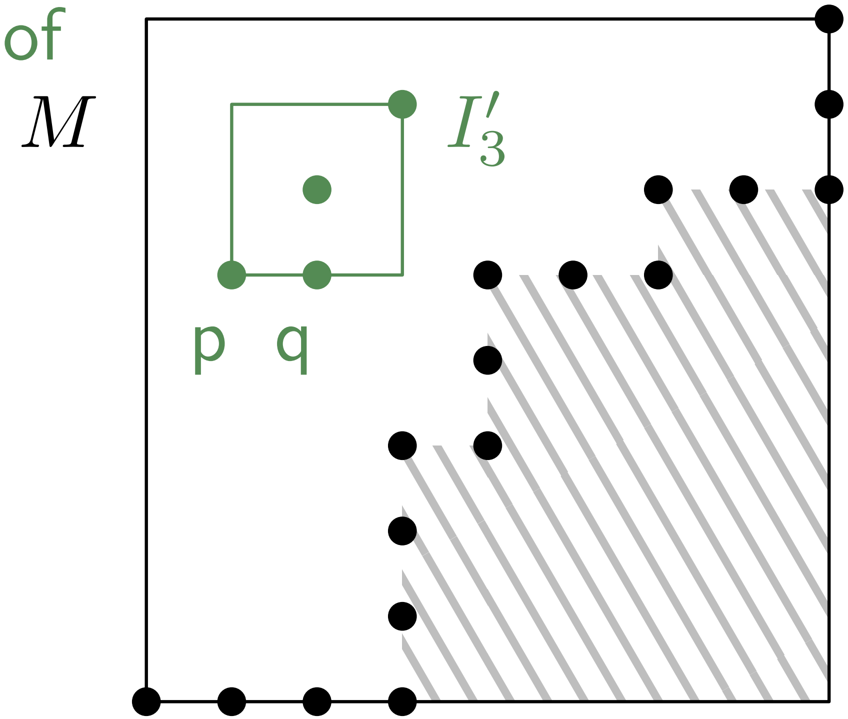

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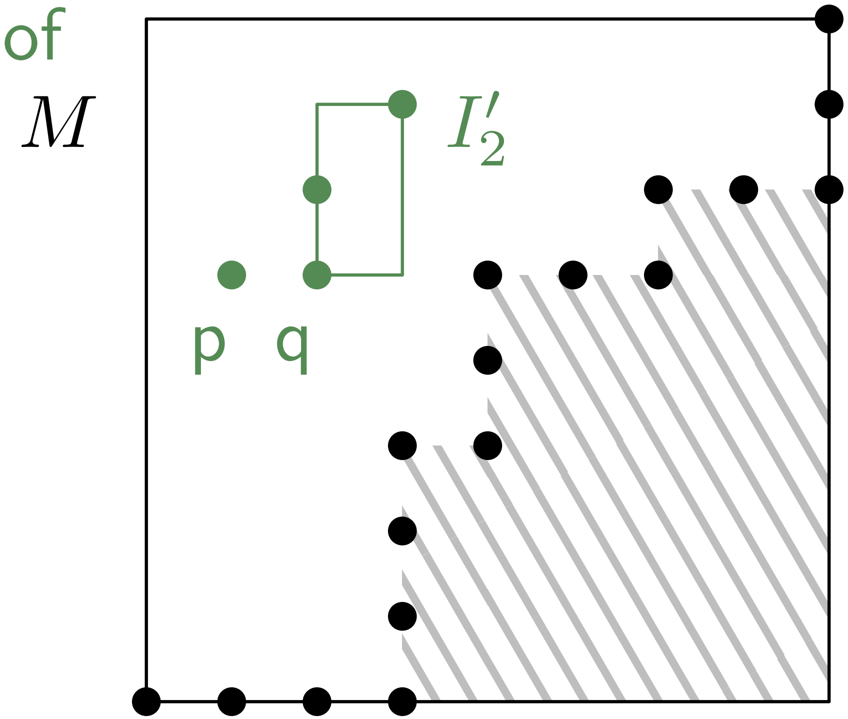

