

VISUAL CHARACTERIZATION OF ASSOCIATIVE QUASITRIVIAL NONDECREASING FUNCTIONS ON FINITE CHAINS

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ABSTRACT. In this paper we provide visual characterization of associative quasitrivial nondecreasing operations on finite chains. We also provide a characterization of bisymmetric quasitrivial nondecreasing binary operations on finite chains. Finally, we estimate the number of functions belonging to the previous classes.

1. INTRODUCTION

Extensive literature deals with aggregation operators on discrete settings and it have been studied by many researchers in the last decades, e.g., [6, 8, 14, 19, 21–26, 28, 32, 33]. These functions play important role in decision making [2–4, 16] and in fuzzy logic. In particular, the investigation of associative quasitrivial functions defined on chains is a topic of increasing interest [7, 10, 29].

The study of n -ary associativity stemmed from the pioneer work of Dörnte [11] and Post [27]. In [12, 13] the reducibility (see Definition 2.2) of n -ary associative functions have been characterized by the existence of neutral elements. In [1] a complete characterization of quasitrivial associative n -ary functions have been presented. In [10] the quasitrivial symmetric nondecreasing associative n -ary operations defined on a nonempty chains have been characterized. Recently, in [18] it was proved that associative idempotent nondecreasing n -ary functions defined on a nonempty chain are reducible.

In this paper we investigate associative quasitrivial nondecreasing functions on finite chains. In [7, 29] idempotent discrete uninorms (i.e. idempotent symmetric nondecreasing associative functions with neutral elements defined on finite chains) have been characterized. Since every idempotent uninorms are quasitrivial, in some sense this paper is a continuation of the works [7, 10, 29] for finite chains where we eliminate the assumption of symmetry of the functions.

Important to note the analogue 'classical' results for the unit interval $[0, 1]$. Czogała-Drewniak proved in [5] that associative monotonic idempotent operations with neutral element are a combination of minimum and maximum, and thus these are quasitrivial. Martin, Mayor and Torrens in [20] gave a complete characterization of associative quasitrivial nondecreasing functions on $[0, 1]$. A refinement of their argument can be found in [30]. (For the n -ary generalization of these results see [17].) We note that in [29] the analogue of the result of Czogała-Drewniak for finite chains has been provided adding the assumption of symmetry of such functions.

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The paper is organized as follows. In Section 2 we present the most important definitions. At the beginning of Section 3 we recall [18, Theorem 4.8], which states that every associative idempotent nondecreasing n -ary function is derived from a binary one. Since every quasitrivial function is idempotent, a characterization for associative quasitrivial nondecreasing binary functions automatically implies a characterization for the n -ary case. Thus, in Section 3 we introduce the basic concept of visualization for quasitrivial monotone binary functions and present some preliminary results due to this concept. Here we discuss an important visual test of non-associativity (Lemma 3.5). This is an analogue of [7, Proposition 18]. Section 4 is devoted to the visual characterization of associative quasitrivial nondecreasing functions with so-called 'downward-right paths' (Theorems 4.12 and 4.13). We also present an Algorithm which provide the contour plot of any associative quasitrivial nondecreasing function. In Section 5 we characterize the bisymmetric quasitrivial nondecreasing binary operations (Theorem 5.3). In Subsection 5.2.1 we show that the analogue of [18, Theorem 4.8] does not hold for bisymmetric quasitrivial nondecreasing n -ary operations (Example 5.4). This answers the question whether a quasitrivial, bisymmetric operation is associative (see [10, Remark 10.(b)]). In Subsection 5.2.2 we provide some special classes of n -ary bisymmetric operations where a reduction to the binary case can be guaranteed (Corollary 5.8). In Section 6 we calculate the number of associative quasitrivial nondecreasing functions defined on a finite chain of given size with and also without the assumption of having neutral elements (Theorem 6.1). We get similar estimations for the number of bisymmetric quasitrivial nondecreasing binary functions defined on a finite chain of given size (Proposition 6.5). Using a slight modification of the proof of [18, Theorem 3.2], in the Appendix we show that every associative quasitrivial monotonic n -ary functions are nondecreasing.

2. DEFINITION

Here we present the basic definitions and some preliminary results. First we introduce the following simplification. For any integer $l \geq 0$ and any $x \in X$, we set $l \cdot x = x, \dots, x$ (l times). For instance, we have $F(3 \cdot x_1, 2 \cdot x_2) = F(x_1, x_1, x_1, x_2, x_2)$.

Definition 2.1. Let X be an arbitrary nonempty set. A function $F : X^n \rightarrow X$ is called

- *idempotent* if $F(n \cdot x) = x$ for all $x \in X$;
- *quasitrivial* (or *conservative*) if

$$F(x_1, \dots, x_n) \in \{x_1, \dots, x_n\}$$

for all $x_1, \dots, x_n \in X$;

- *associative* if

$$\begin{aligned} & F(x_1, \dots, x_{i-1}, F(x_i, \dots, x_{i+n-1}), x_{i+n}, \dots, x_{2n-1}) \\ &= F(x_1, \dots, x_i, F(x_{i+1}, \dots, x_{i+n}), x_{i+n+1}, \dots, x_{2n-1}) \end{aligned}$$

for all $x_1, \dots, x_{2n-1} \in X$ and all $i \in \{1, \dots, n-1\}$;

- *bisymmetric* if

$$F(F(\mathbf{r}_1), \dots, F(\mathbf{r}_n)) = F(F(\mathbf{c}_1), \dots, F(\mathbf{c}_n))$$

for all $n \times n$ matrices $[\mathbf{r}_1 \ \dots \ \mathbf{r}_n] = [\mathbf{c}_1 \ \dots \ \mathbf{c}_n]^T \in X^{n \times n}$.

