Spectral Tile Direction in the Group $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$

Thomas Fallon*, Gergely Kiss†, Azita Mayeli ‡, Gábor Somlai§

September 24, 2023

Abstract

In this paper we investigate the Fuglede's conjecture for $\mathbb{Z}_{p^2q^2r}$ and provide a proof of it under the condition $p^2q^2 \leqslant r$. We develop a new technique by studying the divisibility of the mask polynomial of a given set by some system of cyclotomic polynomials. This combined with a so-called mod-p-method gives a strong tool to study Fuglede's conjecture in this and some other cases.

1 Introduction

Fuglede conjectured that spectral sets coincide with tiles. The conjecture was originally investigated for the Euclidean space but it can be formulated for any locally compact group. Let G be a locally compact group endowed with the Haar measure μ. Let S be a subset of G and denote by $L^2(S)$ the Hilbert space of complex valued square-integrable functions defined on S.

A set $S \subseteq G$ with $0 < \mu(S) < \infty$ is called *spectral* if $L^2(S)$ is spanned by the restrictions of exponential functions that are pairwise orthogonal. A set S is a *tile* if it has a complement T such that for μ-almost every element of G can uniquely be written as $s + t$, where $s \in S$ and $t \in T$. In this case, T is called the *tiling complement* of S. Fuglede [9] proved that for a tile on $\mathbb{R}^n$ if the tiling complement is a lattice, then it is spectral and if the spectrum of a spectral set is a lattice, then it is spectral. A group G is called *Fuglede group* if both directions of Fuglede's conjecture hold on G.

Fuglede's conjecture was disproved by Tao [26] who defined the spectral and tiling properties for finite abelian groups. Using the existence of a Hadamard matrix of size 12 and a complex Hadamard matrix of size 6, Tao constructed spectral sets in $\mathbb{Z}_2^{12}$ and $\mathbb{Z}_3^6$ which do not tile the group, and lifted the latter counterexample to the 5 dimensional Euclidean space. The conjecture was also proved false for dimensions 4 and 3, see [6, 14]. Currently, both directions of the conjecture are open in the 1 and 2 dimensional cases and they fail in higher dimensions.

Dutkay and Lai [3] further developed the connection between the continuous and discrete cases by showing that the tiling-spectral direction of the conjecture holds on $\mathbb{R}$ if and only if it holds on $\mathbb{Z}$ and if and only if it holds on every finite cyclic groups. They also showed (but this is basically the general technology used by Tao and all other people providing counterexamples in this field) that any counterexample in the finite cyclic case for the spectral-tile direction would lead to a counterexample for $\mathbb{R}$. On the other hand, Fuglede's conjecture is wide open on $\mathbb{R}^2$. The equivalence shown by Dutkay and Lai in the one dimensional case is

*The Graduate Center, the City University of New York
e-mail: tfallon@gradcenter.cuny.edu
†Alfréd Rényi Institute of Mathematics, Analysis Department
e-mail: kigergo57@gmail.com
‡The Graduate Center, the City University of New York
e-mail: amayeli@gc.cuny.edu
§Eötvös Loránd University, Department of Algebra and Number Theory, and The Graduate Center of the City University of New York
e-mail: zsomlei@caesar.elte.hu
Keywords: spectral set, tiling, finite geometry, Fuglede's conjecture
AMS Subject Classification (2010): 43A40, 43A70, 52C22, 05B25

not even known on $\mathbb{R}^2$. However a counterexample for any direction of the conjecture on $\mathbb{Z}_m \times \mathbb{Z}_n$, for cyclic groups of order $n, m \in \mathbb{N}$ would provide a counterexample for that direction on $\mathbb{R}^2$. Here just some partial positive results are known. Fuglede's conjecture holds on $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$ [28], $\mathbb{Z}_{pq} \times \mathbb{Z}_{pq}$ [4] and on all of their subgroups, see [10, 13, 22], and the references therein. For the sake of completeness we mention that although the conjecture is known to be false in higher dimensions, it is an active research area to find finite Abelian Fuglede groups. The question is particularly interesting on the direct product of p-groups. Apart from the previously mentioned results that the conjecture holds for $\mathbb{Z}_p$ and $(\mathbb{Z}_p)^2$ [10], it does not hold for $(\mathbb{Z}_p)^n$ where $4 \leqslant n \in \mathbb{N}$ and $p \neq 2$ [21, 8], holds for $(\mathbb{Z}_p)^3$ if $p = 5$ and $p = 7$ [1, 5], and open for $(\mathbb{Z}_p)^3$ in general. For $p = 2$ it is known [8, 7] that the conjecture on $(\mathbb{Z}_2)^n$ holds if $n \geqslant 10$, does not hold if $n \leqslant 6$, and open if $7 \leqslant n \leqslant 9$. The direct product of p-groups are particularly interesting because of the conjecture of R. Shi [22], which states that the tiling-spectral direction holds for $\mathbb{Z}_{p^{l_1}} \times \cdots \times \mathbb{Z}_{p^{l_i}} \times \cdots \times \mathbb{Z}_{p^{l_n}}$, where $n, l_i \in \mathbb{N}$ for $1 \leqslant i \leqslant n$.

In this paper we develop the research on finite cyclic groups seeing as the one dimensional case. Here we just summerize the best available results although many other works precede them. Note that if a group is Fuglede, then its subgroups, as well. In the sequel $n, m \in \mathbb{N}$, p, q, r, s are arbitrary distinct primes.

It is well-known that $\mathbb{Z}_{p^n}$ is a Fuglede group. It was proved by Malikiosis [20] that the spectral-tile direction holds for $\mathbb{Z}_{p^n q^m}$ if $n, m \in \mathbb{N}$ and $m \leqslant 6$, hence combining it with the results of Coven and Meyerowitz [2] (see above) these groups are Fuglede groups. Quite recently, Zhang [27] presented that $\mathbb{Z}_{p^n qr}$ are Fuglede groups. For squarefree order N, the best known result is that $\mathbb{Z}_{pqrs}$ is Fuglede for all choice of p, q, r, s by Kiss, Malikiosis, Somlai and Vizer [11].

In certain groups, we have knowledge that tiles are spectral. Every tile is spectral on the groups $Z_{p^n q^k}$ and $\mathbb{Z}_N$, for squarefree N proved by Meyerowitz and Łaba[1] and independently by Shi [2, 23]. Łaba and Londner [15, 16, 17] demonstrated a series of more general results, confirming that the tile-spectral direction of Fuglede's conjecture holds true for $\mathbb{Z}_{p^2 q^2 r^2}$, where p, q, and r are three distinct primes.

The purpose of this paper is to push the spectral-tile direction of the conjecture a bit closer to the one that is considered by Łaba and Londner [15, 16, 17] for the reverse direction. On the other hand we introduce further techniques that might help understanding a few general cases which are mentioned above.

In order to keep the paper quite a bit shorter we introduce some arithmetic properties on the primes so we assume that one of them is much larger than the others. This helps us to introduce one of techniques that relies on a result of Łaba and Marshall [18].

The main result of this paper follows.

Theorem 1.1. Let p, q, and r be distinct prime numbers. Then every spectral set of the cyclic group $\mathbb{Z}_{p^2 q^2 r}$ is a tile, provided that $p^2 q^2 < r$. Thus $\mathbb{Z}_{p^2 q^2 r}$ is a Fuglede group for all distinct primes p, q, r with $p^2 q^2 < r$.

