Sincov's and other functional equations and negative interest rates

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Abstract. Investigating the future value $F(K, s, t)$ of a capital K invested between dates s and t, the "natural" condition $F(K, s, t) \geq K$ has lost its naturality because of the strange fact of negative interest rates. This leads to the task of describing the possible solutions of the multiplicative SINCOV equation $f(s, u) = f(s, t)f(t, u)$ for $s \leq t \leq u$ where $f(s, t) = 0$ may happen. In this paper we solve this task and discuss connections to the theory of investments.

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1. Introduction and motivation

A rather well known and elegant application of the theory of functional equations is given by the deduction of the formula of theoretical interest compounding. As a starting point some "reasonable" conditions for the *future value* function

$$F\colon [0, \infty) \times [0, \infty) \to \mathbb{R}$$

are given:

$$F(K + L, t) = F(K, t) + F(L, t), \quad K, L, t \geq 0 \tag{1.1}$$

$$F(F(K, t), s) = F(K, t + s), \quad K, s, t \geq 0 \text{ and} \tag{1.2}$$

$$F(K, t) \geq K, \quad K, t \geq 0. \tag{1.3}$$

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Theorem 1.1. *Let $F\colon [0,\infty)\times[0,\infty)\to\mathbb{R}$ be given. Then (1.1), (1.2) and (1.3) are satisfied iff there is some $q\geq 1$ such that*

$$F(K,t)=Kq^t,\quad K,t\geq 0. \tag{1.4}$$

The proof can be found in [1, pp. 105–106], [3] and in [5].

Note that (1.3) together with (1.1) implies that $F(K,t)=K\cdot f(t)$, since (1.1) says that $F(\cdot,t)$ is additive and (1.3) that this function is bounded from below on $[0,\infty)$ (see [1, p. 34, Theorem 1]). (1.2) implies $f(t+s)=f(t)f(s)$ for all $s,t\geq 0$ and (1.3) that $f(t)\geq 1$ for all t. This means that $g:=\ln\circ f$ is additive and ≥ 0 on $[0,\infty)$ and therefore there is some $r\geq 0$ such that $\ln(f(t))=rt$ for all t. So $q:=\exp(g(1))=\exp(r)\geq 1$ and $f(t)=q^t$ for all t.

2. Theoretical rule of interest compounding with negative interest rates allowed

At least for the last decade it has became common in economics to admit zero or even negative interest rates. This clearly contradicts (1.3). So one could ask for a substitute for Theorem 1.1 which allows for the new situation.

Note that

$$F(K,t)\geq cK,\quad K,t\geq 0 \text{ for some } c>0 \tag{2.1}$$

instead of (1.3) does not help. Of course one could get a result as in the theorem with some $q>0$. But taking t large enough shows that (2.1) is only possible when $q\geq 1$. One possibility to characterize the theoretical rule of interest compounding if negative interest rates are admissible could be the following result.

Theorem 2.1. *A function $F\colon [0,\infty)\times[0,\infty)\to\mathbb{R}$ satisfies*

$$F(K+L,t)=F(K,t)+F(L,t),\quad K,L,t\geq 0 \tag{2.2}$$

$$F(F(K,t),s)=F(K,t+s),\quad K,s,t\geq 0 \tag{2.3}$$

$$F(\cdot,t) \text{ is monotonic for all } t \text{ and} \tag{2.4}$$

$$F(K,\cdot) \text{ is monotonic for all } K \tag{2.5}$$

iff there is some $q\geq 0$ such that

$$F(K,t)=Kq^t,\quad K,t\geq 0, \tag{2.6}$$

where for $q=0$ both the cases $q^0=1$ and $q^0=0$ are possible.

Proof. Obviously F with (2.6) satisfies all the conditions (2.2) – (2.5).