We say that $F : X^n \rightarrow X$ has a *neutral element* $e \in X$ if for all $x \in X$ and all $i \in \{1, \dots, n\}$

$$F((i-1) \cdot e, x, (k-i) \cdot e) = x.$$

Let (X, \leq) be a chain (i.e, a totally ordered set). An operation $F : X^n \rightarrow X$ is said to be

- *nondecreasing* (or *order-preserving in each coordinate*) if $F(x_1, \dots, x_n) \leq F(x'_1, \dots, x'_n)$ whenever $x_i \leq x'_i$ for all $i \in \{1, \dots, n\}$,
- *monotone* if it is order-preserving or order-reversing in each of its coordinate.

Definition 2.2. We say that $F : X^n \rightarrow X$ is *derived from* a binary function $G : X^2 \rightarrow X$ if F can be written of the form

$$F(x_1, \dots, x_n) = x_1 \circ \dots \circ x_n,$$

where $x \circ y = G(x, y)$.¹ If such a G exists, then we say that F is *reducible*.

We note that if $n = 2$ we get the 'classical' binary definition of associativity, quasitriviality, idempotency, and neutral element property.

We denote the *diagonal* of X^2 by $\Delta_X = \{(x, x) : x \in X\}$.

Definition 2.3. Let L_k denote $\{1, \dots, k\}$ endowed with the natural ordering (\leq) .

Then L_k is a finite chain. Moreover, every finite chain with k element can be identified with L_k and the domain of an n -variable function defined on a finite chain can be identified with $\underbrace{L_k \times \dots \times L_k}_n = (L_k)^n$ for some $k \in \mathbb{N}$.

For an arbitrary poset X and $a \leq b \in X$ we denote the elements between a and b by $[a, b] \subseteq X$. In particular, for L_k

$$[a, b] = \{m \in L_k : a \leq m \leq b\}.$$

We also introduce the lattice notion of the minimum (\wedge) and the maximum (\vee) as follows

$$\begin{aligned} x_1 \wedge \dots \wedge x_n &= \wedge_{i=1}^n x_i = \min\{x_1, \dots, x_n\}, \\ x_1 \vee \dots \vee x_n &= \vee_{i=1}^n x_i = \max\{x_1, \dots, x_n\}. \end{aligned}$$

3. BASIC CONCEPT AND PRELIMINARY RESULTS

The following general result was published as [18, Theorem 4.8] recently.

Theorem 3.1. *Let X be a nonempty chain and $F : X^n \rightarrow X$ ($n \geq 2$) be an associative idempotent nondecreasing function. Then there exists uniquely an associative idempotent nondecreasing binary function $G : X^2 \rightarrow X$ such that F is derived from G . Moreover, G can be defined by*

$$(1) \quad G(a, b) = F(a, (n-1) \cdot b) = F((n-1) \cdot a, b) \quad (a, b \in X).$$

Remark 1. By the definition of G (equation (1)), it is also clear that if F is quasitrivial, then G is also.

According to Theorem 3.1 and Remark 1, a characterization of associative quasitrivial nondecreasing binary functions automatically implies a characterization for the n -ary case. Thus, it is enough to deal with the binary case ($n = 2$). That is what we do in Section 4 for binary functions that are defined on finite chains.

¹We note that this expression is well-defined if and only if G is associative.

3.1. Visualization of binary functions. In this section we prove and reprove basic properties of quasitrivial associative nondecreasing binary functions in the spirit of visualization.

Lemma 3.2. *Let X be a nonempty chain and let $F : X^2 \rightarrow X$ be a quasitrivial monotone function. If $F(x, t) = x$, then $F(x, s) = x$ for every $s \in [x \wedge t, x \vee t]$. Similarly, if $F(x, t) = t$, then $F(s, t) = t$ for every $s \in [x \wedge t, x \vee t]$.*

Proof. The function F is quasitrivial, thus idempotent. Monotonicity implies the statement as follows.

Without loss of generality we can assume that $x \leq t$. Using monotonicity for every $s \in [x, t]$ one of the following holds

$$x = F(x, t) = F(x, s) = F(x, x),$$

or

$$t = F(x, t) = F(s, t) = F(t, t).$$

□

According to Lemma 3.2, a picture (see Figure 1.) can be drawn using only horizontal and vertical line segments starting from the points of the diagonal which indicates that the value of F along a line is the same. Thus it is clear that these lines do not cross each other according to monotonicity.

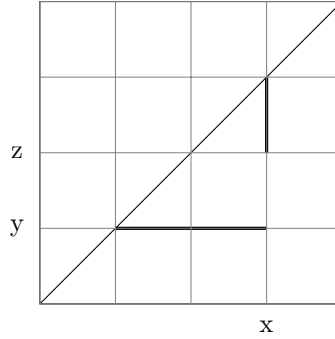


FIGURE 1. $F(x, y) = y$ and $F(x, z) = x$

Here we note that the previous observation can be deduced easily also from [7, Proposition 6]. As a consequence we get the following.