Our paper is organized as follows. In Section 2 we introduce the most relevant notation, tools and theoretic background. Section 3 is devoted to the proof of Theorem 1.1. This section consists of three parts. First, we handle sets of large cardinality (compared to $p^2 q^2 r$, the order of the group). Secondly, we apply the previously mentioned technique working under the condition $r > p^2 q^2$, which leads to the assumption of $r \mid |S|$ and basically simplifies the rest of proof. Thirdly, we develop a new method of proof based on divisibility of a system of cyclotomic polynomials (see Table 1) and combine it with the so-called mod-p-method.

2 Notation and preliminaries

Let S be a subset of a cyclic group $\mathbb{Z}_n$. The *mask polynomial* of S on a cyclic group is defined as $m_S(x) = \sum_{s \in S} x^s$, where this polynomial is considered as an element of the quotient ring $\mathbb{Z}[x]/(x^n - 1)$.

The spectral property of a set is expressed by usually a large amount of Fourier roots. Notice that in the case of a finite abelian group G, the dual group $\hat{G}$ is isomorphic to G, so we will just identify them. We say that a set S in G is spectral if there is a set of representations $\Lambda \subset \hat{G}$ such that the restriction of the elements of Λ to S form an orthogonal basis of $L^2(G)$. This in turn implies that $|S| = |\Lambda|$ so in this case we say that (S, Λ) is a spectral pair. Further using the identification of G and $\hat{G}$, one can observe (Λ, S) is also a spectral pair.

[1] see comments on Tao's blog

A representation being a Fourier root of the characteristic function of a set S can be expressed in terms of the mask polynomial of S. If a representation of order $m \mid n$ is a root of $\hat{1}_s$, then it is equivalent to the fact that the cyclotomic polynomial Φ_m divides m_S. Notice that $\Phi_{p^k} \mid m_S$ implies $p \mid |S|$ for every prime p and $k > 0$.

Another important observation is that each irreducible representation can be factorized in the following way. Let ϕ be a homomorphism from an abelian group G to $\mathbb{C}^*$. Then there is a cyclic group $\mathbb{Z}_m$ such that $\phi = \chi \circ \pi$, where π is a surjective homomorphism from G to $\mathbb{Z}_m$ and χ is a faithful representation of $\mathbb{Z}_m$. In the case of $G = \mathbb{Z}_n$ (and $m \mid n$) there is a unique subgroup of G of order $\frac{n}{m}$, which is isomorphic to $\mathbb{Z}_{\frac{n}{m}}$. Thus there is a natural homomorphism, the quotient map $\pi \colon \mathbb{Z}_n \to \mathbb{Z}_m \cong \mathbb{Z}_n/\mathbb{Z}_{\frac{n}{m}}$, which we call the *projection*.

The following two lemmas provide some inductive tools that were proved in [11].

Lemma 2.1. Suppose that (S, Λ) is a spectral pair. Then the following hold true.

1. Without loss of generality we assume that $0 \in S$ and $0 \in \Lambda$.
2. If S is contained in a subgroup of G, then S is a tile by induction.
3. If Λ is contained in a subgroup of G, then S is a tile by induction.

Lemma 2.2. Suppose that $p \mid |G|$ and S is the union of $\mathbb{Z}_p$ cosets. Then S tiles G.

Another technique relies on the fact that cyclotomic polynomials over a certain finite field simplify which we express in the following lemma. This is formulated in the polynomial ring $\mathbb{Z}[x]$ but the statements clearly hold in $\mathbb{Z}[x]/(x^n - 1)$ as well.

Now we recall and prove some statements on what we call the mod-p-method.

Lemma 2.3. 1. If p is a prime and $m \in \mathbb{N}$ such that $p \nmid m$, then

$$\Phi_{mp}(x) = \Phi_m(x^p)/\Phi_m(x). \tag{1}$$

2. If $n = p^k m$ with prime p and $p \nmid m$, then

$$\Phi_n(x) = \Phi_{pm}(x^{p^{k-1}}). \tag{2}$$

3. If $n = p^k m$ where p is a prime and $p \nmid m$, then

$$\Phi_n \mid m_S \text{ in } \mathbb{Z}[x] \Longrightarrow \Phi_m \mid m_S \text{ in } \mathbb{Z}_p[x].$$

Proof. 1- 2. Well-known identities of cyclomatic polynomials.

3. First we assume $n = pm$ and $p \nmid m$. From equation (1), we get that

$$\Phi_{pm}(x) = \Phi_m(x^p)/\Phi_m(x) = \Phi_m(x)^p/\Phi_m(x) = \Phi_m(x)^{p-1} \text{ in } \mathbb{Z}_p[x].$$

This implies that

$$\Phi_{pm}(x) \mid m_S(x) \Longrightarrow \Phi_m(x) \mid m_S \text{ in } \mathbb{Z}_p[x].$$

If $n = p^k m$, where $k \geqslant 2$, then we apply equation (2) and we get that

$$\Phi_n(x) = \Phi_{pm}(x^{p^{k-1}}) = \Phi_{pm}(x) \text{ in } \mathbb{Z}_p[x],$$

hence

$$\Phi_{p^k m}(x) \mid m_S(x) \Longrightarrow \Phi_{pm}(x) \mid m_S \text{ in } \mathbb{Z}_p[x].$$

This leads to what we have seen earlear, and thus we get the statement that

$$\Phi_n \mid m_S \text{ in } \mathbb{Z}[x] \Longrightarrow \Phi_m \mid m_S \text{ in } \mathbb{Z}_p[x].$$

□

The following consequence of Lemma 2.3 is what we call mod-p-method.

Lemma 2.4. Let $n = p^k m$, where p is a prime and $p \nmid m$, and S is a subset of $\mathbb{Z}_n$. Then the following implication holds. If

$$\Phi_d \mid m_S \text{ in } \mathbb{Z}_p[x] \quad \forall d \mid m, \tag{3}$$

then $|S| = km + lp$ for nonnegative $k, l \in \mathbb{Z}$.

Proof. Let $\bar{S}$ be the projection of S to $\mathbb{Z}_m$. Note that $\bar{S}$ is a multiset. Clearly, the analogue of (3) hold for $\bar{S}$, i.e., $\Phi_d \mid m_{\bar{S}}$ in $\mathbb{Z}_p[x]$ for all $d \mid m$. This implies that $\bar{S} \equiv k \cdot \mathbb{Z}_m \pmod p$, where the equation holds $\pmod p$ at every point of $\mathbb{Z}_m$. The choice of k is not unique but if we choose k to be minimal nonnegative integer, then we obtain $|S| = |\bar{S}| = km + lp$, for some nonnegative $k, l \in \mathbb{Z}$. □

The following is an easy observation that appears in most of the papers written about Fuglede's conjecture for finite abelian groups having finitely many prime different divisors, such as [12, 19].

Lemma 2.5. Assume p and q are prime divisors of the order of a cyclic group $\mathbb{Z}_n$. Assume $0 \in S$ and that S is not contained in any proper subgroup of $\mathbb{Z}_n$. Then there are $s_1, s_2 \in S$ such that $p \nmid s_1 - s_2$ and $q \nmid s_1 - s_2$.

Proof. Since S is not contained in any proper subgroup, then there is $s_1 \in S$ with $p \nmid s_1$. If $q \nmid s_1$, then $s_2 = 0$ satisfies the conditions so we may assume $q \mid s_1$. Using a similar argument, there is $s_2 \in S$ with $q \nmid s_2$ and $p \mid s_2$. Then indeed $p \nmid s_1 - s_2$ and $q \nmid s_1 - s_2$. □

Using the same argument as above, the following lemma was obtained in [11], Lemma 3.6.