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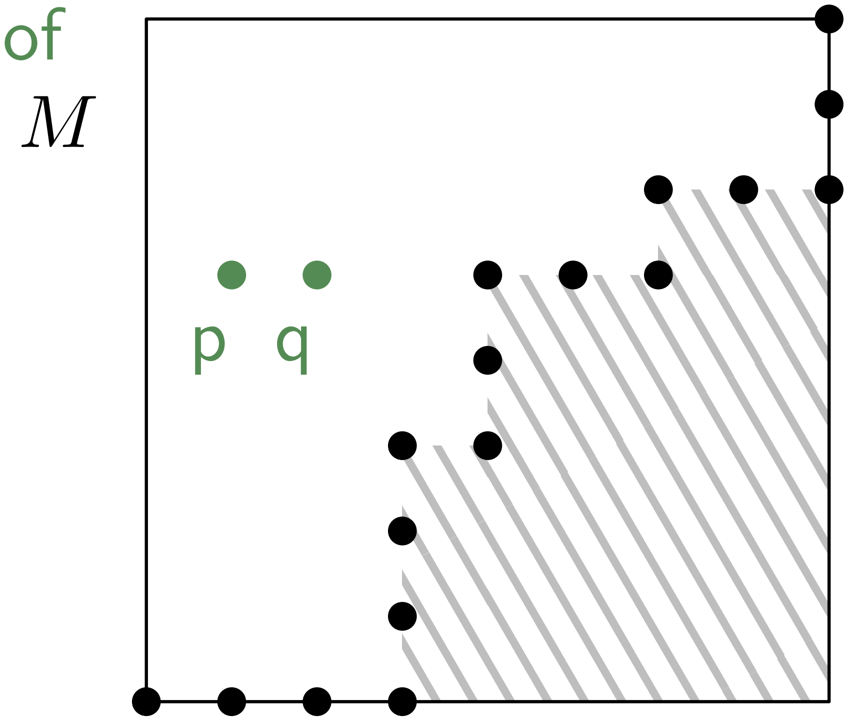

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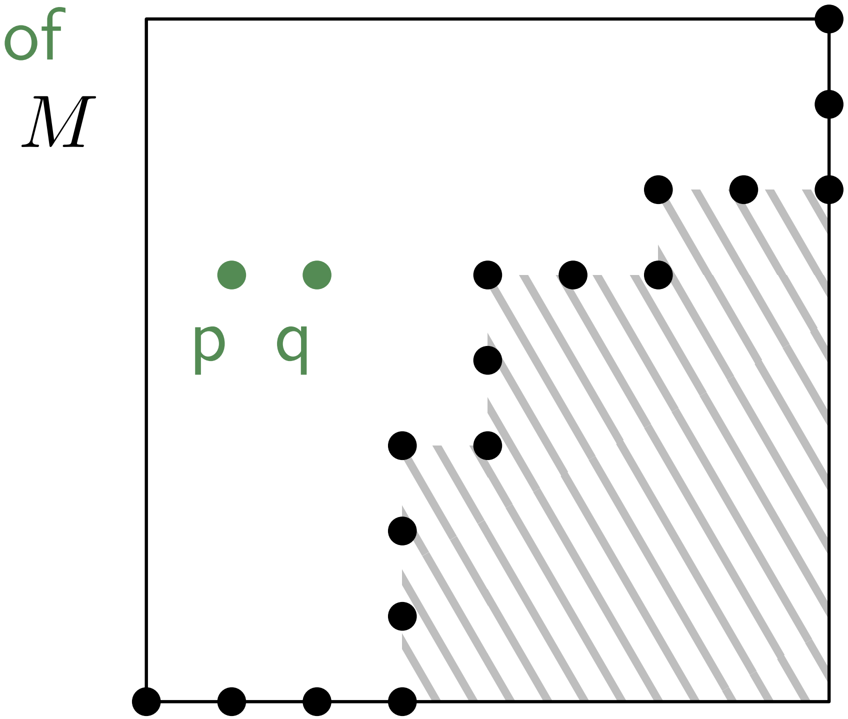

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$\Rightarrow$ There are $\approx n(k-2)$ 1-entries in M above S.