Corollary 3.3. *Let X be a nonempty chain and $F : X^2 \rightarrow X$ be a quasitrivial function.*

$$F \text{ is monotone} \iff F \text{ is nondecreasing.}$$

Proof. We only need to prove that every monotone quasitrivial function is nondecreasing.

As an easy consequence of Lemma 3.2 and the quasitriviality of F , we get $F(s, x) \leq F(t, x)$ and $F(x, t) \leq F(x, s)$ for any $x, s, t \in X$ that satisfies $s \in [x, t]$. This implies that F is nondecreasing in the first variable. Similar argument shows the statement for the second variable.

□

Remark 2. The analogue of Corollary 3.3 holds whenever $n > 2$. The proof is essentially the same as the proof of [18, Theorem 3.10]. Thus we present it in Appendix A.

In the sequel we are dealing with associative, quasitrivial and nondecreasing functions.

There are several know forms of the following proposition. This type of results was first proved by [20]. The form as stated here is [7, Proposition 18].

Proposition 3.4. *Let X be an arbitrary nonempty set and let $F : X^2 \rightarrow X$ be a quasitrivial operation. Then the following assertions are equivalent.*

- (i) F is **not** associative.
- (ii) There exist pairwise distinct $x, y, z \in X$ such that $F(x, y), F(x, z), F(y, z)$ are pairwise distinct.
- (iii) There exists a rectangle in X^2 such that one of the vertices is on Δ_X and the three remaining vertices are in $X^2 \setminus \Delta_X$ and pairwise disconnected.

Remark 3. The (ii) part of the previous statement can be formalized as follows:

- (iv) There exist $x, y, z \in X$ which satisfies one of the following cases:

$$(2) \quad F(x, y) = x, F(x, z) = z, F(y, z) = y \text{ (Case 1),}$$

or

$$(3) \quad F(x, y) = y, F(y, z) = z, F(x, z) = x \text{ (Case 2).}$$

As an important consequence of this form we get that if any two of $x, y, z \in X$ are equal, then F automatically satisfies $F(F(x, y), z) = F(x, F(y, z))$. Therefore we can always assume that $x \neq y \neq z$.

Now we present a visual characterization which is also an equivalent form of the previous statement if F is nondecreasing.

Lemma 3.5. *Let X be chain and let $F : X^2 \rightarrow X$ be a quasitrivial, nondecreasing function. Then F is **not** associative if and only if there are elements $x, y, z \in X$ ($x \neq y \neq z$) that give one of the following pictures.*

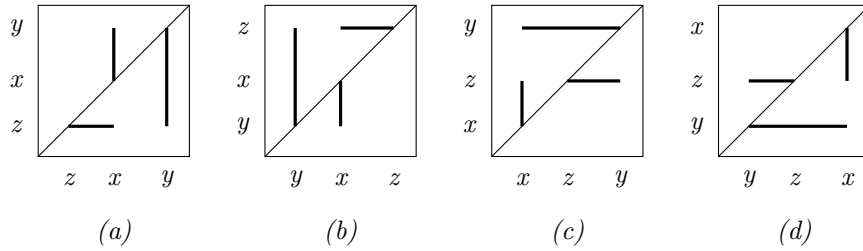


FIGURE 2. Four pictures that prove the non-associativity of F

Proof. By Remark 3, F is not associative if and only if there exists $x, y, z \in X$ satisfying either (2) or (3) and $x \neq y \neq z$. The values $x, y, z \in X$ can be ordered in 6 possible configuration of type $x < y < z$. For each case either (2) or (3) holds. Therefore we have 12 configurations as possible realizations of Case 1 or Case 2.

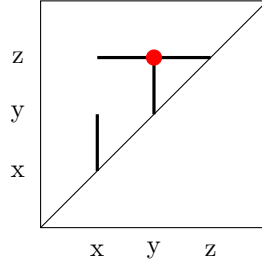


FIGURE 3. Case 1. and $x < y < z$ ('Fake' example)

Let us consider Case 1 (when equation (2) holds) and assume $x < y < z$. Then Lemma 3.2 implies the following (see Figure 3).

The red point signs the problem of this configuration, since two lines with different values cross each other. This is not possible for any quasitrivial monotone function.

Thus this subcase provides 'fake' example to study associativity. From the total, 8 cases are 'fake' in this sense.

The remaining 4 cases are presented in the statement. Figure 2 (a) and (b) represent the case when equation (2) holds, and Figure 2 (c) and (d) represent the case when (3) holds. \square

Since for a 2-element set none of the cases of Figure 2 can be realized, as an immediate consequence of Lemma 3.5 we get the following.

Corollary 3.6. *Every quasitrivial nondecreasing function $F : L_2^2 \rightarrow L_2$ is associative.*

Next consequence was proved first in [20, Proposition 2] for the closed unit interval $[0, 1] \in \mathbb{R}$. For finite chains this is part of [7, Proposition 11.] (see Proposition 3.8).

