

Separately polynomial functions

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Abstract

It is known that if $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ is a polynomial in each variable, then f is a polynomial. We present generalizations of this fact, when \mathbb{R}^2 is replaced by $G \times H$, where G and H are topological Abelian groups. We show, e.g., that the conclusion holds (with generalized polynomials in place of polynomials) if G is a connected Baire space and H has a dense subgroup of finite rank or, for continuous functions, if G and H are connected Baire spaces. The condition of continuity can be omitted if G and H are locally compact or complete metric spaces. We present several examples showing that the results are not far from being optimal.

1 Introduction

It was proved by F. W. Carroll in [2] that if $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ is a polynomial in each variable, then f is a polynomial. Our aim is to find generalizations of this fact, when \mathbb{R}^2 is replaced by the product of two topological Abelian groups.

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On topological Abelian groups we distinguish between the class of polynomials and the wider class of generalized polynomials (see the next section for the definitions). The two classes coincide if the group contains a dense subgroup of finite rank. Now, the scalar product on the square of a Hilbert space is an example of a continuous function which is a polynomial in each variable without being a polynomial (see Example 1 below). Therefore, the appropriate problem is to find conditions on the groups G and H ensuring that whenever a function on $G \times H$ is a generalized polynomial in each variable, then it is a generalized polynomial.

We show that this is the case if G is not the union of countable many zero sets of generalized polynomials, and if H has a dense subgroup of finite rank (Theorem 4). The condition on G is satisfied, for example, if G is a connected Baire space. Note that the continuity of f is not assumed in Theorem 4.

If G and H are both connected Baire spaces, and if a *continuous* function on $G \times H$ is a generalized polynomial in each variable, then it is a generalized polynomial (Corollary 12).

It is not clear if the condition of continuity can be omitted from Corollary 12 (see Question 13). The problem is that a generalized polynomial must be continuous by definition, and a separately continuous function on the product of Baire spaces can be discontinuous everywhere, as it was shown recently in [9]. In our case, however, there are some extra conditions: the spaces are also connected, and the function in question is a generalized polynomial. It is conceivable that continuity follows under these conditions. As for the biadditive case, see [3].

We show that if G and H are connected complete metric spaces, or they are connected and locally compact, then every separately generalized polynomial function on $G \times H$ is a generalized polynomial (Theorem 14).

The proof shows that the conclusion holds whenever G and H are connected Baire spaces and such that every separately continuous function on $G \times H$ has at least one point of joint continuity.

There are several topological conditions implying this property. In fact, the topic has a vast literature starting with the paper [10]. See, e.g., the papers [4], [5], [6], [11].

2 Preliminaries

Let G be a topological Abelian group. We denote the group operation by addition, and denote the unit by 0 . The translation operator T_h and the difference operator Δ_h are defined by $T_h f(x) = f(x + h)$ and $\Delta_h f(x) = f(x + h) - f(x)$ for every $f: G \rightarrow \mathbb{C}$ and $h, x \in G$.

We say that a continuous function $f: G \rightarrow \mathbb{C}$ is a *generalized polynomial*, if there is an $n \geq 0$ such that $\Delta_{h_1} \dots \Delta_{h_{n+1}} f = 0$ for every $h_1, \dots, h_{n+1} \in G$. The smallest n with this property is the *degree of f* , denoted by $\deg f$. The degree of the identically zero function is -1 . We denote by $\mathcal{GP} = \mathcal{GP}_G$ the set of generalized polynomials defined on G .

A function $f: G \rightarrow \mathbb{C}$ is said to be a *polynomial*, if there are continuous additive functions $a_1, \dots, a_n: G \rightarrow \mathbb{C}$ and there is a $P \in \mathbb{C}[x_1, \dots, x_n]$ such that $f = P(a_1, \dots, a_n)$. It is well-known that every polynomial is a generalized polynomial. It is also easy to see that the linear span of the translates of a polynomial is of finite dimension. More precisely, a function is a polynomial if and only if it is a generalized polynomial, and the linear span of its translates is of finite dimension (see [8, Proposition 5]). We denote by $\mathcal{P} = \mathcal{P}_G$ the set of polynomials defined on G .

