Summary of the Ph.D. thesis 'Linear functional equations'

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1 Introduction

We are concerned with the linear functional equation

$$\sum_{i=1}^{n} a_i f(b_i x + c_i y) = 0 \qquad (x, y \in \mathbb{C}) \tag{1}$$

where a_i, b_i, c_i are given complex numbers, and $f : \mathbb{C} \to \mathbb{C}$ is the unknown function.

We shall use the following notations. Let $(G, +)$ be an Abelian group. The difference operator Δ_h is defined by

$$\Delta_h f(x) = f(x + h) - f(x) \qquad (x, h \in G)$$

for every $f : G \to \mathbb{C}$. A function $f\colon G \to \mathbb{C}$ is called a *generalized polynomial* if there is an n such that $\Delta_{h_1} \dots \Delta_{h_{n+1}} f(x) = 0$ for every $h_1, \dots, h_{n+1}, x \in G$. The smallest n for which f satisfies this condition is called the degree of f. We note that every polynomial $p : \mathbb{C} \to \mathbb{C}$ is a generalized polynomial with the same degree but the family of generalized polynomials are wider. We say that the function $f : G \to \mathbb{C}$ is *additive*, if f is a homomorphism of G into the additive group of $\mathbb{C}$. A function f is a generalized polynomial of degree 1 if and only if there is an additive function a such that $f - a$ is constant.

By a well-known result of L. Székelyhidi [10], under some mild conditions on the equation (see (2) below), every solution of equation (1) is a generalized polynomial. But the finer structure of the solutions has been investigated only recently. The description of the space of solutions is the main object of the dissertation.

Let $\mathbb{C}^G$ denote the linear space of all complex valued functions defined on G equipped with the product topology. By a *variety* on G we mean a translation invariant closed linear subspace of $\mathbb{C}^G$. A function is a *polynomial* if it belongs to the algebra generated by the constant functions and the additive functions. A nonzero function $m \in \mathbb{C}^G$ is called an *exponential* if m is multiplicative; that is, if $m(x+y) = m(x) \cdot m(y)$ for every $x, y \in G$. An *exponential monomial* is the product of a polynomial and an exponential, a *polynomial-exponential function* is a finite sum of exponential monomials. If a variety contains an exponential element, then we say that *spectral analysis* holds on this variety. If a variety is spanned by exponential monomials, then we say that *spectral synthesis* holds on this variety. If spectral analysis or synthesis holds in every variety on G, then we say that spectral analysis or synthesis holds on G, respectively.

The most important contribution of our results to the theory of linear functional equations is the application of spectral analysis and synthesis to some varieties related to the spaces of solutions of the equations. The idea of the algebraic point of view of the spectral analysis and synthesis on locally compact Abelian groups goes back to the pioneer work of L. Schwartz [8]. The investigation for case of the discrete Abelian groups started by M. Laczkovich, G. Székelyhidi and L. Székelyhidi [5, 6, 11]. The idea of applying spectral analysis to the varieties related to the space of solutions first appeared in [3]. The method of spectral synthesis was first used in [1] and in full generality was proved in [2].

2 Existence of nonzero solutions of linear functional equations

By the result of L. Székelyhidi [10], the following condition on the parameters implies that every solution of (1) is a generalized polynomial.

The numbers $a_1, \ldots, a_n$ *are nonzero, and* **there exists** *an* $1 \leq i \leq n$ *such that* $b_i c_j \neq b_j c_i$ *holds for any* $1 \leq j \leq n,\ j \neq i$. (2)

Hereinafter, we assume this condition hence every solution is a generalized polynomial, although without this assumption there can be found some other solutions of (1). For some special case we can describe the space of solutions but for full generality it is still open.

We shall restrict our attention to the solutions defined on a subfield K of $\mathbb{C}$, more regularly on the field $\mathbb{Q}(b_1, \ldots, b_n, c_1, \ldots, c_n)$. This is justified in that *any function* $f : K \to \mathbb{C}$ *satisfying* $\sum_{i=1}^{n} a_i f(b_i x + c_i y) = 0$ *for every* $x, y \in K$ *can be extended to a solution on* $\mathbb{C}$.

The idea of applying spectral analysis to varieties on $K^* = \{x \in K : x \neq 0\}$ (which is an Abelian groups with the multiplication) and on $(K^*)^k$ was introduced in [3] .

For the existence of non-constant solutions it needs two ingredients. First, we use the fact that if there is a non-constant solution of (1), then there exists a nonzero additive solution, as well. The second one is following:

Theorem 2.1. *There is a nonzero additive solution of* (1) *if and only if there exists a solution of* (1) *which is an automorphism* $\phi \colon \mathbb{C} \to \mathbb{C}$ *or, equivalently,* *an automorphism satisfying*

$$\sum_{i=1}^{n} a_i\phi(b_i) = 0 \text{ and } \sum_{i=1}^{n} a_i\phi(c_i) = 0. \tag{3}$$

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Theorem 2.1 has many applications. We show a generalization of the theorem of A. Varga [12].