$\Rightarrow$ $\approx nk$ 1-entries including the ones in S $\Rightarrow$ Theorem

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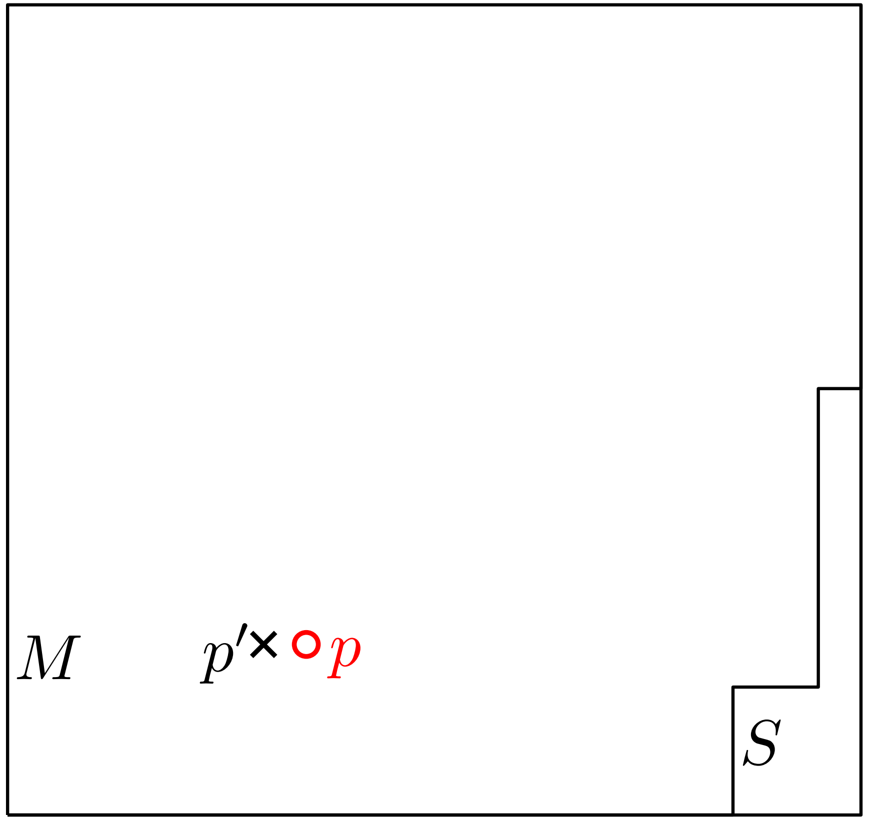

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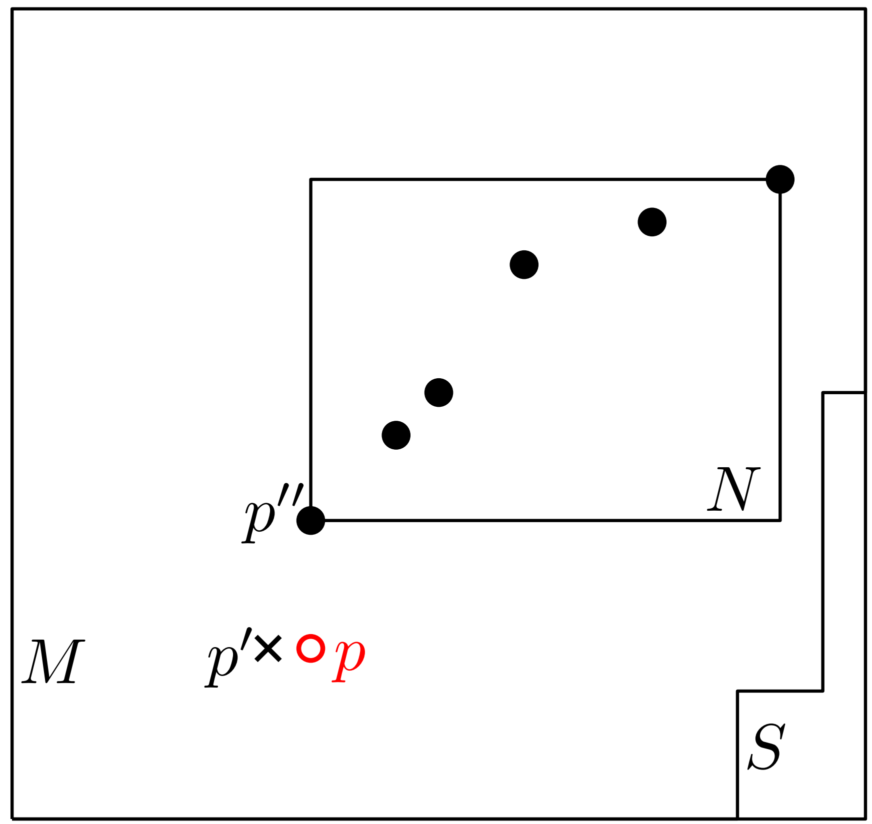

