ASSOCIATIVE IDEMPOTENT NONDECREASING FUNCTIONS ARE REDUCIBLE

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Abstract. An n-variable associative function is called reducible if it can be written as a composition of a binary associative function. In this paper we summarise the known results when the function is defined on a chain and nondecreasing. The main result of this paper shows that associative idempotent and nondecreasing functions are uniquely reducible.

1. Introduction

In this paper we investigate the class of functions $F : X^n \to X$ $(n \geq 2)$ defined on a chain (i.e., totally ordered set) X that are nondecreasing, idempotent and associative. For arbitrary set X the study of associativity stemmed back to the pioneering work of Dörnte [5] and Post [9]. Dudek and Mukhin [6, 7] gave a characterization of reducibility using the terminology of a neutral element (see Theorem 3.5). However their result is essential from a theoretic point of view, it is not really applicable for a given situation unless the function originally has a neutral element (for further details see also [8]). Ackerman [1] made a complete characterization of quasitrivial associative operations. In his paper it was shown that every quasitrivial associative function is derived from a binary or a ternary operation. We recall a simplification of his main result in Theorem 3.11.

Couceiro and Marichal showed in [7] that continuous symmetric cancellative and associative n-ary operations defined on a nonempty real interval are reducible (see Remark 4. of [7]). Although their result is slightly connected to our investigation, it also shows that reducibility is an essential property in the study of associative n-ary operations.

The paper is organised as follows. Section 2 contains the basic definitions and notation. In Section 3.1 we collect the preliminary results in the case when $F : X^n \to X$ is idempotent, monotone, associative and has a neutral element. This part based on the works [7] and [8]. Recently, the reducibility and extremality[1] of quasitrivial associative symmetric nondecreasing functions was shown in [4]. We recall these results in Theorem 3.10 and in equations (3) and (4). We note that

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[1]The definition of extremality for symmetric functions stems from [10].

some basic notation that we use frequently appeared in [4]. In Section 3.2 we complete the study of reducibility of quasitrivial, nondecreasing, associative n-ary functions (without the assumption of symmetry). Section 4 we present the main results about the reducibility of idempotent, nondecreasing, associative functions. Because of its simplicity we present the symmetric case with useful lemmas (see Lemma 4.1 and 4.2) in Section 4.1. In Section 4.2 we prove the general result. The main technicality is that we have to divide the proof into two subcases. Theorem 4.4 can be used only for $n = 3$, and another inductive proof (Theorem 4.8) works for $n > 3$. In Section 5 we discuss extremality which holds in many special case but in full generality it cannot be fulfilled, and monotonicity as a relaxation of the property of the nondecreasingness. In the Appendix we present the proof of Lemma 3.9, which is probably well-known and not so hard to deduce from the paper of Czogała-Drewniak [3], however we could not find it in this generality in the literature.

2. Definitions and notation

Let X be an arbitrary set and $F : X^n \to X$ be an n-ary operation. We denote by S_n the symmetric group on the set $\{1, \dots, n\}$. Now we give a sequence of definitions:

Definition 2.1. The function $F : X^n \to X$ is called

(i) *idempotent* if $F(x, \dots, x) = x$ for every $x \in X$,
(ii) *symmetric* if $F(x_1, \dots, x_n) = F(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ for every $x_1, \dots, x_n \in X$ and every permutation $\sigma \in S_n$,
(iii) *quasitrivial* (or *conservative*) if for any $x_1, \dots, x_n \in X$

$$F(x_1, \dots, x_n) \in \{x_1, \dots, x_n\},$$

(iv) (n-)*associative* if for every $x_1, \dots, x_{2n-1} \in X$ and $1 \le i \le n-1$ we have

$$\begin{aligned} &F(F(x_1, \dots, x_n), x_{n+1}, \dots, x_{2n-1}) = \\ &F(x_1, \dots, x_i, F(x_{i+1}, \dots, x_{i+n}), x_{i+n+1}, \dots, x_{2n-1}). \end{aligned} \tag{1}$$

We will usually say that $F : X^n \to X$ is associative and we will only write that F is n-associative, if we would like to emphasize the number of variables in F.