Let f be a complex valued function defined on $X \times Y$. The *sections* $f_x: Y \rightarrow \mathbb{C}$ and $f^y: X \rightarrow \mathbb{C}$ of f are defined by $f_x(y) = f^y(x) = f(x, y)$ ($x \in X, y \in Y$).

Let G, H be topological Abelian groups. A function $f: (G \times H) \rightarrow \mathbb{C}$ is a *separately polynomial function* if $f_x \in \mathcal{P}_H$ for every $x \in G$ and $f^y \in \mathcal{P}_G$ for every $y \in H$. Similarly, we say that $f: (G \times H) \rightarrow \mathbb{C}$ is a *separately generalized polynomial function* if $f_x \in \mathcal{GP}_H$ for every $x \in G$ and $f^y \in \mathcal{GP}_G$ for every $y \in H$.

In general we cannot expect that every separately polynomial function on $G \times H$ is a polynomial; not even if $G = H$ is a Hilbert space.

Example 1. Let G be the additive group of an infinite dimensional Hilbert space. Then the scalar product $f(x, y) = \langle x, y \rangle$ on G^2 is a separately polynomial function, since its sections are continuous additive functions. In fact, f^y is a linear functional and f_x is a conjugate linear functional for every

$x, y \in G$. Thus the sections of f are polynomials.

Now, while the scalar product is a generalized polynomial (of degree 2) on G^2 , it is not a polynomial on G^2 , because the dimension of the linear span of its translates is infinite. Indeed, let $g(x) = \langle x, x \rangle = \|x\|^2$ for every $x \in G$. Then $\Delta_h g(x) = 2\langle h, x \rangle + \|h\|^2$ for every $h \in G$. It is easy to see that the functions $\langle h, x \rangle$ ($h \in G$) generate a linear space of infinite dimension, and then the same is true for the translates of g and then for those of f as well.

Therefore, the best we can expect is that, under suitable conditions on G and H , every separately generalized polynomial function on $G \times H$ is a generalized polynomial.

We denote by $r_0(G)$ the torsion free rank of the group G ; that is, the cardinality of a maximal independent system of elements of G of infinite order. Thus $r_0(G) = 0$ if and only if G is torsion. In the sequel by the rank of a group we shall mean the torsion free rank. It is known that if G has a dense subgroup of finite rank, then the classes of polynomials and of generalized polynomials on G coincide (see [8, Theorem 9]).

The set of roots of a function $f: G \rightarrow \mathbb{C}$ is denoted by Z_f . That is, $Z_f = \{x \in G: f(x) = 0\}$. We put

$$\mathcal{N}_P = \mathcal{N}_P(G) = \{A \subset G: \exists p \in \mathcal{P}_G, p \neq 0, A \subset Z_p\}$$

and

$$\mathcal{N}_{GP} = \mathcal{N}_{GP}(G) = \{A \subset G: \exists p \in \mathcal{GP}_G, p \neq 0, A \subset Z_p\}.$$

It is easy to see that \mathcal{N}_P and \mathcal{N}_{GP} are proper ideals of subsets of G . Let \mathcal{N}_P^σ and \mathcal{N}_{GP}^σ denote the σ -ideals generated by \mathcal{N}_P and \mathcal{N}_{GP} , respectively. Note that $\mathcal{N}_P \subset \mathcal{N}_{GP}$ and $\mathcal{N}_P^\sigma \subset \mathcal{N}_{GP}^\sigma$.

If G is discrete, then \mathcal{N}_P^σ and \mathcal{N}_{GP}^σ are not proper σ -ideals (except when G is torsion), according to the next observation.