Corollary 2.6. Let (S, Λ) be a spectral pair in $\mathbb{Z}_n$. Assume p and q are different prime divisors of n and $p^k \mid n$ and $q^l \mid n$ while $p^{k+1} \nmid n$ and $q^{l+1} \nmid n$. Then $\Phi_{p^k q^l m} \mid m_S$ and $\Phi_{p^k q^l m'} \mid m_\Lambda$ for some $m, m' \mid \frac{n}{p^k q^l}$ or S is a tile.

One of the key tools in the investigation of Fuglede's conjecture is the so-called cube rule. This and following concepts have been introduced in [12]. Let n be a positive integer and m its square-free radical, i.e. m is the product of the different prime divisors of n. Let $\mathcal{P}_n$ denote the set of prime divisors of n. Let χ be a faithful representation of $\mathbb{Z}_n$ and $S\colon \mathbb{Z}_n \to \mathbb{N}$ be a (multi)set on $\mathbb{Z}_n$ If χ is the root of the Fourier transform $\widehat{(1_S)}$ of the weighted characteristic function 1_S of S, then $\Phi_n \mid m_S$. By [12, Proposition 3.2.] this implies that S is the integer coefficient sum of $\mathbb{Z}_{p_i}$ cosets, where $p_i \in \mathcal{P}_n$ are the different prime divisors of n. As a consequence we obtain (see [12, Proposition 3.5.]) that if $C = \bigoplus A_i$, where $A_i \subset \mathbb{Z}_{p_i}$ of cardinality 2, then the alternating sum on the vertices of C of the weights of S is zero. The sign of each weight is given by the parity of the Hamming distance from a given point. Such a set C is called a cube with respect to the cyclic group $\mathbb{Z}_n$, although C is a product of subsets of $\mathbb{Z}_{p_i}$.

Generally, the following are equivalent and hold, if $\Phi_m \mid m_S$ for some $m \mid n$, where n is square free, see [12] and [25]. Let $\bar{S}$ be the projection of S to $\bigoplus \mathbb{Z}_{p_i}$, where $p_i \in \mathcal{P}_m$.

1. The multiset $\bar{S}$ is the integer coefficient sum of $\mathbb{Z}_{p_i}$ cosets, where $p_i \in \mathcal{P}_m$.
2. For every cube $C = \prod C_i \subset \bigoplus \mathbb{Z}_{p_i}$ ($p_i \in \mathcal{P}_m$) and $c_0 \in C$ the following holds

$$\sum_{c \in C} (-1)^{d_H(c_0, c)} S(c) = 0, \tag{4}$$

   where $S(c)$ denotes the weight of c in S, and d_H denotes the Hamming distance and $\omega(m)$ denotes the number of distinct prime factors of m.

3 Proof of the main theorem

In order to prove Fuglede's conjecture for $\mathbb{Z}_{p^2 q^2 r}$, where $r > p^2 q^2$ it is enough to prove the spectral to tile direction of Fuglede's conjecture for $G = \mathbb{Z}_{p^2 q^2 r}$ since Łaba and Londner [15, 17] proved the reverse.

Let S be a spectral set. We will do some kind of a case by case analysis but we will try to limit the number of different cases we have to consider.

We distinguish cases according to the largest divisor of $p^2 q^2 r$ that divides the size of S, so we introduce the following notation.

$$d \mid\mid |S| \Longleftrightarrow \gcd(|G|, |S|) = d.$$

3.1 Large sets

- $\mathbf{p^2q^2r} \mid\mid |\mathbf{S}|$. Then $S = G$ and we are done.
- $\mathbf{p^2qr} \mid\mid |\mathbf{S}|$ or $\mathbf{pq^2r} \mid\mid |\mathbf{S}|$. We will only prove it when $p^2qr \mid\mid |S|$ since the role of p and q is symmetric.

  If there are $q+1$ points in the same $\mathbb{Z}_{q^2}$ coset, then by pigeonhole principle $\Phi_q \mid m_\Lambda$ and $\Phi_q^2 \mid m_\Lambda$ hold, which implies that $q^2 \mid |S|$, a contradiction. In particular we see that $p^2qr \mid\mid S$ implies that $\Phi_q \nmid m_\Lambda$ or $\Phi_q^2 \nmid m_\Lambda$. Further we can also see that each $\mathbb{Z}_{q^2}$ coset contains q elements of both S and Λ.

  By duality, (Λ, S) is a spectral pair. Hence from $\Phi_q \nmid m_\Lambda$ we would get that each $\mathbb{Z}_q$ coset contains at most one point of S. By cardinality reasons this implies that each $\mathbb{Z}_q$ coset contains exactly 1 point of S and hence S is a tile.

  If $\Phi_q^2 \nmid m_\Lambda$, then the intersection of each $\mathbb{Z}_{q^2}$ coset with S is contained in one $\mathbb{Z}_q$ coset. Then again, since $|S| = p^2qr$ we have that each $\mathbb{Z}_{q^2}$ coset contains exactly one $\mathbb{Z}_q$ coset which is contained in S. Then S is clearly a tile.
- $\mathbf{p^2r} \mid\mid |\mathbf{S}|$ or $\mathbf{p^2q^2} \mid\mid |\mathbf{S}|$ or $\mathbf{q^2r} \mid\mid |\mathbf{S}|$ or $|\mathbf{S}| > \min(\mathbf{p^2r}, \mathbf{p^2q^2}, \mathbf{q^2r})$. The same type of argument can be adapted to each case so we will only handle one of them. Let $p^2r \mid\mid |S|$ so that $|S| = kp^2r$. We show that $k = 1$, and S is a tile.

  Take the projection $\bar{S}$ of S on $\mathbb{Z}_{p^2r}$. As $q \nmid |S|$ we have $\Phi_q \nmid m_\Lambda$ and $\Phi_q^2 \nmid m_\Lambda$. Therefore each $\mathbb{Z}_{q^2}$ coset contains at most 1 point of S. In other words, the projection $\bar{S} \subset \mathbb{Z}_{p^2r}$ is a set. Hence $|S| \leqslant p^2r$. By assumption $p^2r \mid\mid |S|$ we have $|S| = p^2r$. Then $\bar{S}$ covers $\mathbb{Z}_{p^2r}$ once, whence S itself is a tile.

Remark 1. Note that a similar argument can be applied to show that if S is a spectral set such that $|G| = p^k \gcd(|G|, |S|)$ for some prime p and $k \in \mathbb{N}$, then S is a tile. It is proved in the Appendix, see Proposition 4.2.

3.2 Applying arithmetic conditions

The purpose of this subsection is to introduce a technique that is useful when one of the prime divisors of $|G|$ is very large. This dramatically reduces the number of cases we need to investigate.

We learned the following idea from Ruxi Shi during the Fourier bases 2018 meeting in Crete[2].

Lemma 3.1. Let u be a prime divisor of n and let S be a spectral set in $\mathbb{Z}_n$. Assume that for every divisor $d < n$ of n the spectral-tile direction of Fuglede's conjecture holds for $\mathbb{Z}_d$. Suppose further that $u \nmid |S|$ and $\Phi_{mu} \mid m_S$ implies $\Phi_m \mid m_S$ whenever $m \in \mathbb{N}$ is coprime to u. Then S is a tile.