Let on the other hand F satisfy these conditions. Since $F(\cdot,t)$ is additive and monotonic it is bounded from one side on some interval which implies (see [1, p. 34, Theorem 1]) that $F(K,t)=K\cdot f(t)$ with $f(t)=F(1,t)$. Condition (2.3) implies with $K=1$ that

$$f(t)f(s)=f(s+t),\quad s,t\geq 0.$$

To solve this we follow [1, p. 38, Theorem 1] and assume that $f(t_0)=0$ for some $t_0>0$. Then $f(t)=f(t_0+(t-t_0))=f(t_0)f(t-t_0)=0$ for all $t\geq t_0$.

Since moreover $0 = f(t_0) = f\left(n\frac{t_0}{n}\right) = f\left(\frac{t_0}{n}\right)^n$, t_0 may be chosen arbitrarily close to 0. So $f(t) = 0$ for all $t > 0$. $f(0) = f(0+0) = f(0)^2$ implies $f(0) \in \{0,1\}$ and therefore $f(t) = 0^t$ for all t with $0^0 \in \{0,1\}$. In the remaining case $f(t)$ must be different from 0 for all $t > 0$. By $f(t) = f\left(\frac{t}{2}\right)^2$ the value $f(t)$ must even be > 0. Note also that $f(0) = 1$ since $f(t) = f(t+0) = f(t)f(0)$. Moreover f is monotonic. So using the remarks following Theorem 1.1 there is some $q > 0$ such that $f(t) = q^t$ for all t. □

3. Future value formulas depending on the interval of investment

In [5, Theorem 2] a situation is discussed where for $\Delta := \Delta_{\mathbb{R}} := \{(s,t) \in \mathbb{R}^2 \mid s \leq t\}$ the value of the function $F\colon [0,\infty) \times \Delta \to [0,\infty)$ at (K,s,t) denotes the future value of the capital K at time t when invested at time s. Theorem 2 in [5] reads as follows.

Theorem 3.1. *The function* $F\colon [0,\infty) \times \Delta \to [0,\infty)$ *satisfies the conditions*

$$F(K+L,s,t) = F(K,s,t) + F(L,s,t), \quad K,L \geq 0, (s,t) \in \Delta \tag{3.1}$$

$$F(F(K,s,t),t,u) = F(K,s,u), \quad K \geq 0, (s,t),(t,u) \in \Delta \textit{ and} \tag{3.2}$$

$$F(K,s,t) \geq K, \quad K \geq 0, (s,t) \in \Delta, \tag{3.3}$$

iff there is some non decreasing function $\varphi\colon \mathbb{R} \to (0,\infty)$ *such that*

$$F(K,s,t) = K\frac{\varphi(t)}{\varphi(s)}, \quad K \geq 0, (s,t) \in \Delta. \tag{3.4}$$

Remark 3.2. (3.4) is the result of solving the multiplicative SINCOV equation $f(s,t)f(t,u) = f(s,u)$. Moreover, choosing some fixed t_0, the function φ is given by

$$\varphi(t) = \begin{cases} f(t_0,t) & \text{, if } t \geq t_0 \\ \frac{1}{f(t,t_0)} & \text{, if } t < t_0. \end{cases} \tag{3.5}$$

Now we want to investigate the situation when (3.3) is weakened in order to treat (generalized) negative interest rates also in the situation when intervals of investments themselves are considered rather than their length only.

Theorem 3.3. *The function* $F\colon [0,\infty) \times \Delta \to [0,\infty)$ *satisfies the conditions* (3.1) *and* (3.2) *and*

$$F(\cdot,s,t) \textit{ is monotonic on some interval for all } (s,t) \in \Delta, \tag{3.6}$$

iff there is some solution $f\colon \Delta \to [0,\infty)$ *of the Sincov equation*

$$f(s,t)f(t,u) = f(s,u), \quad (s,t),(t,u) \in \Delta \tag{3.7}$$

such that

$$F(K,s,t) = Kf(s,t), \quad K \geq 0, (s,t) \in \Delta. \tag{3.8}$$

Proof. (3.1) and (3.6) imply that F has the form (3.8) with $f(s,t) = F(1,s,t)$. Accordingly (3.2) results in (3.7).