Proposition 2. *Let G be a discrete Abelian group. If G is not torsion, then $G \in \mathcal{N}_P^\sigma$.*

Proof. Let $a \in G$ be an element of infinite order. Then $\phi(na) = n$ ($n \in \mathbb{Z}$) defines a homomorphism from the subgroup generated by a into \mathbb{Q} , the

additive group of the rationals. Since \mathbb{Q} is divisible, ϕ can be extended to G as a homomorphism from G into \mathbb{Q} . Let ψ be such an extension.

Then $p_r = \psi + r$ is a nonzero polynomial on G for every $r \in \mathbb{Q}$. If $x \in G$, then x is the root of p_r , where $r = -\psi(x) \in \mathbb{Q}$. Therefore, $G = \bigcup_{r \in \mathbb{Q}} Z_{p_r} \in \mathcal{N}_P^\sigma$. \square

A simple sufficient condition for $G \notin \mathcal{N}_{GP}^\sigma$ is given by the next result.

Lemma 3. *If G is a connected Baire space, then the σ -ideals \mathcal{N}_P^σ and \mathcal{N}_{GP}^σ are proper; that is, $G \notin \mathcal{N}_P^\sigma$ and $G \notin \mathcal{N}_{GP}^\sigma$.*

Proof. It is enough to prove that every element of \mathcal{N}_{GP} is nowhere dense. Suppose $A \in \mathcal{N}_{GP}$ is dense in a nonempty open set U . Let $p \in \mathcal{GP}(G)$ be a nonzero generalized polynomial vanishing on A . Since $A \subset Z_p$ and Z_p is closed, we have $U \subset Z_p$. Since G is connected, every neighbourhood of the origin generates G . It is known that in such a group, if a generalized polynomial vanishes on a nonempty open set, then it vanishes everywhere (see [12, Theorem 3.2, p. 33]). This implies that p is identically zero, which is impossible. \square

3 Main results

Our next result generalizes Carroll's theorem [2].

Theorem 4. *Let G, H be topological Abelian groups, and suppose that*

- (i) $\mathcal{N}_{GP}^\sigma(G)$ is a proper σ -ideal in G , and
- (ii) H has a dense subgroup of finite rank.

If $f: (G \times H) \rightarrow \mathbb{C}$ is a separately generalized polynomial functions, then f is a generalized polynomial on $G \times H$.

Remark 5. By Lemma 3, (i) of Theorem 4 can be replaced by the condition that G is a connected Baire space.

Lemma 6. *Let H be a topological Abelian group, and suppose that H has a dense subgroup of finite rank. Then, for every positive integer d , there are finitely many points $x_1, \dots, x_s \in H$ and there are generalized polynomials $q_1, \dots, q_s \in \mathcal{GP}_H$ of degree $< d$ such that $p = \sum_{i=1}^s p(x_i) \cdot q_i$ for every $p \in \mathcal{GP}_H$ with $\deg p < d$.*

Proof. Let $\mathcal{GP}^{<d}$ denote the set of generalized polynomials $f \in \mathcal{GP}_H$ of degree $< d$. Clearly, $\mathcal{GP}^{<d}$ is a linear space over \mathbb{C} .

Let K be a dense subgroup of H with $r_0(K) = N < \infty$. Let $\{h_1, \dots, h_N\}$ be a maximal set of independent elements of K of infinite order, and let L denote the subgroup of K generated by the elements h_1, \dots, h_N . If $k = (k_1, \dots, k_N) \in \mathbb{Z}^N$, then we put $\|k\| = \max_{1 \leq i \leq N} |k_i|$. We abbreviate the sum $\sum_{i=1}^N k_i \cdot h_i$ by $\langle k, h \rangle$. Then we have $L = \{\langle k, h \rangle : k \in \mathbb{Z}^N\}$. We put

$$A = \{\langle k, h \rangle : k \in \mathbb{Z}^N, \|k\| \leq [d/2]\}.$$

First we prove that if $p \in \mathcal{GP}^{<d}$ vanishes on A , then $p = 0$.