Proof. It follows from $u \nmid |S|$ that the uS is a set that is contained $\mathbb{Z}_{\frac{n}{u}}$. In this case if (S, Λ) is a spectral pair, then under the assumption on the mask polynomial we obtain that (uS, Λ) is a spectral pair. Further uS is contained in a proper subgroup of $\mathbb{Z}_n$. Hence by Lemma 2.1 uS has a tiling partner T in $\mathbb{Z}_{\frac{n}{u}}$. Then $T \bigoplus \mathbb{Z}_u$ is a tiling partner of S in $\mathbb{Z}_n$. □

From now on we assume that $\mathbf{p^2q^2} \leqslant \mathbf{r}$. Suppose that the condition of Lemma 3.1 does not hold for prime r. Otherwise S is a tile and we are done.

Most of the following statement was originally proved by Łaba and Marshall [18]. We consider is super useful to reduce the cases we need to separately handle so we provide a self-contained but compact proof for a slightly stronger statement. We prove the following theorem.

Lemma 3.2. Assume r is a prime divisor of the order of the cyclic group $\mathbb{Z}_n$ and assume there is $m \mid n$ such that $\gcd(m, r) = 1$ and $\Phi_{mr} \mid m_S$ but $\Phi_m \nmid m_S$. Then $|S| \geqslant r$. Moreover, each $\mathbb{Z}_{\frac{n}{r}}$ coset contains at least one point of S.

Proof. For a divisor k of n, let $\bar{S}_k$ denote the projection of S to $\mathbb{Z}_k$. Our conditions mean that the cube-rule holds for $\bar{S}_{mr}$, but does not hold for all cube for $\bar{S}_m$.

Notice that the $\mathbb{Z}_m$ cosets in $\mathbb{Z}_{mr}$ can be identified with $\mathbb{Z}_m$ so we may say that the cube rule holds in a $\mathbb{Z}_m$ coset for a multiset defined on $\mathbb{Z}_{mr}$. Using this terminology we obtain that since the cube-rule holds for

[2] http://fourier.math.uoc.gr/fb18/

all cube in $\bar{S}_{mr}$, we have either that the cube-rule holds for the $\bar{S}_{mr}$ in each $\mathbb{Z}_m$ coset or it fails for $\bar{S}_{mr}$ in each $\mathbb{Z}_m$ coset. In our case it has to fail for each $\mathbb{Z}_m$ coset since $\Phi_m \nmid m_S$. Hence each $\mathbb{Z}_m$ coset contains at least one point of $\bar{S}_{mr}$, hence $|\bar{S}_{mr}| = |S| \geqslant r$, otherwise the intersection with S is empty for some $\mathbb{Z}_m$ coset, when the cube-rule holds.

Indeed, if there is a $\mathbb{Z}_m$ coset in $\mathbb{Z}_{mr}$ which does not contain elements of S, then the cube-rule holds for $\bar{S}_{mr}$ in that coset, which we have already excluded. □

Lemma 3.3. Assume that the assumptions of Theorem 1.1 hold, i.e., $S \subset \mathbb{Z}_{p^2q^2r}$ is spectral and p, q, r are distinct primes with $p^2q^2 < r$. Then $\Phi_r \mid m_S$.

Proof. By Lemma 3.2, we have already seen that $|S| \geqslant r$ (otherwise S is automatically a tile), and since $p^2q^2 < r$, it follows that there are at least two points in the same $\mathbb{Z}_r$ coset. Since S is spectral, this leads to the conclusion that $\Phi_r \mid \Lambda$, and consequently $r \mid |\Lambda| = |S|$. □

From now on we assume $\mathbf{\Phi_r} \mid \mathbf{m_S}$.

Remark 2. The fact that $|S| \geqslant r$ was also proved by Łaba and Marshall. The novelty here is that the conditions of Lemma 3.2 implies that the projection of S to $\mathbb{Z}_r$ is surjective.

As in the whole argument we can change the role of S and Λ. We get the following corollary.

Corollary 3.4. Let $p^2q^2 \leqslant r$ and (S, Λ) be a spectral pair. Then we have one of the following alternatives.

1. S (resp. Λ) is a tile.

2. There are $m_1 \mid |G|$ ($m_2 \mid |G|$) such that $\gcd(m_1, r) = \gcd(m_2, r) = 1$ and $\Phi_{m_1 r} \mid m_S$ ($\Phi_{m_2 r} \mid m_\Lambda$) but $\Phi_{m_1} \nmid m_S$, ($\Phi_{m_2} \nmid m_\Lambda$). We then have by Lemma 3.3 that $\Phi_r \mid m_S$ and $\Phi_r \mid m_\Lambda$ and hence $r \mid |S| = |\Lambda|$.

3.3 Remaining cases

As a consequence we get that it remains to prove the statement for the following subcases $r \,||\, |S|$, $pr \,||\, |S|$, $qr \,||\, |S|$, and $pqr \,||\, |S|$, and we may assume $\Phi_r \mid m_S$. For the sake of completeness we list at first those cases that are excluded by this assumption. Note however that without this assumption the argument seems to be true but very long that we wish to avoid.

- $k \,||\, |S|$**, when** $k \in \{p^2q, pq^2, p^2, q^2, pq, p, q, 1\}$**.** If $p^2q^2 \leqslant r$, then by Corollary 3.4 we either have that the spectral set S is a tile or $r \mid |S|$, which leads to a contradiction.

- $\mathbf{r} \,||\, |\mathbf{S}|$. Suppose first that $\Phi_{p^2q^2r} \mid m_S$, hence the cube rule holds for every cube in every $\mathbb{Z}_{pqr}$ coset (that can be identified with $\mathbb{Z}_{pqr}$) in $\mathbb{Z}_{p^2q^2r}$.

  We can show that S is the union of $\mathbb{Z}_r$ cosets and then S is a tile. Now we can directly refer to the proof of [24, Lemma 3.5], but for the sake of completeness we present a short argument for this fact.

  Claim 3.5. Let $n = p^kq^lr^m$ and $S \subset \mathbb{Z}_n$ be a spectral set. Suppose further that $\Phi_n \mid m_S$ and $\Phi_p \nmid m_\Lambda$, $\Phi_q \nmid m_\Lambda$. Then S is the union of $\mathbb{Z}_r$ cosets and hence S is a tile.

  Proof. We distinguish two cases. Either there are $x \in S$ and $y \notin S$ and they belong to the same $\mathbb{Z}_r$ coset, or S is the union of $\mathbb{Z}_r$ cosets and then Lemma 2.2 implies that S is a tile. Thus we only have to consider the former case.

  Let $u \neq x$, where u is on the same $\mathbb{Z}_p$ coset as x and $v \neq x$ be an element of the $\mathbb{Z}_q$ coset containing x. Since $\Phi_p \nmid m_\Lambda$, $\Phi_q \nmid m_\Lambda$, it follows that $u, v \notin S$. Otherwise, if for instance $u \in S$, then $p \,||\, x - u$ would imply $\Phi_p \mid m_\Lambda$ and then $p \mid |S|$, a contradiction.

  In this case the cube-rule implies that the diagonal element s of the cube built on the vertices x, y, u, v belongs to S. Since the choice of u (resp. v) was arbitrary on the $\mathbb{Z}_p$ (resp. $\mathbb{Z}_q$) coset containing x, this implies the diagonal elements of such cubes belong to S. At least one of the primes p and q is larger than 2, so there are elements $s_1, s_2 \in S$ such that either $p \,||\, s_1 - s_2$ or $q \,||\, s_1 - s_2$ implying that $p \mid |S|$ or $q \mid |S|$ as above, a contradiction. □

Now we assume that $\Phi_{p^2q^2r} \mid m_\Lambda$ then repeating the same argument for Λ using the fact that (Λ, S) is also a spectral pair, we get that Λ is a tile. Hence $r = |S| = |\Lambda|$ and $\Phi_r \mid m_\Lambda$ guarantees that S is a system of coset representatives of the $\mathbb{Z}_{p^2q^2}$ cosets in $\mathbb{Z}_{p^2q^2r}$. Hence S is a tile.