On the other hand (3.7) and (3.8) imply (3.1), (3.2) and (3.6). □

Remark 3.4. The rest of our considerations is devoted to the solution of (3.7) in the slightly generalized situation that the function

$$f\colon \Delta_J \to \mathbb{R} \tag{3.9}$$

is defined on $\Delta = \Delta_J := \{(s,t) \in J^2 \mid s \leq t\}$ for some non-trivial interval J, has $\mathbb{R}$ as the codomain and solves (3.7). A special case has been considered in [2, Theorem 14]. The problem in its general form was posed by Detlef Gronau as Problem 2.1 in [4].

From now on we assume that $f\colon \Delta \to \mathbb{R}$ satisfies the Sincov equation (3.7) and we use $\Delta^\circ := \{(x,y) \in \Delta \mid x < y\}$. And we proceed with some lemmata.

Lemma 3.5. *Assume that $(x,y) \in \Delta^\circ$ and that $f(x,y) \neq 0$. Then*

$$I_{(x,y)} := \bigcup_{\substack{x' \leq x < y \leq y', \\ (x',y') \in \Delta^\circ, f(x',y') \neq 0}} [x', y'] \tag{3.10}$$

is an interval, and

$$f(u,v) \neq 0, \qquad \forall (u,v) \in \Delta \text{ satisfying } u,v \in I_{(x,y)}.$$

Moreover, $f(u,v) = 1$, if $u = v$.

Proof. Of course $I_{(x,y)}$ is an interval since it is the union of a set of intervals with non empty intersection. Note also that

$$0 \neq f(x',y') = f(x',u)f(u,v)f(v,y') \qquad (x' \leq u \leq v \leq y')$$

implies $f(u,v) \neq 0$.

Now let $u,v \in I_{(x,y)}$, $u \leq v$ and $x_0 \leq u \leq y_0$, $x_1 \leq v \leq y_1$ where $(x_i,y_i) \in \Delta^\circ$, $x_i \leq x < y \leq y_i$ and $f(x_i,y_i) \neq 0$ for $i = 0,1$. Put $x_2 := \min(x_0,x_1), y_2 := \max(y_0,y_1)$. Then $x_2 \leq u \leq v \leq y_2$. Thus it is enough to show $f(x_2,y_2) \neq 0$. Let, for example $x_0 \leq x_1$. Then $f(x_2,x) \neq 0$ since $0 \neq f(x_0,y_0) = f(x_0,x)f(x,y_0)$. Analogously we have $f(y,y_2) \neq 0$ and therefore $f(x_2,y_2) = f(x_2,x)f(x,y)f(y,y_2) \neq 0$.

$f(u,u) = f(v,v) = 1$, since $0 \neq f(u,v) = f(u,u)f(u,v) = f(u,v)f(v,v)$. □

Lemma 3.6. *Let x,y be as in the previous lemma. Then*

$$I_{(x,y)} = \bigcup_{I \in \mathcal{J}} I, \text{ where} \tag{3.11}$$

$$\mathcal{J} := \mathcal{J}_{(x,y)} := \{I \subseteq J \mid I \text{ is an interval}, \tag{3.12}$$

$$x,y \in I, f(u,v) \neq 0 \text{ for all } u,v \in I \text{ such that } u < v\}.$$

Proof. Note that by Lemma 3.5 $I_{(x,y)} \in \mathcal{J}$ implying $I_{(x,y)} \subseteq \bigcup_{J\in\mathcal{J}} J$.

Let, on the other hand $I \in \mathcal{J}$. Then there are sequences $(a_n), (b_n)$ such that (a_n) is decreasing, (b_n) is increasing, $a_n \le x < y \le b_n$ and $I = \bigcup_{n\in\mathbb{N}}[a_n, b_n]$. Since $a_n, b_n \in I$ the value $f(a_n, b_n)$ has to be $\neq 0$. So $[a_n, b_n] \subseteq I_{(x,y)}$. Therefore $I \subseteq I_{(x,y)}$ for all $I \in \mathcal{J}$. □

Lemma 3.7. *Let x, y be as above and assume that $u \in J \setminus I_{(x,y)}$. Then either*

$$u < v \text{ for all } v \in I_{(x,y)} \text{ or} \tag{3.13}$$

$$v < u \text{ for all } v \in I_{(x,y)}. \tag{3.14}$$

Moreover in case (3.13) $f(u, v) = 0$ and in case (3.14) $f(v, u) = 0$.