Corollary 3.7. *Let X be nonempty chain and $F : X^2 \rightarrow X$ be a quasitrivial symmetric nondecreasing function then F is associative.*

Proof. Each cases presented in Figure 2 have crossing lines if we add the assumption of symmetry of F . For instance, we present Case 1/a in Figure 4. The equation $F(y, z) = F(z, y) = y$ and $z < x < y$ implies that $F(x, y) = y$. Thus Figure 2 (a) is not possible for quasitrivial, nondecreasing functions. For other cases similar argument works. Therefore, none of the cases of Figure 2 is possible, hence F is automatically associative. \square

For finite chains more can be stated.

Proposition 3.8 ([7, Proposition 11.]). *If $F : L_k^2 \rightarrow L_k$ is quasitrivial symmetric nondecreasing then it is associative and has a neutral element.*

Remark 4. The conclusion that F has a neutral element is not necessarily true when $X = [0, 1]$ according to [20]. This fact is one of the main difference between the cases $X = L_k$ and $X = [0, 1]$.

If we assume that F has a neutral element (as, for instance, in Proposition 3.8), then as a consequence of Lemma 3.2 we get the following pictures (Figure 5) for

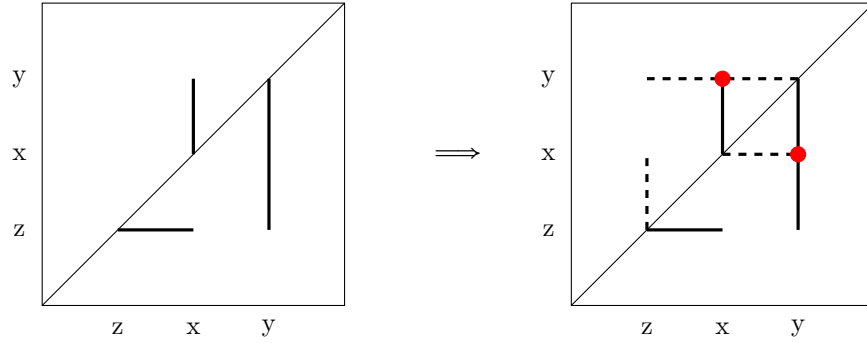


FIGURE 4. The symmetric case

idempotent monotone functions having neutral elements. In Figure 5 the neutral element is denoted by e .

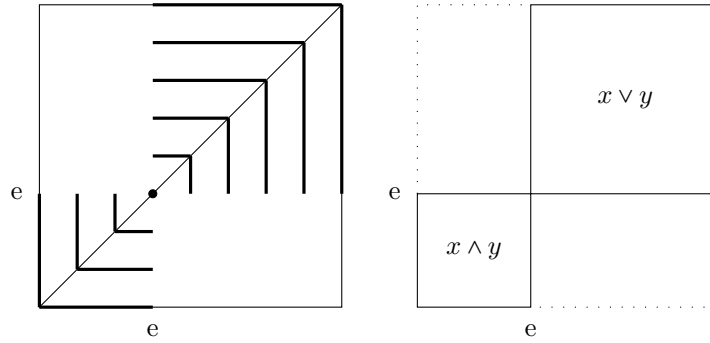


FIGURE 5. Partial description of a symmetric idempotent monotone function

4. VISUAL CHARACTERIZATION OF ASSOCIATIVE QUASITRIVIAL NONDECREASING FUNCTIONS DEFINED ON L_k

From now on we denote the upper and the lower 'triangle' by

$$T_1 = \{(x, y) : x, y \in L_k, x \geq y\}, \quad T_2 = \{(x, y) : x, y \in L_k, x \leq y\},$$

respectively, as in Figure 6. We note that $T_1 \cap T_2$ is the diagonal Δ_{L_k} .

Definition 4.1. For a function $F : L_k^2 \rightarrow L_k$ there can be defined the *upper symmetrization* F_1 and *lower symmetrization* F_2 of F as

$$F_1(x, y) = \begin{cases} F(x, y) & \text{if } (x, y) \in T_1 \\ F(y, x) & \text{if } (y, x) \in T_1 \end{cases} \quad \text{and} \quad F_2(x, y) = \begin{cases} F(x, y) & \text{if } (x, y) \in T_2 \\ F(y, x) & \text{if } (y, x) \in T_2, \end{cases}$$

Briefly, $F_1(x, y) = F(x \wedge y, x \vee y)$, $F_2(x, y) = F(x \vee y, x \wedge y) \quad \forall x, y \in L_k$.

Fodor [15] (see also [31, Theorem 2.6]) shown the following statement.

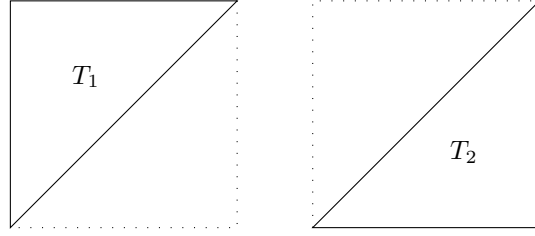


FIGURE 6. The upper and lower 'triangles' T_1 and T_2

Proposition 4.2. *Let X be a nonempty chain and $F : X^2 \rightarrow X$ be an associative function. Then F_1 and F_2 , the lower and the upper symmetrization of F , are also associative.*

This idea makes it possible to investigate the two 'parts' of a non-symmetric associative function as one-half of two symmetric associative functions.