From now on, we may assume $\Phi_{p^2q^2r} \nmid m_S$ and $\Phi_{p^2q^2r} \nmid m_\Lambda$. Then Lemma 2.5 gives that $\Phi_{p^2q^2} \mid m_S$ so the projection of S to $\mathbb{Z}_{p^2q^2}$ is the sum of $\mathbb{Z}_p$ and $\mathbb{Z}_q$ cosets.

Further since $\Phi_{p^2q^2r} \nmid m_\Lambda$, if $x, y \in S$ with $p \nmid x - y$ and $q \nmid x - y$, then $r \mid x - y$. Therefore if there is such a pair in S, then for all elements $x' \in S \cap (\mathbb{Z}_{pqr} + x)$ we have $r \mid x - x'$, i.e., all such $x' \in S \cap \mathbb{Z}_{pqr} + x$ contained in the same $\mathbb{Z}_{pq}$ coset as x.

As the projection of S to $\mathbb{Z}_{p^2q^2}$ is the sum of $\mathbb{Z}_p$ cosets and $\mathbb{Z}_q$ cosets, the same is true in $S \cap (\mathbb{Z}_{pqr} + x)$ by the property that all element of $x' \in S \cap (\mathbb{Z}_{pqr} + x)$ we have that $r \mid x - x'$. As S is a set we obtain that x and x' are contained in a $\mathbb{Z}_p$ coset or a $\mathbb{Z}_q$ coset, when $p \mid |S|$ or $q \mid |S|$, respectively. Both of these contradict the assumption that $r \mid\mid |S|$.

If there is no pair of elements $x, y \in S$ with $p \nmid x - y$ and $q \nmid x - y$, then it can be easily checked that S is contained in a proper subgroup of $\mathbb{Z}_{p^2q^2r}$. Then by Lemma 2.1.2 we get that S is a tile.

- **pr** $\mid\mid$ $|\mathbf{S}|$ or **qr** $\mid\mid$ $|\mathbf{S}|$. As the role of p and q is symmetric, it is enough to handle the case **pr** $\mid\mid$ $|\mathbf{S}|$.

  Clearly, in this case $\Phi_q \nmid m_S$, $\Phi_{q^2} \nmid m_S$ and $\Phi_q \nmid m_\Lambda$, $\Phi_{q^2} \nmid m_\Lambda$.

  Now we project S to $\mathbb{Z}_{p^2r}$. This projection $\bar{S}$ is a set, otherwise there are points of S in the same $\mathbb{Z}_{q^2}$ coset, implying $\Phi_q \mid m_\Lambda$ or $\Phi_{q^2} \mid m_\Lambda$. Then $q \mid |\Lambda| = |S|$ leads to a contradiction.

  Since the projection is a set we have $|S| \leqslant p^2r$, but as $p^2r \nmid S$, we get $|S| < p^2r$. Since $pr \mid\mid |S|$ there is a $\mathbb{Z}_{p^2}$ coset in $\mathbb{Z}_{p^2r}$ containing at least p elements of $\bar{S}$.

  Assume first that there is a $\mathbb{Z}_{p^2}$ coset containing at least $p+1$ element of $\bar{S}$, i.e., there exists an $x \in \mathbb{Z}_{p^2r}$ so that $|\bar{S} \cap (\mathbb{Z}_{p^2} + x)| \geqslant p + 1$. (Note that the same holds for each $\mathbb{Z}_{p^2}$ coset, since $\Phi_r \mid m_S$.)

  As $\bar{S}$ is a projection to $\mathbb{Z}_{p^2r}$, we get that either $\Phi_p \mid m_\Lambda$ or $\Phi_{pq} \mid m_\Lambda$ or $\Phi_{pq^2} \mid m_\Lambda$, and similarly, either $\Phi_{p^2} \mid m_\Lambda$ or $\Phi_{p^2q} \mid m_\Lambda$ or $\Phi_{p^2q^2} \mid m_\Lambda$.

  It also follows from $r > p$ and $\Phi_r \mid m_{\bar{S}}$ and our assumption that each $\mathbb{Z}_{p^2}$ coset contains more than p elements of $\bar{S}$ that either $\Phi_{pr} \mid m_\Lambda$ or $\Phi_{pqr} \mid m_\Lambda$ or $\Phi_{pq^2r} \mid m_\Lambda$. Similarly, either $\Phi_{p^2r} \mid m_\Lambda$ or $\Phi_{p^2qr} \mid m_\Lambda$ or $\Phi_{p^2q^2r} \mid m_\Lambda$.

  This implies that

  $$\Phi_p \mid m_\Lambda,\ \Phi_{p^2} \mid m_\Lambda,\ \Phi_r \mid m_\Lambda,\ \Phi_{pr} \mid m_\Lambda,\ \Phi_{p^2r} \mid m_\Lambda \tag{5}$$

  in the polynomial ring $\mathbb{Z}_q[x]/(x^n - 1)$ by Lemma 2.3.

  If every one of the divisibility relations in (5) hold, then we can applying Lemma 2.4 and we get that $|S| = kp^2r + lq$, for some nonnegative integers $k, l \in \mathbb{N}$. Clearly, $k < 1$, as $|S| < p^2r$. Thus $k = 0$, but then $q \mid |S|$, a contradiction.

  Therefore it remains to handle following case. There is a $\mathbb{Z}_{p^2}$ coset in $\mathbb{Z}_{p^2r}$ containing exactly p elements of $\bar{S}$. As $pr \mid\mid |S|$ and each $\mathbb{Z}_{p^2}$ coset contains the same amount of elements of $\bar{S}$ which is p, so we have $|S| = pr$. Moreover, we may assume that at least one of the relations of (5) fails in $\mathbb{Z}_q[x]/(x^n - 1)$.

  If $\Phi_p \nmid m_\Lambda$ in $\mathbb{Z}_q[x]$, then $\Phi_p \nmid m_\Lambda$ and $\Phi_{pq} \nmid m_\Lambda$ and $\Phi_{pq^2} \nmid m_\Lambda$. In this case each $\mathbb{Z}_p$ coset of $\mathbb{Z}_{p^2r}$ contains exactly one element of $\bar{S}$ and hence S is a tile.

  If $\Phi_p^2 \nmid m_\Lambda$ in $\mathbb{Z}_q[x]$, then $\Phi_{p^2} \nmid m_\Lambda$ and $\Phi_{p^2q} \nmid m_\Lambda$ and $\Phi_{p^2q^2} \nmid m_\Lambda$. Now every element of S is contained in a $\mathbb{Z}_{pq^2r}$ coset, when we are done by Lemma 2.1. So at least one of these polynomial divisibility conditions holds.

  If $\Phi_{pr} \nmid m_\Lambda$ in $\mathbb{Z}_q[x]$, then $\Phi_{pr} \nmid m_\Lambda$ and $\Phi_{prq} \nmid m_\Lambda$ and $\Phi_{prq^2} \nmid m_\Lambda$. Again we get that each $\mathbb{Z}_{pr}$ coset contains at most p elements of $\bar{S}$ that are in one $\mathbb{Z}_p$ coset. It follows that $|S| \leqslant p^2 < pr$, a contradiction.