Suppose $p \neq 0$. Since p is continuous and K is dense in H , there is an $x_0 \in K$ such that $p(x_0) \neq 0$. The maximality of the system $\{h_1, \dots, h_N\}$ implies that $nx_0 \in L$ with a suitable nonzero integer n . It is easy to see that there is a polynomial $P \in \mathbb{C}[x]$ such that $p(mx_0) = P(m)$ for every integer m . Since $P(1) = p(x_0) \neq 0$, it follows that $P \neq 0$, hence P only has a finite number of roots. Thus $p(mnx_0) = P(mn) \neq 0$ for all but a finite number of integers m . Fix such an m . Then $mnx_0 \in L$, and thus $mnx_0 = \langle k, h \rangle$ with a suitable $k \in \mathbb{Z}^N$. We find that $p(\langle k, h \rangle) \neq 0$ for some $k \in \mathbb{Z}^N$.

Let $k \in \mathbb{Z}^N$ be such that $p(\langle k, h \rangle) \neq 0$ and $\|k\|$ is minimal. If $\|k\| \leq [d/2]$, then $\langle k, h \rangle \in A$, and we have $p(\langle k, h \rangle) = 0$ by assumption. Thus we have $\|k\| > [d/2]$. Then there is an $\ell = (\ell_1, \dots, \ell_N)$ such that $\ell_i \in \{-1, 0, 1\}$ for every $i = 1, \dots, N$, and $\|k - j\ell\| < \|k\|$ for every $j = 1, \dots, d$. By the minimality of $\|k\|$ we have $p(\langle k - j\ell, h \rangle) = 0$ for every $j = 1, \dots, d$.

Put $v = \langle \ell, h \rangle$. Since $\deg p < d$, it follows that $\Delta_v^d p(x) = 0$ for every

$x \in H$. Now we have

$$\begin{aligned}
0 &= \Delta_v^d p(\langle k, h \rangle) = \sum_{j=0}^d (-1)^{d-j} \binom{d}{j} p(\langle k, h \rangle - jv) = \\
&= (-1)^d p(\langle k, h \rangle) + \sum_{j=1}^d (-1)^{d-j} \binom{d}{j} p(\langle k - j\ell, h \rangle) = \\
&= (-1)^d p(\langle k, h \rangle),
\end{aligned}$$

which is impossible. This proves $p = 0$.

The set of functions $V = \{p|_A : p \in \mathcal{GP}^{<d}\}$ is a linear space over \mathbb{C} . The map $p \mapsto p|_A$ is linear from $\mathcal{GP}^{<d}$ onto V and, as we proved above, it is injective. Therefore, $\mathcal{GP}^{<d}$ is of finite dimension.

Let b_1, \dots, b_s be a basis of $\mathcal{GP}^{<d}$. Since the functions b_1, \dots, b_s are linearly independent, there are elements x_1, \dots, x_s such that the determinant $\det |b_i(x_j)|$ is nonzero (see [1, Lemma 1, p. 229]). Put $X = \{x_1, \dots, x_s\}$. Then $b_1|_X, \dots, b_s|_X$ are linearly independent, and thus the map $f \mapsto f|_X$ is bijective and linear from $\mathcal{GP}^{<d}$ onto the set of functions $f: X \rightarrow \mathbb{C}$.

Then there are functions $q_1, \dots, q_s \in \mathcal{GP}^{<d}$ such that $q_i(x_i) = 1$ and $q_i(x_j) = 0$ for every $i, j = 1, \dots, s$, $i \neq j$.

Let $p \in \mathcal{GP}^{<d}$ be given. Then $p - \sum_{i=1}^s p(x_i)q_i$ is a generalized polynomial of degree $< d$ vanishing on X , hence on H . That is, we have $p = \sum_{i=1}^s p(x_i)q_i$. \square

Proof of Theorem 4. Let $f: (G \times H) \rightarrow \mathbb{C}$ be a separately generalized polynomial function. Put $G_n = \{x \in G : \deg f_x < n\}$ ($n = 1, 2, \dots$). Since $\mathcal{N}_{GP}^\sigma(G)$ is a proper σ -ideal in G , there is an n such that $G_n \notin \mathcal{N}_{GP}(G)$. Fix such an n .