By Proposition 3.8, both symmetrization of a nondecreasing quasitrivial function $F : L_k^2 \rightarrow L_k$ has a neutral element.

Definition 4.3. We call an element *upper (or lower) half-neutral element* of F if it is the neutral element of the upper (or the lower) symmetrization. For simplicity we always denote the upper and lower half-neutral element of F by e and f , respectively.

Summarizing the previous results we get following partial description.

Proposition 4.4. *Let $F : L_k^2 \rightarrow L_k$ be an associative quasitrivial nondecreasing function. Then it has an upper and an lower half-neutral element denoted by e and f . Moreover, if $e \leq f$ then*

$$F(x, y) = \begin{cases} x \wedge y & \text{if } x \vee y \leq e \\ y & \text{if } e \leq x \leq f \\ x \vee y & \text{if } x \wedge y \leq f \end{cases}$$

Analogously, if $f \leq e$ then

$$F(x, y) = \begin{cases} x \wedge y & \text{if } x \vee y \leq f \\ x & \text{if } f \leq x \leq e \\ x \vee y & \text{if } x \wedge y \leq e \end{cases}$$

We note that $e = f$ iff F has a neutral element.

Proposition 4.4 is illustrated in Figure 7 when $e \leq f$.

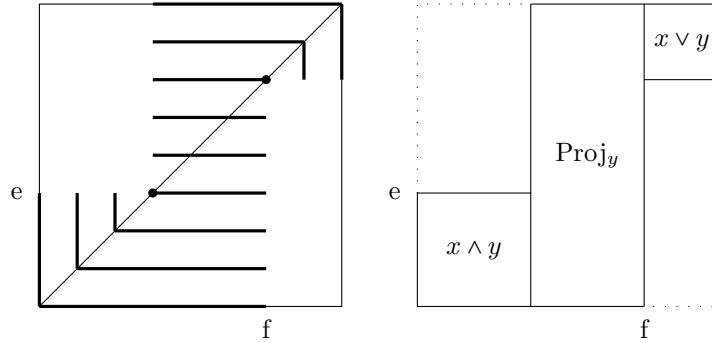


FIGURE 7. Partial description of idempotent monotone functions when $e \leq f$

The following lemma is essential for the visual characterization.

Lemma 4.5. *Let $F : L_k^2 \rightarrow L_k$ be an associative quasitrivial nondecreasing function. Assume that there exists $a < b \in L_n$ such that $F(a, b) = a$ and $F(b, a) = b$. Then one of the following holds:*

(a) *If $F(a + 1, a) = a$, then*

$$F(x, b) = b \text{ and } F(y, a) = a$$

for every $x \in [a + 1, b]$ and $y \in [a, b - 1]$.

(b) *If $F(a + 1, a) = a + 1$, then $F(x, y) = x$ ($= Proj_x$) for all $x, y \in [a, b]^2$.*

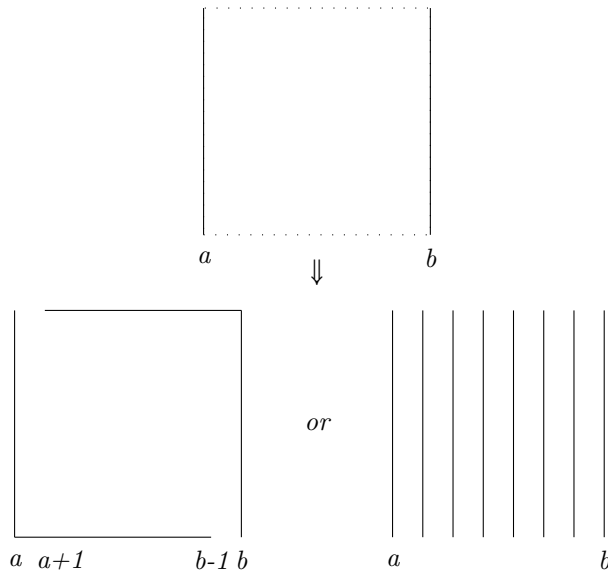


FIGURE 8. Graphical interpretation of Lemma 4.5

Proof. Assume first that $F(a + 1, a) = a$. Then it follows that $F(a + 1, b) = b$, otherwise we get Figure 2 (a). Using Lemma 3.2 we have that $F(x, b) = b$ for every

$x \in [a + 1, b]$. The equation $F(b - 1, b) = b$ implies that $F(b - 1, a) = a$, otherwise we are in the situation of Figure 2 (b). Similarly, as above we get that $F(y, a) = a$ for every $y \in [a, b - 1]$. Here we note that an analogue argument gives the same result if we assume originally that $F(b - 1, b) = b$.

Now assume that $F(a + 1, a) = a + 1$. This immediately implies that $F(x, a) = x$ for every $x \in [a, b]$ by quasitriviality, since it cannot be a by the nondecreasingness of F . Using Lemma 3.2 again, it follows that $F(x, y) = x$ for all $y \in [a, x]$. Since $F(b - 1, b) = b$ also implies the previous case, the assumption $F(a + 1, a) = a + 1$ implies $F(b - 1, b) = b - 1$. Similarly as above, this condition implies that $F(x, b) = x$ for all $x \in [a, b]$ and, by Lemma 3.2, it follows that $F(x, y) = x$ for every $y \in [x, b]$. Altogether we get that $F(x, y) = x = \text{Proj}_x(x, y)$ as we stated. \square

Remark 5. Analogue of Lemma 4.5 can be formalized as follows.