  Assume finally $\Phi_{p^2r} \nmid m_\Lambda$ in $\mathbb{Z}_q[x]$. Using Lemma 2.5 we get that $\Phi_{p^2r} \mid m_\Lambda$ or $\Phi_{p^2qr} \mid m_\Lambda$ or $\Phi_{p^2q^2r} \mid m_\Lambda$. Lemma 2.3 leads to $\Phi_{p^2r} \mid m_\Lambda$ in $\mathbb{Z}_q[x]$ in any of these cases, a contradiction.

$\Phi_p \mid m_\Lambda$	or	$\Phi_{pq} \mid m_\Lambda$	or	$\Phi_{pq^2} \mid m_\Lambda$	$\Longrightarrow$	$\Phi_p \mid m_\Lambda$ in $\mathbb{Z}_q[x]/(x^n-1)$
$\Phi_{pr} \mid m_\Lambda$	or	$\Phi_{pqr} \mid m_\Lambda$	or	$\Phi_{pq^2r} \mid m_\Lambda$	$\Longrightarrow$	$\Phi_{pr} \mid m_\Lambda$ in $\mathbb{Z}_q[x]/(x^n-1)$
$\Phi_{p^2} \mid m_\Lambda$	or	$\Phi_{p^2q} \mid m_\Lambda$	or	$\Phi_{p^2q^2} \mid m_\Lambda$	$\Longrightarrow$	$\Phi_{p^2} \mid m_\Lambda$ in $\mathbb{Z}_q[x]/(x^n-1)$
$\Phi_{p^2r} \mid m_\Lambda$	or	$\Phi_{p^2qr} \mid m_\Lambda$	or	$\Phi_{p^2q^2r} \mid m_\Lambda$	$\Longrightarrow$	$\Phi_{p^2r} \mid m_\Lambda$ in $\mathbb{Z}_q[x]/(x^n-1)$

Table 1: System of divisibility relations

- **pqr** $||$ $|\mathbf{S}|$. Suppose that $|S| = upqr$ for some $0 \neq u \in \mathbb{N}$, where $\gcd(u, pqr) = 1$. Now have that $\Phi_r \mid m_S$ and $\Phi_r \mid m_\Lambda$. The following table describes the main strategy of the proof.

  Case 1. If for each line of Table 1 at least one divisibility of the polynomials holds, then

  $$\Phi_r \mid m_\Lambda, \Phi_p \mid m_\Lambda, \Phi_{pr} \mid m_\Lambda, \Phi_{p^2} \mid m_\Lambda, \Phi_{p^2r} \mid m_\Lambda \text{ in } \mathbb{Z}_q[x]/(x^n-1).$$

  Hence, by Lemma 2.4 we obtain that

  $$|S| = upqr = kp^2r + lq,$$

  where k and l are nonnegative natural numbers. Equivalently, $kp^2r = upqr - lq$, which implies that $k \mid q$.

  If $k > 0$, then $|S| \geqslant p^2qr$. If there is $\mathbb{Z}_{q^2}$ coset containing more than q points of S, then $\Phi_q\Phi_{q^2} \mid m_S$ so $q^2 \mid |S|$, a contradiction. As $|S| \geqslant p^2qr$ we have that each $\mathbb{Z}_{q^2}$ coset contains exactly q elements of S. In other words we cannot have $\Phi_q\Phi_{q^2} \mid m_\Lambda$. Therefore in each $\mathbb{Z}_{q^2}$ coset of $\mathbb{Z}_{p^2q^2r}$ the set S is either a system a coset representatives of $\mathbb{Z}_q$, or in each $\mathbb{Z}_{q^2}$ coset S is a $\mathbb{Z}_q$ coset. In both cases, S is a tile.

  If $k = 0$, then the intersection of S with each $\mathbb{Z}_{q^2}$ coset is either empty or of cardinality q since none of the $\mathbb{Z}_{q^2}$ can contain more than q elements of S. The same argument gives that S has a very rigid structure. For each $\mathbb{Z}_{q^2}$ coset, either there is no point in that coset, or it contain exactly q elements of S. In the latter case, either S is a $\mathbb{Z}_q$ coset in that $\mathbb{Z}_{q^2}$ coset, or a full $\mathbb{Z}_q$ coset representative system in $\mathbb{Z}_{q^2}$. Moreover, for each $\mathbb{Z}_{q^2}$ coset that intersects S non-trivially the same holds from these two options. Indeed, we cannot have a $\mathbb{Z}_{q^2}$ coset containing a $\mathbb{Z}_q$ coset full of the elements of S and another one having q elements of S as coset representatives since we would again get $\Phi_q\Phi_{q^2} \mid m_S$. If the set is union of $\mathbb{Z}_q$ cosets then we are done by Lemma 2.2. Therefore we may assume that if S intersects a $\mathbb{Z}_{q^2}$ coset, then it is a system of full $\mathbb{Z}_q$ coset representatives in that coset.

  Now we assume that the same is true if we change the role of p and q. Therefore if there are points of S on a given $\mathbb{Z}_{p^2}$ coset, then they form a system $\mathbb{Z}_p$ coset representatives.

  Now for a given point first we take the full $\mathbb{Z}_q$ coset representatives system belongs to S and for each of its point we take the corresponding $\mathbb{Z}_p$ coset representative system. This subsystem of S contains at most one point in each $\mathbb{Z}_{pq}$ coset. Moreover, if a $\mathbb{Z}_{pq}$ coset in a $\mathbb{Z}_{p^2q^2}$ coset contains k elements of S, then this $\mathbb{Z}_{p^2q^2}$ coset contains k complete set of coset representatives for $\mathbb{Z}_{pq}$ cosets. In particular we obtain $\Phi_{q^2}\Phi_{p^2} \mid m_\Lambda$.

  Suppose that $|S| = pqr$, then since $\Phi_r \mid m_S$ each $\mathbb{Z}_{pq}$ coset contains exactly one element of S. Then S is a tile with a tiling complement $\mathbb{Z}_p \times \mathbb{Z}_q$.

  Now we assume that $|S| > pqr$. Since $\Phi_r \mid m_S$, this implies that all $\mathbb{Z}_{p^2q^2}$ cosets contain at least two full systems of coset representatives of $\mathbb{Z}_{pq}$. This implies that $\Phi_{p^2q^2r} \mid m_S$ and the cube-rule holds for all $\mathbb{Z}_{pqr}$ cosets. As $\Phi_{q^2}\Phi_{p^2} \mid m_\Lambda$, and $p^2 \nmid |S|, q^2 \nmid |S|$ we get that $\Phi_p \nmid m_\Lambda, \Phi_q \nmid m_\Lambda$.

  Then by Claim 3.5 (see also [24, Lemma 3.5]), S is the union of $\mathbb{Z}_r$ cosets and by Lemma 2.2, S is a tile.

  Case 2. There is a 'line' of Table 1, for which none of the conditions hold.

  The following fact we use frequently in the sequel. As $p^2 \nmid |S|$ we have that $\Phi_p\Phi_{p^2} \nmid m_\Lambda$ This means that S has at most p elements in each $\mathbb{Z}_{p^2}$ cosets. Similarly, $q^2 \nmid |S|$ implies that $\Phi_q\Phi_{q^2} \nmid m_\Lambda$. Hence, S has at most q elements in each $\mathbb{Z}_{q^2}$ coset.

  Since there is at least one 'line' of Table 1, for which none of the conditions hold, we distinguish 4 cases.