By Lemma 3, there are points $y_1, \dots, y_s \in H$ and generalized polynomials $q_1, \dots, q_s \in \mathcal{GP}_H$ such that $p = \sum_{i=1}^s p(y_i) \cdot q_i$ for every $p \in \mathcal{GP}_H$ with $\deg p < n$. Therefore, we have

$$f(x, y) = \sum_{i=1}^s f(x, y_i)q_i(y)$$

for every $x \in G_n$ and $y \in H$. If $y \in H$ is fixed, then $f(x, y) - \sum_{i=1}^s f(x, y_i)q_i(y)$ is a generalized polynomial on G vanishing on G_n . Since $G_n \notin \mathcal{N}_{GP}(G)$, it follows that $f(x, y) = \sum_{i=1}^s f(x, y_i)q_i(y)$ for every $(x, y) \in G \times H$. By $f^{y_i} \in \mathcal{GP}_G$ and $q_i \in \mathcal{GP}_H$, we obtain $f \in \mathcal{GP}_{G \times H}$. \square

Next we show that in Theorem 4 none of the conditions on G and H can be omitted. First we show that without condition (i) the conclusion of Theorem 4 may fail. We shall need the easy direction of the following result.

Lemma 7. *Let G, H be discrete Abelian groups. A function $f: (G \times H) \rightarrow \mathbb{C}$ is a generalized polynomial if and only if the sections f_x ($x \in G$) and f^y ($y \in H$) are generalized polynomials of bounded degree.*

Proof. Suppose $f: (G \times H) \rightarrow \mathbb{C}$ is a generalized polynomial of degree $< d$. Then $\Delta_{(x_1,0)} \dots \Delta_{(x_d,0)} f = 0$ for every $x_1, \dots, x_d \in G$. Then, for every $y \in H$, we have $\Delta_{(x_1,0)} \dots \Delta_{(x_d,0)} f^y = 0$ for every $x_1, \dots, x_d \in G$, and thus f^y is a generalized polynomial of degree $< d$ for every $y \in H$. A similar argument shows that f_x is a generalized polynomial of degree $< d$ for every $x \in G$, proving the “only if” statement.

Now suppose that $f: (G \times H) \rightarrow \mathbb{C}$ is such that f_x ($x \in G$) and f^y ($y \in H$) are generalized polynomials of degree $< d$. Then we have

$$\Delta_{(h_1,0)} \dots \Delta_{(h_d,0)} f^y = 0 \tag{1}$$

for every $h_1, \dots, h_d \in G$ and $y \in H$, and

$$\Delta_{(0,k_1)} \dots \Delta_{(0,k_d)} f_x = 0 \tag{2}$$

for every $k_1, \dots, k_d \in H$ and $x \in G$. In order to prove that f is a generalized polynomial of degree $< 2d$, it is enough to show that

$$\Delta_{(a_1,b_1)} \dots \Delta_{(a_{2d},b_{2d})} f = 0 \tag{3}$$

for every $(a_i, b_i) \in G \times H$ ($i = 1, \dots, 2d$). The identity $\Delta_{u+v} = T_u \Delta_v + \Delta_u$ gives

$$\Delta_{(a_i,b_i)} = T_{(a_i,0)} \Delta_{(0,b_i)} + \Delta_{(a_i,0)}$$

for every i . Therefore, the left hand side of (3) is the sum of terms of the form $T_c \Delta_{c_1} \dots \Delta_{c_{2d}} f$, where $c \in G \times \{0\}$, and $c_i \in (G \times \{0\}) \cup (\{0\} \times H)$ for

every i . If there are at least d indices i with $c_i \in (G \times \{0\})$, then (1) gives $\Delta_{c_1} \dots \Delta_{c_{2d}} f = 0$. Otherwise there are at least d indices i with $c_i \in (\{0\} \times H)$, and then (2) gives $\Delta_{c_1} \dots \Delta_{c_{2d}} f = 0$. This proves (3). \square

Now we turn to the first example.