Proof. Assume $v \le u \le w$ for some $v, w \in I_{(x,y)}$. Then $u \in [v, w] \subseteq I_{(x,y)}$, a contradiction.

So, let $u < v$ for all $v \in I_{(x,y)}$, and suppose that $f(u, v_0) \neq 0$ for some $v_0 \in I_{(x,y)}$. Then there are $x_0, y_0 \in I_{(x,y)}$ such that $x_0 \le v_0 \le y_0$, $x_0 \le x < y \le y_0$ and $f(x_0, y_0) \neq 0$. Then $f(u, y_0) = f(u, v_0)f(v_0, y_0) \neq 0$ since $f(u, v_0), f(v_0, y_0) \neq 0$. But then $[u, y_0] \subseteq I_{(x,y)}$ contradicting $u \notin I_{(x,y)}$.

The other case, $v < u$ for all $v \in I_{(x,y)}$ may be treated similarly. □

Lemma 3.8. *Let $(x, y) \in \Delta^\circ$ be such that $f(x, y) \neq 0$. Then $I_{(u,v)} = I_{(x,y)}$ for all $u, v \in I_{(x,y)}$ satisfying $u < v$.*

Proof. By Lemma 3.5 we have $f(u', v') \neq 0$ for all $u', v' \in I_{(x,y)}$ with $u' < v'$. Thus using Lemma 3.6 results in $I_{(x,y)} \subseteq I_{(u,v)}$. Therefore $x, y \in I_{(u,v)}$, which analogously implies $I_{(u,v)} \subseteq I_{(x,y)}$. □

Lemma 3.9. *Let I_1, I_2 be intervals such that $I_1 \cap I_2 = \{a\}$. Then either $a = \min(I_1) = \max(I_2)$ or $a = \min(I_2) = \max(I_1)$.*

Proof. The (simple) considerations are left to the reader. □

Lemma 3.10. *Let $(x, y), (u, v) \in \Delta^\circ$ with $f(x, y), f(u, v) \neq 0$. Then either $I_{(x,y)} = I_{(u,v)}$ or $I_{(x,y)} \cap I_{(u,v)} = \emptyset$.*

Proof. Let $I_{(x,y)} \cap I_{(u,v)} \neq \emptyset$. If this intersection contains two different points a, b with, say, $a < b$, then $I_{(x,y)} = I_{(a,b)} = I_{(u,v)}$ by Lemma 3.8. Otherwise $I_{(x,y)} \cap I_{(u,v)} = \{a\}$. Using Lemma 3.9 we may assume, without loss of generality that $a = \max I_{(x,y)} = \min I_{(u,v)}$. Therefore $x < y \le a \le u < v$ and $f(x, a), f(a, v) \neq 0$. Thus also $f(x, v) = f(x, a)f(a, v) \neq 0$ which implies by Lemma 3.6 that $[x, v] \subseteq I_{(x,y)} \cap I_{(u,v)}$. So $I_{(x,y)} \cap I_{(u,v)} = \{a\}$ is not possible. Accordingly $I_{(x,y)} \cap I_{(u,v)} \neq \emptyset$ implies $I_{(x,y)} = I_{(u,v)}$. □

Now we are able to formulate necessary conditions for the solutions f of the Sincov equation on $\Delta = \Delta_J$.