1. $\Phi_p \nmid m_\Lambda, \Phi_{pq} \nmid m_\Lambda$ and $\Phi_{pq^2} \nmid m_\Lambda$. These conditions imply that in each $\mathbb{Z}_{pq^2}$ coset at most one $\mathbb{Z}_{q^2}$ coset may contain elements of S. As each $\mathbb{Z}_{q^2}$ coset contains at most q elements of S, we get that each $\mathbb{Z}_{pq^2}$ coset contains at most q elements of S. This implies that each $\mathbb{Z}_{p^2q^2}$ coset contains at most pq elements and $\mathbb{Z}_{p^2q^2r}$ contains at most pqr. As $pqr \mid |S|$ we have $|S| = pqr$.

   We obtain that each $\mathbb{Z}_{pq^2}$ coset contains exactly one $\mathbb{Z}_{q^2}$ coset having q elements of S. Since $q^2 \nmid |S|$ we have that S is a tile.

2. $\Phi_{pr} \nmid m_\Lambda$, $\Phi_{prq} \nmid m_\Lambda$ and $\Phi_{prq^2} \nmid m_\Lambda$.

   In this case $0 \in S \cap \mathbb{Z}_{prq^2} \subset \mathbb{Z}_{pq^2} \cap \mathbb{Z}_{q^2r}$. It follows from the conditions that $S \cap \mathbb{Z}_{pq^2r} + x$ is either a subset of a $\mathbb{Z}_{pq^2}$ coset, when $S \cap \mathbb{Z}_{pq^2r} + x$ has at most pq elements, or $S \cap \mathbb{Z}_{pq^2r} + x$ is a subset of $\mathbb{Z}_{q^2r}$ coset and then S has at most qr elements, for every $x \in \mathbb{Z}_{p^2q^2r}$. All in all we get that

   $$|S| \leqslant lpq + (p-l)qr,$$

   where $l \leqslant p$ denotes the number of $\mathbb{Z}_{pq^2r}$ cosets in which S is contained in a $\mathbb{Z}_{pq^2}$ coset. As $|S| = kpqr$ for some $\gcd(k, pqr) = 1$ and $p < r$, we get that $kpqr \leqslant lpq + (p-l)qr \leqslant pqr$. Equality holds if and only if $k = 1$ and $l = 0$ and each $\mathbb{Z}_{pq^2r}$ coset contains exactly qr elements of S located in a $\mathbb{Z}_{q^2r}$ coset. Further using the fact that $q^2 \nmid |S|$, this gives a structural description of S implying that S is a tile.

3. $\Phi_{p^2} \nmid m_\Lambda$, $\Phi_{p^2q} \nmid m_\Lambda$ and $\Phi_{p^2q^2} \nmid m_\Lambda$. The proof in this case is similar to the one of Case 1. Now S intersects each $\mathbb{Z}_{p^2q^2}$ coset in at most one $\mathbb{Z}_{pq^2}$ coset. As before, in each $\mathbb{Z}_{q^2}$ coset S has at most q elements. Hence a $\mathbb{Z}_{pq^2}$ coset as well as each $\mathbb{Z}_{p^2q^2}$ coset contains at most pq elements of S. Finally, this with $pqr \mid\mid |S|$ implies that S has exactly pqr elements in G and each $\mathbb{Z}_{q^2}$ coset that intersects S nontrivially contains exactly q elements of S. Furthermore, since $|S| = pqr$, we also get that S intersects each $\mathbb{Z}_{p^2q^2}$ coset in exactly one $\mathbb{Z}_{pq^2}$ coset and S has exactly pq elements in that coset (exactly q elements in each $\mathbb{Z}_{q^2}$ coset of it). Since $q^2 \nmid |S|$ we have that in each $\mathbb{Z}_{q^2}$ cosets of those $\mathbb{Z}_{pq^2}$ cosets that intersects S nontrivially, S is either a $\mathbb{Z}_q$ coset or in each of them S is a coset representative of the $\mathbb{Z}_q$ cosets. Both indicates that S is a tile. Indeed, in both cases, for all $x \in \mathbb{Z}_{p^2q^2r}$ if $S \cap (\mathbb{Z}_{q^2} + x) \neq \varnothing$, then the sets $(S - x) \cap \mathbb{Z}_{q^2}$ have a common tiling complement T in $\mathbb{Z}_{q^2}$. Hence, $S' = S + T$ covers once those $\mathbb{Z}_{pq^2}$ cosets which has nonempty intersection with S. This means that S' contains exactly one $\mathbb{Z}_{pq^2}$ coset from each $\mathbb{Z}_{p^2q^2}$ coset. Therefore, it is clear that there exists a T' such that $(\mathbb{Z}_{pq^2} + x') + T' = (\mathbb{Z}_{p^2q^2} + x')$ for all $x' \in S'$. Since S' has intersection with all $\mathbb{Z}_{p^2q^2}$ cosets, this implies that $\mathbb{Z}_{p^2q^2r} = S' + T' = (S + T) + T' = S + (T + T')$. Thus S tiles $\mathbb{Z}_{p^2q^2r}$.

4. $\Phi_{p^2r} \nmid m_\Lambda$, $\Phi_{p^2qr} \nmid m_\Lambda$ and $\Phi_{p^2q^2r} \nmid m_\Lambda$.

   The assumptions contradict Lemma 2.5.

Acknowledgement: The research of the first author is partly supported by the AMS-Simons Research Enhancement Grant.

The research of the second author is supported by projects no. K142993 of the National Research, Development and Innovation Fund of Hungary. The author is supported by Bolyai János Research Fellowship of the Hungarian Academy of Sciences and ÚNKP-22-5 New National Excellence Program of the Ministry for Culture and Innovation.

Gábor Somlai is a Fulbright Fellow at the Graduate Center of City University of New York and holds the Hungarian Scholarship of the Rosztoczy Foundation, whose research is also supported by ÚNKP.

The second and fourth authors also express sincere gratitude to the Graduate Center for their unwavering academic support, which greatly contributed to the successful completion of this research.

4 Appendix

The main purpose of the Appendix is to prove Proposition 4.2. In order to do that we consider the following statement.

Lemma 4.1. Let S be a subset of $\mathbb{Z}_{p^n}$.

- Assume $(S-S)\backslash\{0\}$ has k elements of mutually different orders, then S has at least p^k elements.
- Assume $(S-S)\backslash\{0\}$ has k elements of mutually different orders and $|S| = p^k$. Identify the elements of $\mathbb{Z}_{p^n}$ with $\{0,1,\dots,p^n-1\}$. Write all these integers in base p. Then there is a subset A_S of $\{0,1,\dots,n-1\}$ such that the elements of S have the same i'th coordinate for every $i \in \{0,1,\dots,n-1\}\backslash A_S$ and can have arbitrary value at the coordinates indexed by the elements of A_S.

Proof. It is clear that the order of the difference of two elements of $\mathbb{Z}_{p^n}$ only depends on the coordinate the two numbers differ when we write them in base p. Thus we have k coordinates where these numbers can differ. After the last one these coordinates they all need to coincide.

In the last coordinate where they can differ, the elements of S can clearly have at most p different values. Using simple inductive argument for the remaining coordinates proves both statements. $\square$

Proposition 4.2. Let (S,Λ) be a spectral pair such that $\frac{|G|}{\gcd(|G|,|S|)} = p^k$ for some prime p and $k \in \mathbb{N}$, then S is a tile.