Example 8. Let G, H be discrete Abelian groups. We show that if none of G and H is torsion, then there is a separately polynomial function $f: (G \times H) \rightarrow \mathbb{C}$ such that f is not a generalized polynomial on $G \times H$.

By Proposition 2, $\mathcal{N}_P^\sigma(G)$ is not a proper σ -ideal; that is, $G = \bigcup_{n=1}^{\infty} A_n$, where $A_n \neq \emptyset$ and $A_n \in \mathcal{N}_P(G)$ for every n . Let $p_n \in \mathcal{P}_G$ be such that $p_n \neq 0$ and $A_n \subset Z_{p_n}$. Then p_n is not constant; that is, $\deg p_n \geq 1$.

Let $P_n = p_1 \dots p_n$; then $P_n(x) = 0$ for every $x \in \bigcup_{i=1}^n A_i$, and we have $\deg P_1 < \deg P_2 < \dots$. (Here we use the fact that $\deg pq = \deg p + \deg q$ for every $p, q \in \mathcal{G}\mathcal{P}_G$, $p, q \neq 0$.) Note that for every $x \in G$ we have $P_n(x) = 0$ for all but a finite number of indices n .

Similarly, we find polynomials $Q_n \in \mathcal{P}_H$ such that $\deg Q_1 < \deg Q_2 < \dots$, and for every $y \in H$ we have $Q_n(y) = 0$ for all but a finite number of indices n .

We put $f(x, y) = \sum_{n=1}^{\infty} P_n(x)Q_n(y)$ for every $x \in G$ and $y \in H$. If $y \in H$ is fixed, then the sum defining f is finite, and thus $f^y \in \mathcal{P}_G$. Similarly, we have $f_x \in \mathcal{P}_H$ for every $x \in G$.

The degrees $\deg f^y$ ($y \in H$) are not bounded. Indeed, for every N , there is an $y \in H$ be such that $Q_N(y) \neq 0$. Then $f^y = \sum_{n=1}^M Q_n(y) \cdot P_n$ with an $M \geq N$, where the coefficients $Q_n(y)$ are nonzero if $n \leq N$. Therefore, $\deg f^y \geq \deg P_N \geq N$, proving that the set $\{\deg f^y: y \in H\}$ is not bounded. By Lemma 3, it follows that f is not a generalized polynomial.

By the example above, if G and H are discrete Abelian groups of positive and finite rank, then the conclusion of Theorem 4 fails. That is, $G \notin \mathcal{N}_{GP}^\sigma(G)$ cannot be omitted from the conditions of Theorem 4.

Next we show that the condition on H cannot be omitted either.

Example 9. Let H be a discrete Abelian group of infinite rank. We show that if G is a topological Abelian group such that \mathcal{P}_G contains nonconstant

polynomials, then there is a continuous separately polynomial function f on $G \times H$ such f is not a generalized polynomial.

Let h_α ($\alpha < \kappa$) be a maximal set of independent elements of H of infinite order, where $\kappa \geq \omega$. Let K denote the subgroup of H generated by the elements h_α ($\alpha < \kappa$). Every element of K is of the form $\sum_{\alpha < \kappa} k_\alpha h_\alpha$, where $k_\alpha \in \mathbb{Z}$ for every α , and all but a finite number of the coefficients k_α equal zero.

Let $p \in \mathcal{P}_G$ be a nonconstant polynomial. We define $f(x, y) = \sum_{i=1}^{\infty} k_i \cdot p^i(x)$ for every $x \in G$ and $y \in K$, $y = \sum_{\alpha < \kappa} k_\alpha h_\alpha$. (Note that the sum only contains a finite number of nonzero terms for every x and y .) In this way we defined f on $G \times K$ such that f_x is additive on K for every $x \in G$.