Theorem 3.11. *Let $f\colon \Delta = \Delta_J \to \mathbb{R}$ be a solution of (3.7). Then there is a countable (possibly empty) system $\mathcal{S}$ of pairwise disjoint non-trivial intervals $I \subseteq J$ and there is a function $d\colon \bigcup_{I\in\mathcal{S}} I \to \mathbb{R}^\times := \mathbb{R} \setminus \{0\}$, such that*

$$f(x, y) = \frac{d(y)}{d(x)}, \quad x, y \in I \in \mathcal{S}, x \le y. \tag{3.15}$$

Moreover for any $x \in I \in \mathcal{S}$

$$f(x,y) = f(z,x) = 0, \quad I \not\ni z < x < y \notin I, \tag{3.16}$$

and

$$f(x,x) = \begin{cases} 1 & \text{if } x \in \bigcup_{I\in\mathcal{S}} I, \\ 0 \text{ or } 1 & \text{otherwise.} \end{cases} \tag{3.17}$$

Proof. Let $\mathcal{S} := \{I_{(x,y)} \mid (x,y) \in \Delta^\circ, f(x,y) \neq 0\}$. Then the intervals in $\mathcal{S}$ are pairwise disjoint by Lemma 3.10. Moreover $\mathcal{S}$ is countable since every $I_{(x,y)}$ equals $I_{(r,s)}$ with $x \leq r < s < y$ and $r, s \in \mathbb{Q}$ by Lemma 3.8.

Let $I \in \mathcal{S}$ and $x_0 \in I$. Then $d \colon I \to \mathbb{R}^\times$,

$$d(x) = \begin{cases} f(x_0, x) & \text{, if } x \geq x_0 \\ \frac{1}{f(x,x_0)} & \text{, if } x < x_0, \end{cases} \tag{3.18}$$

is well defined and satisfies $f(x,y) = \frac{d(y)}{d(x)}$ for all $x, y \in I, x \leq y$. This can be easily seen by distinguishing the cases $x \geq x_0$, $y < x_0$ and $x < x_0 \leq y$.

(3.16) follows from Lemma 3.7.

Since $f(x,x)f(x,y) = f(x,y) \neq 0$ for $x, y \in I \in \mathcal{S}$, we have $f(x,x) = 1$ which is the first case of (3.16). The second case of (3.16) follows from $f(x,x) = f(x,x)f(x,x)$ and from Lemma 3.7. (Note also that $f(x,y) = 0$ for all $x, y \notin \bigcup_{I\in\mathcal{S}} I$ with $x < y$.) □

Finally we prove that all Sincov functions on $\Delta = \Delta_J$ may be obtained by using the just derived necessary conditions.

Theorem 3.12. *Let J be a non-trivial interval and $\mathcal{S}$ the at most countable (maybe empty) set of disjoint subintervals of J. Let furthermore $d \colon \bigcup_{I\in\mathcal{S}} I \to \mathbb{R}^\times$ be an arbitrary function. Then $f \colon \Delta = \Delta_J \to \mathbb{R}$ defined by*

$$f(x,y) := \begin{cases} \frac{d(y)}{d(x)}, & \text{if } x, y \in I \in \mathcal{S}, \\ 0, & \text{if } x, y \text{ not in the same } I \in \mathcal{S}, \\ 0 \text{ or } 1 \text{ arbitrarily}, & \text{if } x = y \notin \bigcup_{I\in\mathcal{S}} I \end{cases} \tag{3.19}$$

satisfies (3.7).

Proof. Let $x \leq y \leq z$. If $x, z \in I (\in \mathcal{S})$ then also $y \in I$. Moreover

$$f(x,y)f(y,z) = \frac{d(y)}{d(x)}\frac{d(z)}{d(y)} = \frac{d(z)}{d(x)} = f(x,z).$$

If $x \in I$ and $z \notin I$ we have $f(x,z) = 0$. Assuming $y \in I$ implies $f(y,z) = 0$ and therefore $0 = f(x,z) = f(x,y)f(y,z)$. This also holds true when $y \notin I$.