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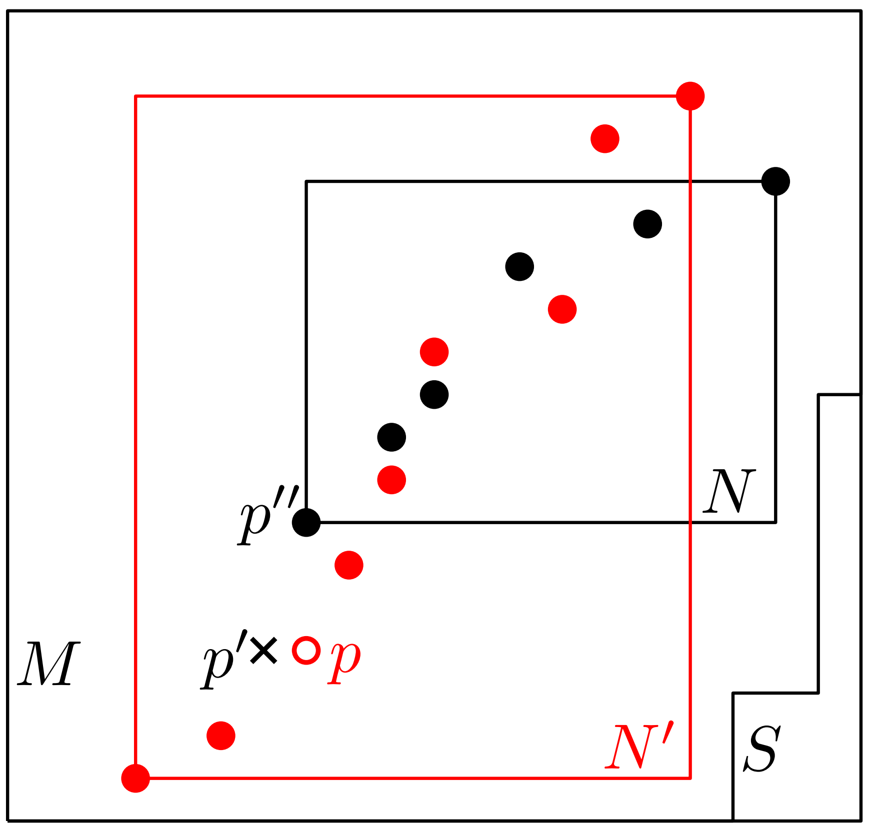
Using saturation at p there exists $N' = I_{k-2}$ above S which would be an I_{k-1} with p



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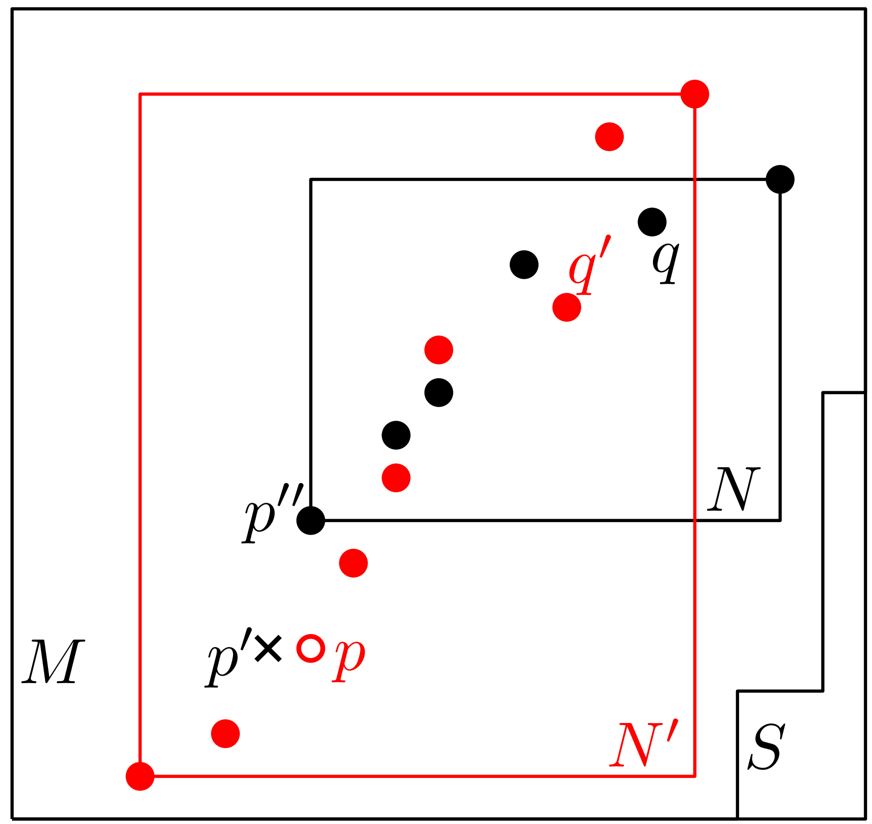