We say that $F : X^n \to X$ has a *neutral element* denoted by $e \in X$ if for every $x \in X$ and $1 \le i \le k$ we have $F(e, \dots e, x, e, \dots, e) = x$, where x is in the i'th coordinate of F.

For any integer $k \ge 0$ and any $x \in X$, we set $k \cdot x = \underbrace{x, \dots, x}_{k \text{ times}}$. For instance, idempotency of F can be written of the form $F(n \cdot x) = x$.

The function F is called *nondecreasing* if

$$F(a_1, \dots, a_n) \ge F(b_1, \dots, b_n),$$

for every n-tuple $(a_1, \dots, a_n), (b_1, \dots, b_n)$ with $a_i \ge b_i$ for $1 \le i \le n$.

The function F is called *monotone in the i-th variable* if for all fixed elements $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n$ of X, the 1-variable function defined as

$$f_i(x) := F(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n)$$

is order-preserving or order-reversing. The function F is called *monotone* if it is monotone in each of its variables.

We use the lattice notion of for the minimum ($\wedge$) and for maximum ($\vee$) of a set. Hence we introduce the notation

$$\wedge_{i=1}^{n} x_i = \min\{x_1, \dots, x_n\},$$
$$\vee_{i=1}^{n} x_i = \max\{x_1, \dots, x_n\}.$$

3. Preliminary results

Definition 3.1. We say that $F : X^n \to X$ *is derived from* $G : X^2 \to X$ if F can be written of the form

$$F(x_1, \dots, x_n) = x_1 \circ \dots \circ x_n,$$

where $x \circ y = G(x, y)$. We note that this expression is well-defined if and only if G is associative. If such a G exists, then we say that F is *reducible*.

We note that if $n = 2$ then the function F is derived from itself.

The previous definition only deals with the existence of such an n-associative function that can be derived from a binary one. The uniqueness of the binary operation follows from certain conditions. The following result was proved first in [4, Proposition 3.8].

Proposition 3.2. *Assume that the operation $F : X^n \to X$ is associative and derived from an associative idempotent operation. Then the binary operation is unique.*

In our case, when X is a totally ordered set and F is monotone, we can strengthen the previous statement. The result presented here follows from [8, Lemma 3.4] when F is chosen to be monotone.

Proposition 3.3. *Let X be a totally ordered set and $F : X^n \to X$ be an associative, idempotent, monotone function F which is derived from an associative binary function G. Then G is idempotent as well.*

Combining the previous statements we get:

Corollary 3.4. *Let X be a totally ordered set. Let F be an associative, idempotent, monotone function which derived from a binary function $G : X^2 \to X$, then G is uniquely determined by F.*

3.1. Neutral element.

Suppose that $F : X^n \to X$ is an associative function having the neutral element $e \in X$, then one can define $G : X^2 \to X$ by

$$G(a, b) = F(a, (n-2) \cdot e, b) \tag{2}$$

for every $a, b \in X$. The following theorem of Dudek and Mukhin [7] shows a general result for arbitrary set X.

Theorem 3.5. *Let X be a nonempty set. Let $F : X^n \to X$ be an associative function. Then F is derived from G if and only if F has a neutral element or one can be adjoin a neutral element to X for F. In this case such a G can be defined by* (2).

We note that the previous statement also holds for $n = 2$. Indeed, every associative binary function is reducible and if an associative function F has no neutral element, then we can adjoin one. Let $e \notin X$ be an element and F be defined as $F(x, e) = F(e, x) = x$ for every $x \in X \cup \{e\}$. It is easy to check that F is also associative on $X \cup \{e\}$.

The following statement was proved in [8, Proposition 3.13] applying the previous structural theorem.

Proposition 3.6. *Let X be a totally ordered set and $F : X^n \to X$ be an associative, monotone, idempotent function with a neutral element e. Let G be defined by* (2)*. Then F is derived from the binary function G, which is also associative, idempotent, monotone and has the same neutral element e.*

Since every monotone, idempotent associative binary function is nondecreasing by [8, Lemma 3.10], the previous statement immediately has a simple consequence.