Proof. Let $|G| = p^n d$, where $n \geqslant k$ and $p \nmid d$. By assumption $|G| = p^k \gcd(|G|,|S|)$ we get that $p^{n-k}d \mid\mid |S|$. Let $\bar{S}$ denote the multiset obtained as the projection of S on $\mathbb{Z}_d$. Then by pigeonhole principle there is an element in $\mathbb{Z}_d$ having multiplicity in $\bar{S}$ at least p^{n-k}. Hence there is a $\mathbb{Z}_{p^n}$ coset in G containing at least p^{n-k} points of S. This implies that at least $n-k$ relations of the following system hold by Lemma 4.1

$$\Phi_p \mid m_\Lambda, \quad \Phi_{p^2} \mid m_\Lambda, \quad \dots \quad \Phi_{p^n} \mid m_\Lambda.$$

On the other hand, each relation $\Phi_{p^i} \mid m_\Lambda$ implies a p factor of $|S|$, hence if more then $n-k$ relation holds, then $p^{n-k+1} \mid |S|$, which is a contradiction. This implies that exactly $n-k$ relation holds of the previous system, and there are $0 \leqslant i_1,\dots,i_k \leqslant n$ such that $\Phi_{p^{i_j}} \nmid m_\Lambda$ for all $j \in \{1,\dots,k\}$. This means that there is no elements in S such that their difference is of order p^{i_j}. Hence in all $\mathbb{Z}_{p^n}$ cosets of G there can be at most p^{n-k} elements of S. By cardinality reason, as $p^{n-k}d \mid\mid |S|$, all $\mathbb{Z}_{p^n}$ coset contains exactly p^{n-k} elements, hence $S = p^{n-k}d$. Moreover, one can see using the proof of Lemma 4.1 that these elements are fully determined. This in turn implies that S is a tile of G. $\square$

References

[1] P. Birklbauer. "The Fuglede Conjecture Holds in $\mathbb{Z}_5^3$". In: *Exp. Math.* (2017), pp. 1–5.

[2] E. M. Coven and A. Meyerowitz. "Tiling the integers with translates of one finite set". In: *Journal of Algebra* 212.1 (1999), pp. 161–174.

[3] D. E. Dutkay and C.-H. Lai. "Some reductions of the spectral set conjecture to integers". In: *Math. Proc. Cambridge Philos. Soc.* 156.1 (2014), pp. 123–135.

[4] T. Fallon, G. Kiss, and G. Somlai. "Spectral sets and tiles in $\mathbb{Z}_p^2 \times \mathbb{Z}_q^2$". In: *J. Funct. Anal.* 282.12 (2022), Paper No. 109472, 16. ISSN: 0022-1236,1096-0783.

[5] T. Fallon, A. Mayeli, and D. Villano. "The Fuglede Conjecture holds in $\mathbb{F}_p^3$ for $p = 5,7$". In: *Proc. Amer. Math. Soc.* (). to appear.

[6] B. Farkas, M. Matolcsi, and P. Móra. "On Fuglede's conjecture and the existence of universal spectra". In: *J. Fourier Anal. Appl.* 12.5 (2006), pp. 483–494.

[7] S. J. Ferguson and N. Sothanaphan. "Addendum to "Fuglede's conjucture fails in 4 dimensions over odd prime fields"". In: (2019). URL: https://arxiv.org/pdf/1910.04212.pdf.

[8] S. J. Ferguson and N. Sothanaphan. "Fuglede's conjecture fails in 4 dimensions over odd prime fields". In: *Discrete Math.* 343.1 (2019), p. 111507.

[9] B. Fuglede. "Commuting self-adjoint partial differential operators and a group theoretic problem". In: *J. Funct. Anal.* 16.1 (1974), pp. 101–121.

[10] A. Iosevich, A. Mayeli, and J. Pakianathan. "The Fuglede conjecture holds in $\mathbb{Z}_p \times \mathbb{Z}_p$". In: *Anal. & PDE* 10.4 (2017), pp. 757–764.

[11] G. Kiss, R. D. Malikiosis, G. Somlai, and M. Vizer. "Fuglede's conjecture holds for cyclic groups of order $pqrs$". In: *J. Fourier Anal. Appl.* 28.5 (2022), Paper No. 79, 23. ISSN: 1069-5869,1531-5851.

[12] G. Kiss, R. D. Malikiosis, G. Somlai, and M. Vizer. "On the discrete Fuglede and Pompeiu problems". In: *Anal. & PDE* 13.3 (2020), pp. 765–788.

[13] G. Kiss and S. Somlai. "Fuglede's conjecture holds on $\mathbb{Z}_p^2 \times \mathbb{Z}_q$". In: *Proc. Amer. Math. Soc.* 149.10 (2021), pp. 4181–4188.

[14] M. N. Kolountzakis and M. Matolcsi. "Complex Hadamard matrices and the spectral set conjecture". In: *Collect. Math.* Extra (2006), pp. 281–291.

[15] I. Łaba and I. Londner. "Combinatorial and harmonic-analytic methods for integer tilings". In: *Forum Math. Pi* 10 (2022), Paper No. e8, 46. ISSN: 2050-5086.

[16] I. Łaba and I. Londner. "Splitting for integer tilings and the Coven-Meyerowitz tiling conditions". In: *Arxiv preprint* (2023). URL: https://arxiv.org/pdf/2207.11809.pdf.

[17] I. Łaba and I. Londner. "The Coven-Meyerowitz tiling conditions for 3 odd prime factors". In: *Invent. Math.* 232.1 (2023), pp. 365–470. ISSN: 0020-9910,1432-1297.

[18] I. Łaba and C. Marshall. "Vanishing sums of roots of unity and the Favard length of self-similar product sets". In: *Discrete Anal.* (2022), Paper No. 19, 31. ISSN: 2397-3129.

[19] R. D. Malikiosis. "On the structure of spectral and tiling subsets of cyclic groups". In: *Forum Math. Sigma* 10 (2022), Paper No. e23, 42. ISSN: 2050-5094.

[20] R. D. Malikiosis and M. N. Kolountzakis. "Fuglede's conjecture on cyclic groups of order $p^n q$". In: *Discrete Analysis* 2017:12 (2017). 16pp, p. 16.

[21] S. Mattheus. "A counterexample to Fuglede's conjecture in $(\mathbb{Z}/p\mathbb{Z})^4$ for all odd primes". In: *Bull. Belg. Math. Soc. Simon Stevin* 27.4 (2020), pp. 481–488. ISSN: 1370-1444,2034-1970.

[22] R. Shi. "Equi-distribution on planes and spectral set conjecture on $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$". In: *J. Lond. Math. Soc* 102.2 (2020), pp. 1030–1046.

[23] R. Shi. "Fuglede's conjecture holds on cyclic groups $\mathbb{Z}_{p_1 p_2 p_3}$". In: *Discrete Analysis* 2019:14 (2019). 14pp, p. 14.

[24] G. Somlai. "Spectral sets in $\mathbb{Z}_{p^2 qr}$". In: *Discrete Anal.* (2023), Paper No. 5, 10. ISSN: 2397-3129.

[25] J. Steinerberger. "Minimal vanishing sums of roots of unity with large coefficients". In: *Proc. London Math.* 97.3 (2008), pp. 689–717.

[26] T. Tao. "Fuglede's conjecture is false in 5 and higher dimensions". In: *Math. Res. Lett.* 11.2 (2004), pp. 251–258.

[27] T. Zhang. "A group ring approach to Fuglede's conjecture in cyclic groups". In: *Arxiv preprint* (2023). URL: https://arxiv.org/pdf/2210.15174.pdf.

[28] T. Zhang. "Fuglede's conjecture holds in $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$". In: *SIAM Journal on Discrete Mathematics* 37.2 (2023).