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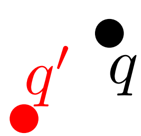
Using saturation at p there exists $N' = I_{k-2}$ above S which would be an I_{k-1} with p

Assume exist 1-entries of N and N' in this arrangement:



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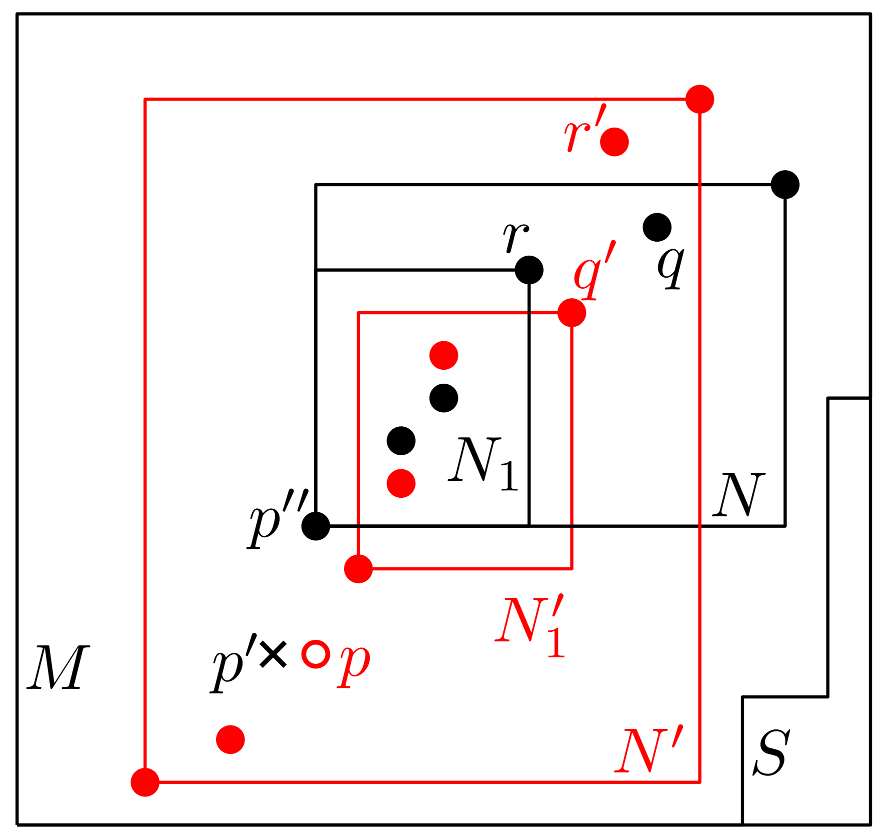

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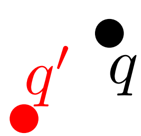