Corollary 3.7. *Let X and F as in the previous case. Then F is nondecreasing.*

Observation 3.8. *Let X and F as in Proposition 3.6. If F is symmetric, then G defined by* (2) *is also symmetric.*

Lemma 3.9 shows a connection between the existence of a neutral element and quasitriviality. The base of the idea appears in the classical Czogała-Drewniak's theorem [3] where $X = [0, 1]$. For the sake of completeness we present a short proof in the Appendix.

Lemma 3.9. *Let X be a totally ordered set and $F : X^n \to X$ be an associative, idempotent, monotone function having a neutral element e. Then F is quasitrivial.*

3.2. **Quasitriviality.** In [4, Theorem 3.3 and Corollary 3.4] the authors proved the following characterisation for quasitrivial, symmetric, nondecreasing and associative functions.

Theorem 3.10. *Let X be a totally ordered set. Let $F : X^n \to X$ be a quasitrivial, symmetric, nondecreasing, associative function. Then F is reducible. More precisely, F is derived from $G : X^2 \to X$ defined by*

$$G(x, y) = F((n-1) \cdot x, y) = F(x, (n-1) \cdot y). \tag{3}$$

It is easy to see that function G defined by (3) is quasitrivial, symmetric and nondecreasing. In [4, Theorem 3.3] it was also proved that in this case

$$F(x_1, \dots, x_n) = G(\wedge_{i=1}^n x_i, \vee_{i=1}^n x_i). \tag{4}$$

This means that F is extremal (see Definition 5.1).

One can prove that F remains reducible if we eliminate the symmetry condition of F. The result is weaker in the sense that it is only shows the existence of such a decomposition (see Theorem 3.12). We note that the analogue of (4) does not hold (for further details see Section 5.1).

The following result is an easy consequence of [1, Theorem 1.4] using the statement therein for $A_2 = \varnothing$.

Theorem 3.11. *Let X be an arbitrary set. Suppose $F : X^n \to X$ be a quasitrivial, n-associative function. Then F is not derived from a binary function G if n is odd and there exist b_1, b_2 $(b_1 \neq b_2)$ such that for any $a_1, \dots, a_n \in \{b_1, b_2\}$*

$$F(a_1, \dots, a_n) = b_i \ (i = \{1, 2\}), \tag{5}$$

where b_i occurs odd number of times.

As a consequence of this theorem we prove the following:

Theorem 3.12. *Let X be a totally ordered set and let $F : X^n \to X$ be an associative, quasitrivial, nondecreasing function. Then F is reducible.*

Proof. By contradiction we assume that F is not derived from a binary function. Now we apply the previous theorem since we intend to show that in this case the conditions for b_1, b_2 cannot be satisfied. Thus every associative, quasitrivial, nondecreasing function defined on a totally ordered set X is reducible.

According to Theorem 3.11, if F is not reducible, then n is odd. Hence $n \geq 3$ and there exist b_1, b_2 satisfying equation (5). Since $b_1 \neq b_2$, we may assume that $b_1 < b_2$ (the case $b_2 < b_1$ can be handled similarly). By our assumption on b_1 and b_2 we have

$$F(n \cdot b_1) = b_1, \;\; F(b_2, (n-1) \cdot b_1) = b_2, \;\; F(2 \cdot b_2, (n-2) \cdot b_1) = b_1. \tag{6}$$

Since F is nondecreasing we have

$$F(n \cdot b_1) \leq F(b_2, (n-1) \cdot b_1) \leq F(2 \cdot b_2, (n-2) \cdot b_1).$$

This implies $b_1 = b_2$, a contradiction. □

4. Main results

In this section we prove that every (n-)associative ($n \geq 2$), idempotent, nondecreasing function defined on a totally ordered set X is derived from a binary function G. As it was shown in Corollary 3.4, this G is also unique. This result generalises the previous results on reducibility. As a consequence of Theorem 3.5 this means that if an associative idempotent nondecreasing function F defined on a totally ordered set X, then either there is a neutral element of F or we can adjoin an element to X which acts as a neutral element of F. We note that all of our statements also hold for $n = 2$ but bring no information in this case. Practically, we just deal with the cases when $n \geq 3$.

4.1. **Symmetric case.** The symmetric case (as usual) is much simpler than the general one but we present a separate argument here. Our result is based on the following two lemmas.