Let $F : L_k^2 \rightarrow L_k$ be an associative quasitrivial nondecreasing function. Assume that there exists $a < b \in L_k$ such that $F(b, a) = a$ and $F(a, b) = b$. Then one of the following holds:

(a) If $F(a, a + 1) = a$, then

$$F(b, x) = b \text{ and } F(a, y) = a$$

for every $x \in [a + 1, b]$ and $y \in [a, b - 1]$.

(b) If $F(a, a + 1) = a + 1$, then $F(x, y) = y (= \text{Proj}_y)$ for all $x, y \in [a, b]^2$.

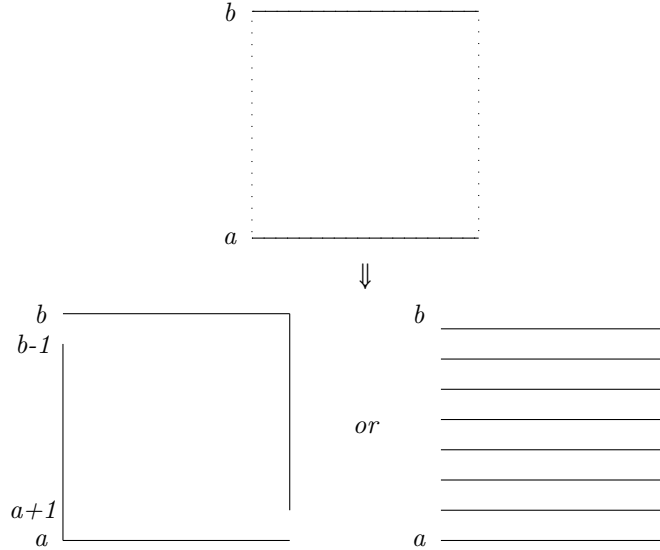


FIGURE 9. Graphical interpretation of Remark 5

The proof of this statement is analogue to Lemma 4.5 using Figure 2(c) and (d) instead of Figure 2(a) and (b), respectively.

From the previous results we conclude the following.

Lemma 4.6. *Let $F : L_k^2 \rightarrow L_k$ be an associative quasitrivial and nondecreasing function and e and f be the upper and the lower half-neutral elements, respectively, and let $a, b \in L_k$ ($a < b$) be given. If $F(x, y) = x$ for every $x, y \in [a, b]$ (i.e., Lemma 4.5 (b) holds), then $f < e$ and $[a, b] \subseteq [f, e]$. Similarly, if $F(x, y) = y$ for every $x, y \in [a, b]$ (i.e., Remark 5 (b) holds), then $e < f$ and $[a, b] \subseteq [e, f]$.*

Proof. This is a direct consequence of Proposition 4.4. It can be easily checked that $\tilde{F} = F|_{[a,b]^2}$, contains a part where \tilde{F} is a minimum or a maximum if a or b is not in $[e \wedge f, e \vee f]$. Moreover, it is also easily follows that if $F(x, y) = x$ for every $x, y \in [a, b]$, then $f < e$ must hold. Similarly, $F(x, y) = y$ for every $x, y \in [a, b]$ implies $e < f$. \square

Corollary 4.7. *Let F, e, f be as in Lemma 4.6 and assume that $a, b \in X$ such that $a < b$ and $F(a, b) \neq F(b, a)$. Then*

- (i) *Lemma 4.5(b) holds iff $f < e$ and $a, b \in [f, e]$,*
- (ii) *Remark 5(b) holds iff $e < f$ and $a, b \in [e, f]$.*
- (iii) *Lemma 4.5(a) or Remark 5(a) holds iff $a, b \notin [e \wedge f, e \vee f]$.*

With other words we have:

Corollary 4.8. *Let F, e, f be as in Lemma 4.6. Then $F(a, b) = F(b, a)$, if $a \notin [e \wedge f, e \vee f]$ and $b \in [e \wedge f, e \vee f]$, or $b \notin [e \wedge f, e \vee f]$ and $a \in [e \wedge f, e \vee f]$.*

This form makes it possible to extend the partial description. (See Figure 10 for the case $e < f$.)

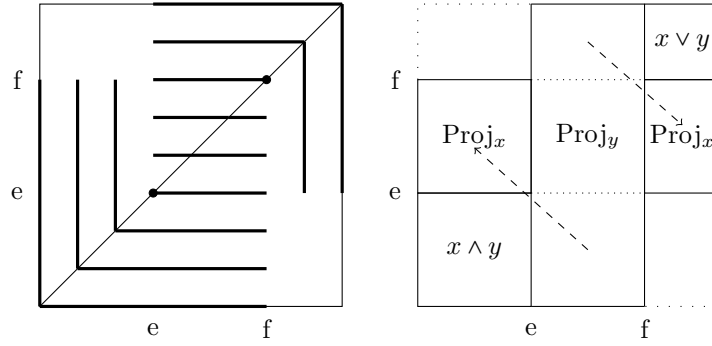


FIGURE 10. Extended partial description of idempotent monotone functions when $e < f$

Using Lemma 4.5 and Remark 5 we can provide a visual characterization of associative quasitrivial nondecreasing functions. The characterization based on the following algorithm which outputs the contour plot of F .