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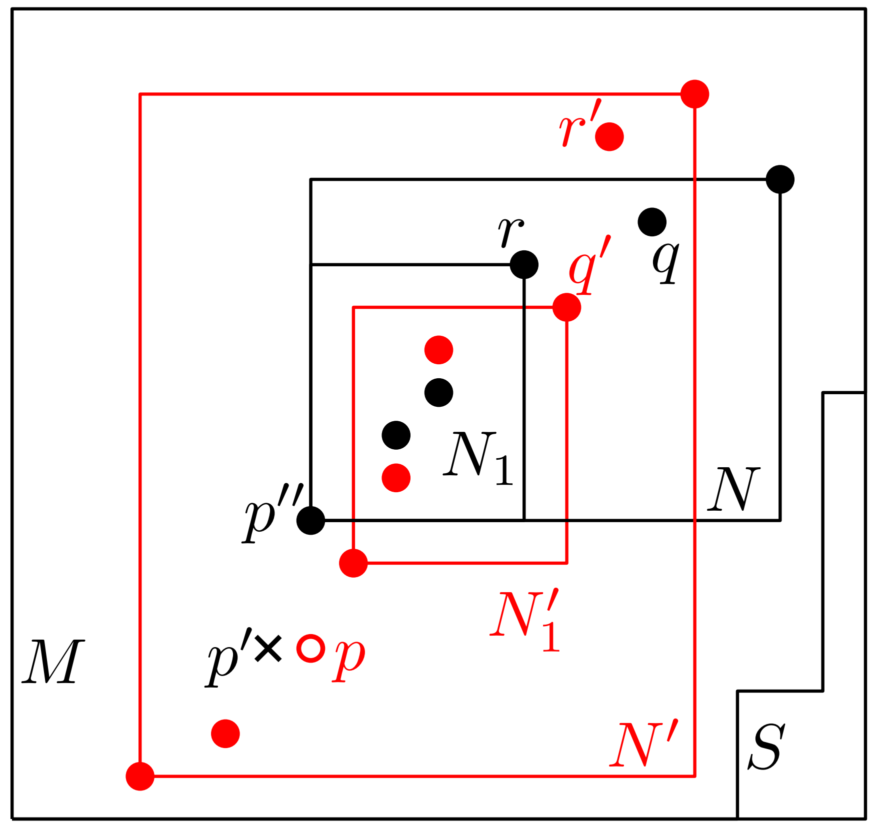

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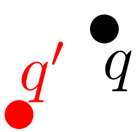

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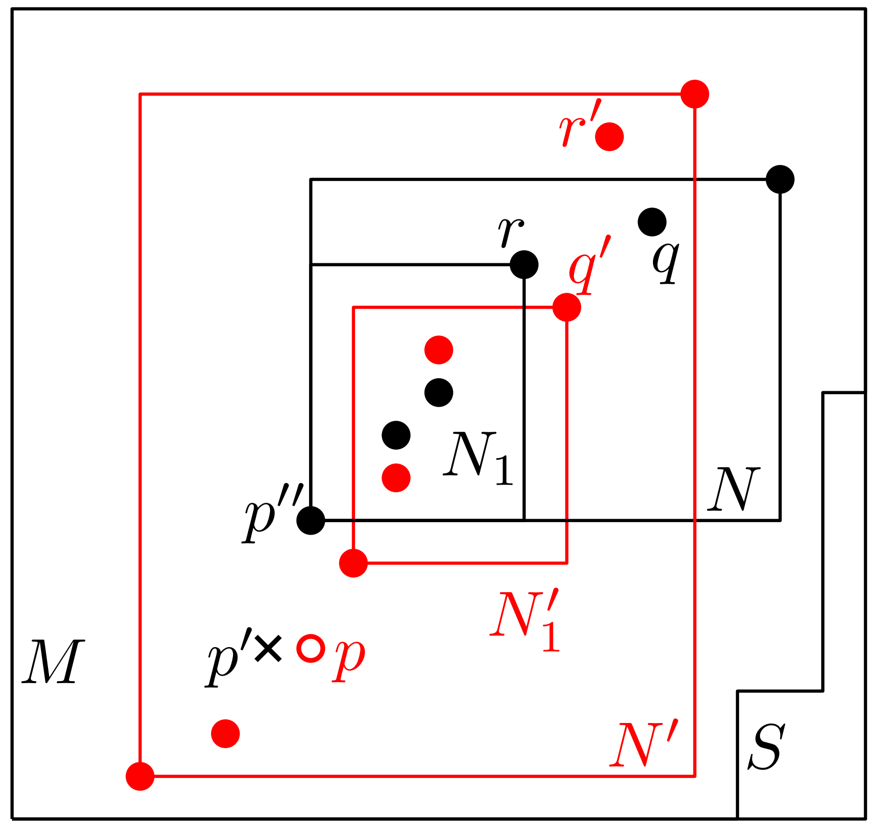

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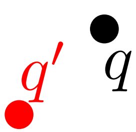

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Otherwise, replacing N_1 with N_1' we get an I_l' from N showing that p has level at least $l+1$, contradiction.

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Thank you for your attention