Lemma 4.1. *Let X be a totally ordered set and $F : X^n \to X$ be an associative, nondecreasing, idempotent function. Then for every $a, c \in X$*

$$F(a, (n-1) \cdot c) = F((n-1) \cdot a, c).$$

Proof. If $a = c$, then the statement trivially follows from the idempotency of F. We assume that $a < c$. (The case $a > c$ can be handled similarly.) We denote $F((n-1) \cdot a, c)$ by θ. Since F is nondecreasing and idempotent we have $a \leq \theta \leq c$.

$$\begin{aligned} &\theta = F((n-1) \cdot a, c) \leq F(a, (n-1) \cdot c) \leq F(\theta, (n-1) \cdot c) = \\ &F(F((n-1) \cdot a, c), (n-1) \cdot c) = F((n-1) \cdot a, F(n \cdot c)) = \\ &F((n-1) \cdot a, c) = \theta. \end{aligned}$$

Thus, we get $F(a, (n-1)c) = F((n-1)a, c)$. □

Remark 1. As a consequence of the previous lemma we obtain that if F is an associative, idempotent, nondecreasing function, then $F(k \cdot a, (n-k) \cdot c)$ is the same for every $1 \leq k \leq n-1$ since $F((n-1) \cdot a, c) \leq F(k \cdot a, (n-k) \cdot c) \leq F(a, (n-1) \cdot c)$.

Lemma 4.2. *Let X be a totally ordered set and $F : X^n \to X$ be an associative, idempotent and nondecreasing function. Then the function G defined by*

$$F(a, (n-1) \cdot c) = F((n-1) \cdot a, c) = G(a, c). \tag{7}$$

is associative, idempotent and nondecreasing.

We note that by Lemma 4.1 and Remark 1, G is well-defined and $F(a,(n-1)\cdot c) = F(k\cdot a,(n-k)\cdot c)$ for every $k = 1,\dots,n-1$.

Proof. It is clear that G is idempotent and nondecreasing. The following equation shows that G is associative.

$$G(a,G(b,c)) = F((n-1)\cdot a, F(b,(n-1)\cdot c) =$$
$$F(F((n-1)\cdot a,b),(n-1)\cdot c) = G(G(a,b),c).$$

□

Now we investigate the question of reducibility for the symmetric case.

Theorem 4.3. *Let X be a totally ordered set and let $F : X^n \to X$ be an associative, symmetric, nondecreasing, idempotent function. Then F is derived from a unique binary function $G : X^2 \to X$ which can be obtained as*

$$G(a,c) = F(a,(n-1)\cdot c). \tag{8}$$

Moreover

$$F(x_1,\dots,x_n) = G(\wedge_{i=1}^n x_i, \vee_{i=1}^n x_i). \tag{9}$$

Remark 2. Equation (9) means that F is extremal (see Section 5.1).

Proof. Applying Lemma 4.1, we can define G for any $a,c\in X$ by

$$G(a,c) = F((n-1)\cdot a,c) = F(a,(n-1)\cdot c).$$

The uniqueness of the binary function follows from Corollary 3.4 so we only have to verify that G fulfils our requirements.

Since F is nondecreasing we have that

$$G(a,c) = F((n-1)\cdot a,c) \le F(a,x_1,\dots,x_{n-2},c) \le F(a,(n-1)\cdot c) = G(a,c) \tag{10}$$

for every $a \le x_1,\dots,x_{n-2} \le c$. Applying Lemma 4.1 again we get that the inequalities in (10) are equalities. Thus by the symmetry of F, the value of $F(x_1,\dots,x_n)$ depends only on $\wedge_{i=1}^n x_i$ and $\vee_{i=1}^n x_i$.

Using the symmetry of F we can reorder the entries of F and we get

$$F(x_1,\dots,x_n) = F(\wedge_{i=1}^n x_i,\dots,\vee_{i=1}^n x_i) = G(\wedge_{i=1}^n x_i, \vee_{i=1}^n x_i).$$

This argument shows that F is derived from G (and extremal). □

4.2. **General case.** In this section we do not assume that our functions are symmetric. In Theorem 4.4 and 4.8 we prove the reducibility of associative idempotent nondecreasing n-ary functions for $n \ge 3$ which is the main result of this section. Surprisingly to us, it seems from our argument that the cases $n = 3$ and $n \ge 4$ should be handled in different ways and separately. First we discuss the case $n = 3$.