If $y \in H$, then there is a nonzero integer n such that $ny \in K$. Then we define $f(x, y) = \frac{1}{n} \cdot f(x, ny)$ for every $x \in G$. It is easy to see that $f(x, y)$ is well-defined on $G \times H$, and f_x is additive on H for every $x \in G$. Therefore, f_x is a polynomial on G for every $x \in G$.

If $y \in H$ and $ny \in K$ for a nonzero integer n , then f^y is of the form $\frac{1}{n} \cdot \sum_{i=1}^N k_i \cdot p^i$, and thus $f^y \in \mathcal{P}_G$. Since f^y is continuous for every $y \in H$ and H is discrete, it follows that f is continuous on $G \times H$.

Still, f is not a generalized polynomial on $G \times H$, as the set of degrees $\deg f^y$ ($y \in H$) is not bounded: if $y = h_i$, then $f^y = p^i$, and $\deg p^i = i \cdot \deg p \geq i$ for every $(i = 1, 2, \dots)$.

In the example above we may choose G in such a way that $G \notin \mathcal{N}_{GP}^\sigma(G)$ holds. (Take, e.g., $G = \mathbb{R}$.) In our next example this condition holds for both G and H .

Example 10. Let E be a Banach space of infinite dimension, and let G be the additive group of E equipped with the weak topology τ of E . Then G is a connected topological Abelian group. It is well-known that every ball in E is nowhere dense w.r.t. τ , and thus G is of first category in itself.

Still, we show that $G \notin \mathcal{N}_{GP}^\sigma(G)$. Indeed, the original norm topology of E is stronger than τ , and makes E a connected Baire space. If a function is continuous w.r.t. τ , then it is also continuous w.r.t. the norm topology. Therefore, every polynomial $p \in \mathcal{P}(G)$ is also a polynomial on E , and thus

$\mathcal{N}_P(G) \subset \mathcal{N}_P(E)$ and $\mathcal{N}_P^\sigma(G) \subset \mathcal{N}_P^\sigma(E)$. Since $\mathcal{N}_P^\sigma(E)$ is proper by Lemma 3, it follows that $\mathcal{N}_P^\sigma(G)$ is proper. The same is true for \mathcal{N}_{GP}^σ .

Now let H be an infinite dimensional Hilbert space, and let G be the additive group of H equipped with the weak topology of H . Let f be the scalar product on H^2 . Since the linear functionals and conjugate linear functionals are continuous w.r.t. the weak topology, it follows that f is a separately polynomial function on G^2 (see Example 1).

However, f is not a generalized polynomial on G^2 , since f is not continuous. In order to prove this, it is enough to show that $f(x, x) = \|x\|^2$ is not continuous on H w.r.t. the weak topology. Suppose it is. Then there is a neighbourhood U of 0 such that $\|x\| < 1$ for every $x \in U$. By the definition of the weak topology, there are linear functionals L_1, \dots, L_n and there is a $\delta > 0$ such that whenever $|L_i(x)| < \delta$ ($i = 1, \dots, n$), then $\|x\| < 1$.

Since H is of infinite dimension, there is an $x \neq 0$ such that $L_i(x) = 0$ for every $i = 1, \dots, n$. (Otherwise every linear functional would be a linear combination of L_1, \dots, L_n , and then $H = H^*$ would be finite dimensional.) Then $\lambda x \in U$ for every $\lambda \in \mathbb{C}$ and $\|\lambda x\| < 1$ for every $\lambda \in \mathbb{C}$, which is impossible.

In the example above the function f is a generalized polynomial with respect to the discrete topology, and the only reason why it is not a generalized polynomial is the lack of continuity. We show that this is the case whenever the σ -ideals $\mathcal{N}_{GP}^\sigma(G)$ and $\mathcal{N}_{GP}^\sigma(H)$ are proper.