If $x \notin \bigcup_{I\in\mathcal{S}} I$ we have $f(x,z) = 0$ for $x < z$ and therefore $0 = f(x,z) = 0 \cdot f(y,z) = f(x,y)f(y,z)$ if additionally $x < y$. In case $y = x$ we have $f(y,z) = 0$ implying (3.7). If finally $x = y = z$ we again have $f(x,z) = f(x,y)f(y,z)$. □

Remark 3.13. Gronau in [4] gave two types of solutions. The first one with $f(x,y) = \delta_x(y)$ for all $x \le y$ is the special case $\mathcal{S} = \emptyset, g = 1$ of Theorem 3.12. The second one may be described by $\mathcal{S} = \{[x_0, y_0]\}$, $\sigma = d_{[x_0,y_0]}$ and $g(x) = 0$ for all $x \notin [x_0, y_0]$.

Remark 3.14 (Generalization). The codomain of the function f may be chosen to be much more general without altering the results.

Let $(G, \cdot)$ be an arbitrary not necessarily abelian group with neutral element 1 and add an *absorbing* element 0, such that $0 \notin G$ and in $G' := G \uplus \{0\}$ we have $x \cdot 0 = 0 \cdot x = 0$. Then the only elements in G' with $x^2 = x$ are 0 and 1. This, more or less, implies that Theorems 3.11, 3.12 also hold in the new situation with the modification that the functions d are defined as

$$d(x) = \begin{cases} f(x_0, x) & \text{, if } x \ge x_0 \\ f(x, x_0)^{-1} & \text{, if } x < x_0, \end{cases} \tag{3.20}$$

because then $f(x,y) = d(x)^{-1}d(y)$ in Theorem 3.11 and

$$f(x,y)f(y,z) = d(x)^{-1}d(y)d(y)^{-1}d(z) = d(x)^{-1}d(z) = f(x,z)$$

in the proof of Theorem 3.12.

Examples of groups with added absorbing elements are $G \uplus \{0\}$ where G is a subgroup of $K^\times$ for division algebras K, in particular $\mathbb{R}$ and $[0, \infty)$ and also,for any n and any field K, the union $G \uplus \{0\}$ where G is a subgroup $\mathrm{Gl}_n(K)$ and 0 the null matrix. The last example is itself a special case of $G \uplus \{0\}$ where G is a subgroup of the group of units in a unitary ring R and 0 the zero element in R.

Remark 3.15. In [2] the Sincov equation is considered in the form

$$g(x,z) = g(x,y)g(y,z), \quad x > y > z, x, y, z \in J, \tag{3.21}$$

with $J = (0,1)$. The general solution of (3.21) is easily derived from Theorems 3.11 and 3.12 by

i) considering f defined by $f(y,x) = g(x,y)$ when $x > y$ and
ii) by observing that we may extend f to the pairs (x,x) by choosing $f(x,x) \in \{0,1\}$ as in Theorems 3.11 and 3.12.

As a result of Theorems 3.12 and 3.11 we obtain the following characterization of $F(K,s,t)$ satisfying (3.1), (3.2) and (3.6).

Corollary 3.16. *The function* $F\colon [0,\infty) \times \Delta_J \to [0,\infty)$ *satisfies the conditions* (3.1), (3.2) *and* (3.6) *iff there is a countable (possibly empty) system* $\mathcal{S}$ *of pairwise disjoint non-trivial intervals* $I \subseteq J$ *and there is a function* $d\colon \bigcup_{I \in \mathcal{S}} I \to \mathbb{R}^\times$, *such that*

$$F(K,s,t) = K\frac{d(s)}{d(t)}, \quad s,t \in I \in \mathcal{S}, s \le t. \tag{3.22}$$

Moreover for any $s \in I \in \mathcal{S}$

$$F(K,s,t) = F(K,u,s) = 0, \quad I \not\ni u < s < t \notin I \tag{3.23}$$

and

$$F(K, s, s) = \begin{cases} K & \textit{if } s \in \bigcup_{I \in \mathcal{S}} I, \\ 0 \textit{ or } K & \textit{otherwise.} \end{cases} \tag{3.24}$$

Remark 3.17. This corollary also implies Theorem 3.1 by observing that (3.3) implies $\mathcal{S} = \{\mathbb{R}\}$ and also that d is monotonically increasing.

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