Theorem 4.4. *Let X be a totally ordered set and $F : X^3 \to X$ be an associative, idempotent, nondecreasing function. Then F is derived from a unique binary function denoted by $G : X^2 \to X$. The function G can be defined by*

$$G(a,c) = F(a,c,c) = F(a,a,c). \tag{11}$$

Proof. By Lemma 4.1, G can be defined by (11). Applying Lemma 4.2 we get that G is associative, nondecreasing and idempotent. We need to show that

$$F(a,b,c) = G(a,G(b,c)) = G(G(a,b),c)$$

for every $a,b,c \in X$.

If $a \le b \le c$ (the case $a \ge b \ge c$ can be handled similarly), then we can directly apply (11) and we obtain

$$G(a,c) = F(a,a,c) \le F(a,b,c) \le F(a,c,c) = G(a,c).$$

On the other hand, since G is nondecreasing and idempotent, we have

$$\begin{aligned} &G(a,G(b,c)) \le G(a,G(c,c)) = G(a,c), \\ &G(G(a,b),c) \ge G(G(a,a),c)) = G(a,c). \end{aligned} \tag{12}$$

By the associativity of G and equation (12) we get $G(a,c) \le G(G(a,b),c) = G(a,G(b,c)) \le G(a,c)$. Hence $F(a,b,c) = G(a,c) = G(G(a,b),c) = G(a,G(b,c))$ as required.

Assume $(a \le b,\ c \le b)$ or $(a \ge b,\ c \ge b)$ (i.e. b is the smallest or the largest among a,b,c). We could assume that all of the previous relations are strict inequalities. Otherwise we are in the previous case but the proof works for these cases as well.

We introduce the following notation

$$\begin{aligned} \theta_1 &= G(a,b) = F(a,a,b) = F(a,b,b), \\ \theta_2 &= G(b,c) = F(b,b,c) = F(b,c,c). \end{aligned}$$

Then we get:

$$\begin{aligned} &F(a,b,c) = F(F(3\cdot a),F(3\cdot b),c) = \\ &F(a,F(a,a,b),F(b,b,c)) = F(a,\theta_1,\theta_2). \end{aligned} \tag{13}$$

and

$$\begin{aligned} &F(a,b,c) = F(a,F(3\cdot b),F(3\cdot c)) = \\ &F(F(a,b,b),F(b,c,c),c) = F(\theta_1,\theta_2,c). \end{aligned} \tag{14}$$

Suppose that $b = \max\{a,b,c\}$ ($b = \min\{a,b,c\}$ can be handled similarly). If $\theta_1 \le \theta_2$, then $a \le b$ implies $G(a,a) = a \le G(a,b) = \theta_1 \le \theta_2$, so a,θ_1,θ_2 are in increasing order. Therefore by the previous case

$$F(a,\theta_1,\theta_2) = G(a,\theta_2) = G(a,G(b,c)). \tag{15}$$

Using equation (13) we get that $F(a,b,c) = G(a,G(b,c))$, which equals to $G(G(a,b),c)$ since G is associative.

If $\theta_1 \ge \theta_2$, then by $c \le b$ we get that $c = G(c,c) \le G(b,c) = \theta_2 \le \theta_1$. Now the sequence θ_1,θ_2,c is in decreasing order, hence

$$F(\theta_1,\theta_2,c) = G(\theta_1,c) = G(G(a,b),c).$$

Using equation (14) we get that $F(a,b,c) = G(G(a,b),c)$. Finally, since G is associative, we finish the proof of Theorem 4.4. □

Now we prove the analogous result for $n \ge 4$. The main problem is that in case $n = 3$ we deeply use the fact that every ordered triple is either monotone or one of its extremum is in the middle. Generally, for $n > 3$ these are plenty other cases. Therefore we follow another way to generalize the previous result. We start with two lemmas.