Theorem 2.2. (i) *Suppose that the parameters $b_1, \ldots, b_s$ are algebraic numbers and $b_{s+1}, \ldots, b_n$ are algebraically independent over $\mathbb{Q}$, where $0 \le s < n$. If the parameters $a_1, \ldots, a_n$ are algebraic numbers, then*

$$\sum_{i=1}^{n} a_i f(b_i x) = 0 \tag{4}$$

has no nonzero additive solution.

(ii) *Suppose that the parameters $a_1, \ldots, a_s$ are algebraic numbers and $a_{s+1}, \ldots, a_n$ are algebraically independent over $\mathbb{Q}$, where $0 \le s < n$. If the parameters $b_1, \ldots, b_n$ are algebraic numbers, then* (4) *has no nonzero additive solution.*

We can generalize Theorem 2.1 for the existence of generalized polynomials of degree $k > 1$ in the following way.

Theorem 2.3. *For every positive integer k the following are equivalent.*

(i) *There exists a generalized polynomial of degree k which is a solution of* (1)*.*

(ii) *There exist field automorphisms $\phi_1, \ldots, \phi_k$ of $\mathbb{C}$ such that $\phi_1 \cdot \ldots \cdot \phi_k$ is a solution of* (1)*.*

(iii) *There exist field automorphisms $\phi_1, \ldots, \phi_k$ of $\mathbb{C}$ such that*

$$\sum_{i=1}^{n} a_i \prod_{j\in J} \phi_j(b_i) \prod_{j'\notin J} \phi_{j'}(c_i) = 0$$

for every $J \subseteq \{1, \ldots, k\}$.

3 Space of solutions of (1)

In this section we also assume that the every solution is generalized polynomial, which is true if we assume condition (2).

3.1 Algebraic inner parameters

First we deal with the additive case ($k = 1$), moreover we start with the special case when b_i and c_i are algebraic numbers. The following theorem is a direct application of spectral synthesis proved in [6].

Theorem 3.1. *Let $b_1, \ldots, b_n, c_1, \ldots, c_n$ be algebraic numbers, and put $K = \mathbb{Q}(b_1, \ldots, b_n, c_1, \ldots, c_n)$. Then every additive solution of (1) defined on K is of the form*

$$d_1\phi_1 + \cdots + d_k\phi_k,$$

where $d_1, \ldots, d_k$ are complex numbers and $\phi_1, \ldots, \phi_k : K \to \mathbb{C}$ are injective homomorphisms satisfying

$$\sum_{i=1}^{n} a_i\phi_j(b_i) = 0 \text{ and } \sum_{i=1}^{n} a_i\phi_j(c_i) = 0 \tag{5}$$

for every $j \in \{1, \ldots, k\}$.

This result can be easily generalized composing Theorems 2.3 and 3.1.

Theorem 3.1 might suggest that if there are many injective homomorphisms which are solutions of (1), then the closed linear space generated by these injective homomorphisms contains every additive solution as well. This is not true in general.

Theorem 3.2. *Let $K \subset \mathbb{C}$ be a field which contains a transcendental number. Then there exist linear functional equations of the form*

$$\sum_{i=1}^{n} a_i f(b_i x + c_i y) = 0$$

such that $b_i, c_i \in K$ for every $i = 1, \ldots, n$, and there exists an additive solution d on K which is not contained by the variety generated by the injective homomorphism solutions.

A function $h : K \to K$ is a *derivation on K* if h is additive and satisfies the Leibnitz's rule (i.e.: $h(xy) = h(x)y + xh(y)$ for every $x, y \in K$). We note that the additive solution d in Theorem 3.2 is a derivation on K. This motivates the direction of our investigation of the general case.

3.2 Differential operators on a field

Suppose that the complex numbers $t_1, \ldots, t_n$ are algebraically independent over $\mathbb{Q}$. The elements of the field $\mathbb{Q}(t_1, \ldots, t_n)$ are the rational functions of $t_1, \ldots, t_n$ with rational coefficients. By a *differential operator on* $\mathbb{Q}(t_1, \ldots, t_n)$ we mean an operator of the form

$$D = \sum c_{i_1,\ldots,i_n} \cdot \frac{\partial^{i_1+\cdots+i_n}}{\partial t_1^{i_1} \cdots \partial t_n^{i_n}}, \tag{6}$$

where $\partial/\partial t_i$ are the usual partial derivatives, the sum is finite, in each term the coefficient is a complex number, and the exponents $i_1, \ldots, i_n$ are nonnegative integers. If $i_1 = \ldots = i_n = 0$, then by $\partial^{i_1+\cdots+i_n}/\partial t_1^{i_1} \cdots \partial t_n^{i_n}$ we mean the identity operator on $\mathbb{Q}(t_1, \ldots, t_n)$. The degree of the differential operator D is the maximum of the numbers $i_1 + \ldots + i_n$ such that $c_{i_1,\ldots,i_n} \neq 0$.