Theorem 11. *Let G, H be topological Abelian groups, and suppose that $\mathcal{N}_{GP}^\sigma(G)$ is a proper σ -ideal in G , and $\mathcal{N}_{GP}^\sigma(H)$ is a proper σ -ideal in H . If $f: (G \times H) \rightarrow \mathbb{C}$ is a separately generalized polynomial function, then f is a generalized polynomial with respect to the discrete topology.*

Proof. Suppose f satisfies the conditions. By Lemma 7, it is enough to show that the degrees $\deg f_x$ and f^y are bounded.

Put $A_n = \{x \in G: \deg f_x < n\}$. Then $G = \bigcup_{n=1}^{\infty} A_n$. Since \mathcal{N}_{GP}^σ is a proper σ -ideal, there is an n such that $A_n \notin \mathcal{N}_{GP}^\sigma$. We fix such an n , and prove that

$$\Delta_{(0, h_1)} \cdots \Delta_{(0, h_n)} f = 0 \tag{4}$$

for every $h_1, \dots, h_n \in H$.

Let g denote the left hand side of (4). Then $g(x, y) = \sum_{i=1}^s a_i f(x, y + b_i)$, where $s = 2^n$, $a_i = \pm 1$ and $b_i \in H$ for every i . Let $y \in H$ be fixed. Then $g^y = \sum_{i=1}^s a_i f^{y+b_i}$, and thus g^y is a generalized polynomial on G .

If $x \in A_n$, then $\deg f_x < n$, and thus $g_x = 0$. Therefore $g^y(x) = 0$ for every $x \in A_n$. Since g^y is a generalized polynomial and $A_n \notin \mathcal{N}_{GP}$, it follows that $g^y = 0$. Since y was arbitrary, this proves (4). Thus $\deg f_x < n$ for every $x \in G$.

A similar argument shows that, for a suitable m , $\deg f^y < m$ for every $y \in H$. \square

Corollary 12. *Let G, H be topological Abelian groups, and suppose that G and H are connected Baire spaces. If $f: (G \times H) \rightarrow \mathbb{C}$ is a continuous separately generalized polynomial function, then f is a generalized polynomial.* \square

Question 13. Is the condition of continuity necessary in the statement of Corollary 12? (See the introduction.)

Assuming somewhat stronger than being a Baire space we can omit the condition of continuity from Corollary 12.

Theorem 14. *Let G, H be connected topological Abelian groups, and suppose that either*

- (i) G and H are complete metric spaces, or
- (ii) G and H are locally compact.

If $f: (G \times H) \rightarrow \mathbb{C}$ is a separately generalized polynomial function, then f is a generalized polynomial.

Proof. Under the conditions the groups G, H are connected Baire spaces. By Theorem 11, f is a generalized polynomial with respect to the discrete topology. So we only have to prove that f is continuous.

Suppose (i). Then the function f is Baire 1 on $G \times H$ by [7, p. 378]. Since $G \times H$ is completely metrizable, it follows that f has a point of (joint) continuity. Now [12, Theorem 3.6] states that if f is a discrete generalized polynomial on an Abelian group which is generated by every neighbourhood of the origin, and if f has a point of continuity, then f is continuous everywhere. In our case the group $G \times H$ is connected, so the condition is satisfied, and we conclude that f is continuous everywhere on $G \times H$.

If (ii) holds, then G, H are connected and locally compact Abelian groups, hence they are σ -compact as well. By [10, Theorem 1.2], it follows that f has a point of continuity, and then we can complete the proof as above. \square

Remark 15. The proof of Theorem 14 actually gives the following, more general statement.

Suppose that (i) G and H are connected, (ii) $G \notin \mathcal{N}_{GP}^\sigma(G)$ and $H \notin \mathcal{N}_{GP}^\sigma(H)$, and (iii) every separately continuous function on $G \times H$ has at least one point of joint continuity. Then every separately generalized polynomial function on $G \times H$ is a generalized polynomial.

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