Lemma 4.5. *Let X be a totally ordered set and $F : X^n \to X$ be an associative, idempotent, nondecreasing function. Then*

$$F(x_1,\dots,x_{i-1},2\cdot x_i,x_{i+1},\dots,x_{n-1}) = F(x_1,\dots,x_i,2\cdot x_{i+1},x_{i+2},\dots,x_{n-1}) \tag{16}$$

holds for every $i \in \{1,\dots,n-2\}$ and $x_1,\dots,x_{n-1} \in X$.

Proof. Lemma 4.1 gives $F((n-1)\cdot a,c) = F(a,(n-1)\cdot c)$. Since F is nondecreasing we obtain

$$F((n-1)\cdot a,c) = F(k\cdot a,(n-k)\cdot c) \tag{17}$$

for every $1 \le k \le n-1$ (as in Remark 1). The following direct calculation proves the statement. We use the idempotency of F in the first and last equalities, the associativity of F and in the second and fourth equalities and we use equation (17) for x_i and x_{i+1} in the third equality

$$\begin{aligned}
&F(x_1,\dots,2\cdot x_i,x_{i+1},\dots,x_{n-1})) = F(x_1,\dots,x_i,F(n\cdot x_i),x_{i+1},\dots,x_{n-1})) = \\
&F(x_1,\dots,2\cdot x_i,F((n-1)\cdot x_i,x_{i+1}),\dots,x_{n-1}) = \\
&F(x_1,\dots,2\cdot x_i,F((n-2)\cdot x_i,2\cdot x_{i+1}),\dots,x_{n-1}) = \\
&F(x_1,\dots,F(n\cdot x_i),2\cdot x_{i+1},\dots,x_{n-1}) = F(x_1,\dots,x_i,2\cdot x_{i+1},\dots,x_{n-1}).
\end{aligned}$$

□

Corollary 4.6. *Let X and F be as above. One can define $H : X^{n-1} \to X$ by the following formula*

$$H(x_1,\dots,x_{n-1}) = F(2\cdot x_1,x_2,\dots,x_{n-1}) = \ldots = F(x_1,\dots,x_{n-2},2\cdot x_{n-1}) \tag{18}$$

Remark 3. We note that H defined by (18) is also idempotent and nondecreasing if F has the same properties.

Lemma 4.7. *Let X be a totally ordered set and $F : X^n \to X$ be an associative, idempotent, nondecreasing function. Then $H : X^{n-1} \to X$ which is defined in Corollary 4.6 is also associative.*

Proof. By the previous remark it is enough to show that H is associative. The following equations hold for any $k \in \{3,\dots,n-1\}$

$$\begin{aligned}
&H(x_1,\dots,x_{k-1},H(y_1,\dots,y_{n-1}),x_{k+1},\dots,x_{n-1}) = \\
&F(2\cdot x_1,\dots,x_{k-1},F(2\cdot y_1,\dots,y_{n-1}),x_{k+1},\dots,x_{n-1}) = \\
&F(2\cdot x_1,\dots,x_{k-2},F(x_{k-1},2\cdot y_1,\dots,y_{n-2}),y_{n-1},x_{k+1},\dots,x_{n-1}) = \\
&H(x_1,\dots,x_{k-2},H(x_{k-1},y_1,\dots,y_{n-2}),y_{n-1},x_{k+1}\dots,x_{n-1}).
\end{aligned}$$

For $k = 2$ the previous calculation does not hold. In that case we can get the following equation using (18) for different indices.

$$\begin{aligned}
&H(x_1,H(y_1,\dots,y_{n-1}),x_3,\dots,x_{n-1}) = \\
&F(x_1,F(2\cdot y_1,\dots,y_{n-1}),x_3,\dots,2\cdot x_{n-1}) = \\
&F(F(x_1,2\cdot y_1,\dots,y_{n-2}),y_{n-1},x_3,\dots,2\cdot x_{n-1}) = \\
&H(H(x_1,y_1,\dots,y_{n-2}),y_{n-1},x_3\dots,x_{n-1}).
\end{aligned}$$

□

Since $H : X^{n-1} \to X$ is associative, idempotent and nondecreasing, now we can use induction for $n \ge 3$.