Let K be an arbitrary finitely generated subfield of $\mathbb{C}$. Then it can be written of the form

$$K = \mathbb{Q}(t_1, \ldots, t_n, \alpha)$$

where $t_1, \ldots, t_n$ are algebraically independent over $\mathbb{Q}$ and α is in algebraic over $\mathbb{Q}(t_1, \ldots, t_n)$. It can be proved that if there is a differential operator D on $\mathbb{Q}(t_1, \ldots, t_n)$, then it is uniquely extended as a differential operator (on K).

3.3 Spectral sythesis and the space of additive solutions

Theorem 3.3. *Suppose that the transcendence degree of the field K over $\mathbb{Q}$ is finite. Let $f : K \to \mathbb{C}$ be additive, and let m be an exponential on K^*. Let ϕ be an extension of m to $\mathbb{C}$ as an automorphism of $\mathbb{C}$. Then the following are equivalent.*

(i) *$f = p \cdot m$ on K^*, where p is a generalized polynomial on K^*.*

(i) *$f = p \cdot m$ on K^*, where p is a polynomial on K^*.*

(iii) *There exists a unique differential operator D on K such that $f = \phi \circ D$ on K.*

Theorem 3.4. *Suppose that the transcendence degree of the field K over $\mathbb{Q}$ is finite. Then spectral synthesis holds in every variety on K^* consisting of additive functions (with respect to addition).*

The proof of the Theorem 3.4 is based on relatively new result of (local) spectral synthesis on countably generated Abelian groups [4].

As an application of Theorems 3.3 and 3.4 we describe the additive solutions of the linear functional equation (1). We denote by S_k the set of solutions of degree k of (1). We can show that

$$S_1^* = \{f|_{K^*} : f \in S_1\}$$

is a variety on K^*. For $k > 1$ the analogue statement is not true, we need to extend our attention for k-additive functions.

The next theorem is our main result concerning the additive solutions of linear functional equations and it has many applications

Theorem 3.5. *The linear space S_1 is spanned by the functions $\phi \circ D$, where ϕ and D are as above.*

3.4 Spectral synthesis and the space of solutions of higher degree

As it was mentioned before the analogues theorems of Theorem 3.3 and 3.4 can be proved for the k-additive functions on K^k and $(K^*)^k$ instead of K and K^*, respectively. Finally, we may obtain the following result:

Theorem 3.6. *The linear space S_k is spanned by the functions $\prod_{i=1}^{k} \phi_i \circ D_i$, where ϕ_i are an automorphism of $\mathbb{C}$ and $\prod_{i=1}^{k} \phi_i$ in S_k, and D_i are differential operators on K.*

We remark that most of the cases there is no boundary for the number of terms in D_i's in general, nevertheless it is a finite expression.

3.5 The discrete Pompeiu problem

In the last section we are concerned with the discrete Pompeiu problem and its connection to linear functional equations. The problem is stemmed from the classical Pompeiu problem and from the question asked by L. Pósa.

Question 3.7 (Pósa)**.** *Suppose that the function $f : \mathbb{R}^2 \to \mathbb{R}$ has the property that the sum of the values of f at the vertices of any square of fix size is zero. Is it true that $f \equiv 0$?*

Let D be a finite set of $\mathbb{R}^2$ and let G be a transformation group on $\mathbb{R}^2$. We say that D has the *discrete Pompeiu property with respect to G* if for every function $f : \mathbb{R}^2 \to \mathbb{C}$ the equation

$$\sum_{d \in \sigma(D)} f(d) = 0 \tag{7}$$

for all $\sigma \in G$ implies $f \equiv 0$.

The answer to Pósa's question is affirmative.

Theorem 3.8. *Let D be the vertex set of the unit square. Then D has the discrete Pompeiu property with respect to the congruences of $\mathbb{R}^2$.*

The proof of Theorem 3.8 uses spectral analysis and some results of Euclidean Ramsey theory based on the following theorem of L. E. Shader [9].

Theorem 3.9. *For any 2-coloring of the plane all right triangles are Ramsey.*

Pósa's question can be generalized as follows:

Question 3.10 (Discrete Pompeiu problem)**.** *Let $D \subset \mathbb{R}^2$ be a finite set. Is it true that D has the discrete Pompeiu property with respect to the congruences?*

In full generality this question remains open.

We denote by Σ the similarity group of $\mathbb{R}^2$. it can be shown that the discrete Pompeiu problem with respect to Σ is equivalent to the existence of non-constant solution of a linear functional equation.

Theorem 3.11. *Suppose that D is a nonempty finite subset of $\mathbb{C}$. Let $f : \mathbb{R}^2 \to \mathbb{C}$ be a function which satisfies equation (7) for every $\sigma \in \Sigma$. Then $f \equiv 0$.*

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