Before we present the algorithm we note that the letters indicated in the following figures represent the value of the function F in the corresponding points or lines (not a coordinate of the points itself as usual).

Algorithm

Initial setting: Let $Q_1 = L_k^2$ and $F : L_k^2 \rightarrow L_k$ be an associative quasitrivial nondecreasing function.

Step i. For $Q_i = [a, b]^2$ ($a \leq b$) we distinguish cases according to the values of $F(a, b)$ and $F(b, a)$. Whenever Q_i contains only 1 element ($a = b$) for some i , then we are done.

- I. (a) If $F(a, b) = F(b, a) = a$, then draw straight lines between the points (b, a) and (a, a) and between (a, b) and (a, a) . Let $Q_{i+1} = [a + 1, b]^2$. (See Figure 11.)

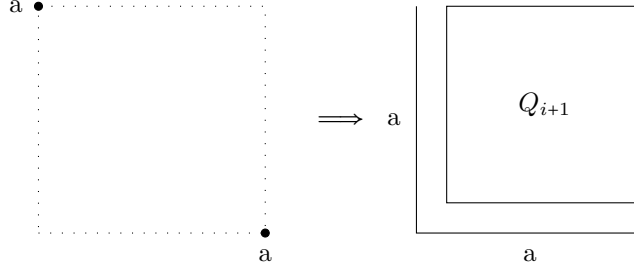


FIGURE 11. Case I.(a)

- (b) If $F(a, b) = F(b, a) = b$, then draw straight lines between the points (a, b) and (a, a) and between (b, a) and (a, a) . Let $Q_{i+1} = [a, b - 1]^2$.
- II. (a) If $F(a, b) = a, F(b, a) = b$ and $F(a + 1, a) = a + 1$, then $F(x, y) = x$ for all $x, y \in [a, b]$ and we are done. (See Figure 12)

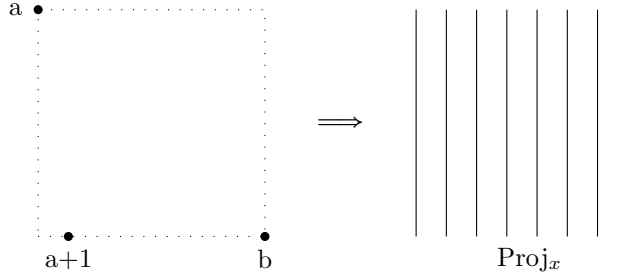


FIGURE 12. Case II(a)

- (b) If $F(a, b) = b, F(b, a) = a$ and $F(a, a + 1) = a + 1$, then $F(x, y) = y$ for all $x, y \in [a, b]$ and we are also done.
- III. (a) If $F(a, b) = a, F(b, a) = b$ and $F(a + 1, a) = a$, then Lemma 4.5 (a) holds and we have Figure 13. Let $Q_{i+1} = [a + 1, b - 1]^2$.
- (b) If $F(a, b) = b, F(b, a) = a$ and $F(a, a + 1) = a$, then Remark 5 (a) holds. Let $Q_{i+1} = [a + 1, b - 1]^2$.

It is clear that the algorithm is finished after finitely many steps. Let us denote this number of steps by $l \in \mathbb{N}$.

We also denote the top-left and the bottom-right corner of Q_i by p_i and q_i ($i = 1, \dots, l$), respectively.

Let \mathcal{P} (and \mathcal{Q}) denote the path containing p_i (and q_i) for $i \in \{1, \dots, l\}$ and line segments between consecutive p_i 's (and q_i 's). Let us denote the line segment between p_i and p_{i+1} by $\overline{p_i, p_{i+1}}$. We set the notation $\mathcal{P} = (p_j)_{j=1}^l$ and $\mathcal{Q} = (q_j)_{j=1}^l$.

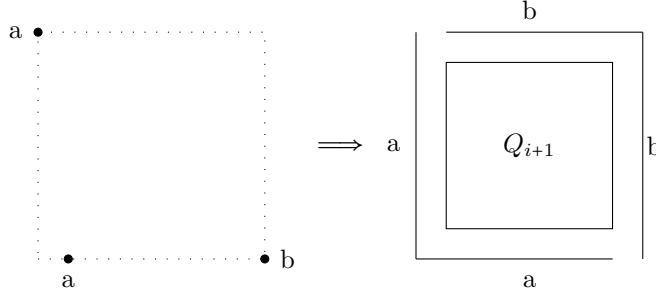
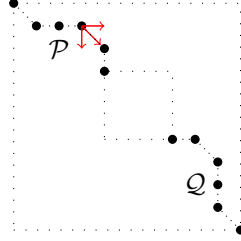


FIGURE 13. Case III.(a)

Clearly, we get the path \mathcal{P} if we start at the top-left corner of L_k^2 and in each step we move either one place to the right or one place downward or one place diagonally downward-right.