Theorem 4.8. *Let X be a totally ordered set and $F : X^n \to X$ $(n \geq 2)$ be an associative, idempotent, nondecreasing function. Then there exists a unique associative, idempotent nondecreasing binary function $G : X^2 \to X$ from which F is derived. Moreover, G can be defined by*

$$G(a,c) = F(a,(n-1)\cdot c) = F((n-1)\cdot a,c). \tag{19}$$

Proof. For $n = 2$ the statement is automatically true. The statement is proved by induction for $n \geq 3$. Theorem 4.4 gives the result for $n = 3$.

Now we assume that $n > 3$. By Lemma 4.1 and Lemma 4.2 the function $G : X^2 \to X$ is well-defined, associative, idempotent and nondecreasing. Let $H : X^{n-1} \to X$ be defined by (18) as in Corollary 4.6. The function H is also associative, nondecreasing, idempotent according to Lemma 4.7.

Now we recall the notation $G(a,b) = a \circ b$ which is well-defined since G is associative by Lemma 4.2.

By induction, H is derived from a binary function. Since

$$a \circ b = G(a,b) = F((n-1)\cdot a,b) = H((n-2)a,b) \tag{20}$$

we have that H is derived from G, i.e:

$$H(x_1,x_2,\ldots,x_{n-1}) = x_1 \circ x_2 \circ \cdots \circ x_{n-1}. \tag{21}$$

Now we show that F is also derived from the same binary function G.

$$\begin{aligned} &F(x_1,x_2\ldots,x_n) = F(F(n\cdot x_1),x_2,\ldots,x_n) = \\ &F((n-2)\cdot x_1,F(2\cdot x_1,x_2,\ldots,x_{n-1}),x_n) = \\ &H((n-3)\cdot x_1,H(x_1,x_2,\ldots,x_{n-1}),x_n) = \\ &x_1 \circ \ldots \circ x_1 \circ (x_1 \circ x_2 \circ \ldots \circ x_{n-1}) \circ x_n = \\ &x_1 \circ x_2 \circ \ldots \circ x_{n-1} \circ x_n. \end{aligned} \tag{22}$$

In the second equation we use the associativity of F, in the third we substitute H using that $n-2 \geq 2$, in the fourth equation we apply (21), in the last equation we use the idempotency and associativity of G. Equation (22) shows that F is also derived from G. By (20), G is of the form (19). The uniqueness of G comes from Corollary 3.4. □

Corollary 4.9. *Let X be a totally ordered set and $n \geq 2$ be an integer. An associative, idempotent, monotone function $F : X^n \to X$ is reducible if and only if F is nondecreasing.*

Proof. ($\Longleftarrow$): This immediately follows from [8, Corollary 3.12] which states that if $F : X^n \to X$ $(n \geq 2)$ is an associative, monotone (at least in the first and the last variables), idempotent and reducible, then F is nondecreasing (in each of its variables).

($\Longrightarrow$): By Theorem 4.8, every associative, idempotent, nondecreasing n-ary function $(n \geq 2)$ is reducible. □

Example 4.10. Let X be a totally ordered Abelian group with respect to the addition and $g : X \to X$ be a monotone bijective function on X. Then the function

$$F(x,y,z) = g^{-1}(g(x) - g(y) + g(z))$$

is idempotent associative monotone but nondecreasing. Thus F is not reducible.

5. Further remarks

5.1. Extremality.

Definition 5.1. We say that $F : X^n \to X$ is *extremal*[2] if there exists a $G : X^2 \to X$ such that for every $x_1, \dots, x_n \in X$ we have that $F(x_1, \dots, x_n)$ equals to either $G(\wedge_{i=1}^n x_i, \vee_{i=1}^n x_i)$ or $G(\vee_{i=1}^n x_i, \wedge_{i=1}^n x_i)$. Particularly, if $F : X^n \to X$ is symmetric and extremal, then there exists a symmetric $G : X^2 \to X$ such that $F(x_1, \dots, x_n) = G(\wedge_{i=1}^n x_i, \vee_{i=1}^n x_i)$.

In [4] it was shown (as we have already stated in (4)) that if $F : X^n \to X$ is associative, quasitrivial, symmetric and nondecreasing defined on the chain X then F is extremal. As a possible generalization it was shown in Theorem 4.3 that instead of quasitriviality it is enough to assume idempotency (see also Remark 2). Namely:

Proposition 5.2. *Let X be a totally ordered set. Then every associative, symmetric, nondecreasing, idempotent function $F : X^n \to X$ is extremal.*

In [8, Theorem 2.6.], it was shown that every associative nondecreasing idempotent function which has a neutral element is extremal.

If $F : X^n \to X$ is associative, quasitrivial and nondecreasing, then F is not necessarily extremal. It can be shown easily that the projection to the i'th coordinate is not extremal. This also gives an example for associative, idempotent, nondecreasing function which is not extremal.

5.2. Monotonicity.

Although in the binary case (and in the multivariate case) it cannot happen. Example 4.10 shows that there exists associative idempotent monotone function, which is not nondecreasing (so it is not reducible by Corollary 4.9). The characterization of these functions are not know yet. We conjecture the following (in the spirit of Aczélian n-ary semigroups [2]):

Conjecture 5.3. Let X be a totally ordered Abelian group with respect to the addition. An associative idempotent monotone function $F : X^n \to X$ is not reducible if and only if n is odd and there exists a monotone bijection $g : X \to X$ such that

$$F(x_1, x_2, \dots, x_n) = g^{-1}\Big(\sum_{i=1}^{n} (-1)^i g(x_i)\Big). \tag{23}$$

The 'if' part of the statement is clear. In the case when $X = \mathbb{R}$ we note that if Conjecture 5.3 holds, then such an F must be automatically continuous, since every monotone bijection on an interval is continuous.

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[2] In [10] a mean $\mu : \mathbb{R}^* \to \mathbb{R}$ was called *extremal* if for all elements $a_1 \le a_2 \le \dots \le a_n \in \mathbb{R}$ we have $\mu(a_1, a_2, \dots, a_n) = \mu(a_1, a_n)$.

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Appendix

Lemma 5.4 (Lemma 3.9)**.** *Let X be a totally ordered set and $F : X^n \to X$ be an associative, idempotent, monotone function having a neutral element e. Then F is quasitrivial.*

Proof. By Corollary 3.7, we can automatically assume that F is nondecreasing. For $n = 2$ and $x, y \in X$, we distinguish two different cases:

(1) $(x \le e, y \le e)$ or $(e \le x, e \le y)$,
(2) $(x \le e \le y)$ or $(y \le e \le x)$.

We show that in each case $F(x,y)$ is either the maximum of the minimum, thus it is quasitrivial. In Case 1 if $x \le e, y \le e$, then by idempotency we get

$$x = F(x,e) \ge F(x,y)$$
$$y = F(e,y) \ge F(x,y).$$

Thus $x \wedge y \ge F(x,y)$.

On the other hand if $x \le y$ (the case $x \ge y$ can be handled similarly), then

$$x = F(x,x) \le F(x,y) \le F(y,y) = y,$$

by monotonicity and idempotency. This implies that $F(x,y) = x \wedge y$.

Similarly if $e \le x, e \le y$, it can be obtained that $F(x,y) = x \vee y$.

In Case 2 the two subcases can be handled similarly. Now we deal with $x \le e \le y$. we denote $F(x,y) = \theta$. Assume that $x \le \theta \le e \le y$, then using associativity, we get

$$F(x,\theta) = F(x, F(x,y)) = F(F(x,x), y) = F(x,y) = \theta \tag{24}$$

On the other hand, since $x \le e, \theta \le e$, we have already proved that

$$F(x,\theta) = x \wedge \theta = x.$$

This shows that $\theta = x$. For $x \le e \le \theta \le y$ similarly we have

$$\theta = F(\theta, y) = y.$$

Thus we get that the binary function F is quasitrivial.

If $n > 2$ and F is an (n-)associative, idempotent, non-decreasing and have a neutral element, then we can use Proposition 3.6. Thus there exists a binary function G which is associative, idempotent, non-decreasing and have a neutral element. By the case $n = 2$ we know that G is quasitrivial and, since F is derived from G, F is also quasitrivial. □