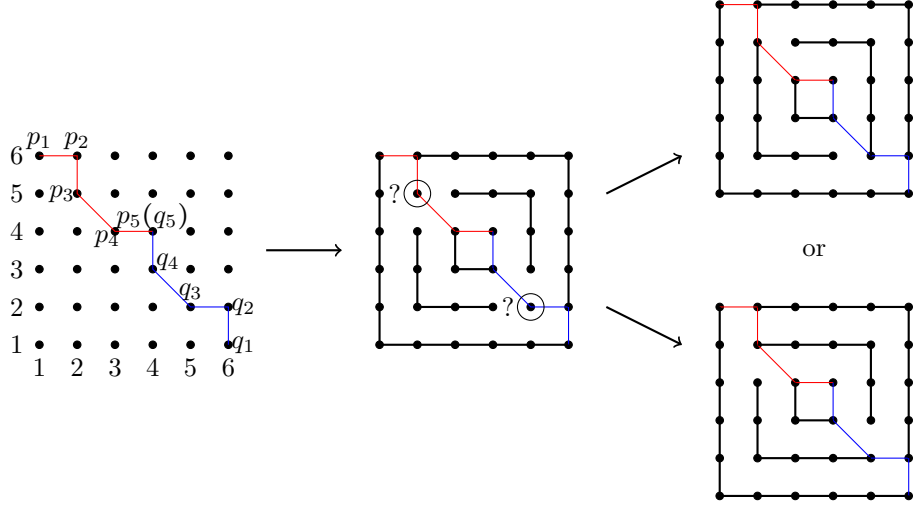
Definition 4.9. We say that a path is a *downward-right path* of L_k if in each step it moves to the nearest point of L_k^2 either one place to the right or one place downward or one place diagonally downward-right.


 FIGURE 14. The path \mathcal{P} is a downward-right path

If $\overline{p_i, p_{i+1}}$ is horizontal or vertical, then the reduction from Q_i to Q_{i+1} is uniquely determined. Moreover, if $\overline{p_i, p_{i+1}}$ is horizontal, then $F(x, y) = F(y, x) = x \wedge y$, where $p_i = (x, y)$ and $q_i = (y, x)$. Similarly if $\overline{p_i, p_{i+1}}$ is vertical, then $F(x, y) = F(y, x) = x \vee y$, where $p_i = (x, y)$ and $q_i = (y, x)$. On the other hand if $\overline{p_i, p_{i+1}}$ is diagonal, then we have a free choice for the value of F in p_i . This is determined by either Lemma 4.5 (a) or Remark 5 (a). Since in this case the value of F in q_i is different from p_i , the value in q_i is automatically defined. It is also clear from the algorithm that the path \mathcal{Q} is the reflection of \mathcal{P} to the diagonal Δ_{L_k} .

Using the previous paragraph and Lemma 3.2 it is possible to reconstruct functions from a given downward-right path \mathcal{P} which starts at $(1, k)$.

Example 4.10. We illustrate the reconstruction on $L_6 \times L_6$. The paths $\mathcal{P} = (p_j)_{j=1}^5$ and $\mathcal{Q} = (q_j)_{j=1}^5$ denoted by red and blue, respectively. According to the previous observations we get the following pictures (see Figure 15). It can be clearly seen that \mathcal{Q} is the reflection of \mathcal{P} to the diagonal Δ_{L_6} , and 4 is the neutral element of the reconstructing function, where \mathcal{P} and \mathcal{Q} touch each other and reach the diagonal Δ_{L_6} . For the precise statement and proof see Theorem 4.13.

FIGURE 15. Reconstruction of F from the path \mathcal{P}

Definition 4.11. Let $\mathcal{P} \subset L_k^2$ be the downward-right path from $(1, k)$ to (a, b) ($a < b$) and let \mathcal{Q} be the reflection of \mathcal{P} to the diagonal Δ_{L_k} .

We say that $(x, y) \in L_k^2 \setminus (\mathcal{P} \cup \mathcal{Q} \cup [a, b]^2)$ is *above* $\mathcal{P} \cup \mathcal{Q}$ if there exists $p = (x, w) \in \mathcal{P}$ such that $y > w$ or $q = (w, y) \in \mathcal{Q}$ such that $x > w$.

Similarly, we say that $(x, y) \in L_k^2 \setminus (\mathcal{P} \cup \mathcal{Q} \cup [a, b]^2)$ is *below* $\mathcal{P} \cup \mathcal{Q}$ if there exists a $p = (x, w) \in \mathcal{P}$ such that $y < w$ or a $q = (w, y) \in \mathcal{Q}$ such that $x < w$.

Using this terminology we can summarize the previous observations and we get the following characterization. The next statement can be seen as the analogue of theorem of Czogała-Drewniak [5, Theorem 3.] for finite chains.

Theorem 4.12. For every associative quasitrivial nondecreasing function $F : L_k^2 \rightarrow L_k$ there exist half-neutral elements $a, b \in L_k$ ($a \leq b$) and a downward-right path $\mathcal{P} = (p_j)_{j=1}^l$ (for some $l \in \mathbb{N}, l < k$) from $(1, k)$ to (a, b) . We denote the reflection of \mathcal{P} to the diagonal Δ_{L_k} by $\mathcal{Q} = (q_j)_{j=1}^l$. Then for every $(x, y) \notin \mathcal{P} \cup \mathcal{Q}$

$$F(x, y) = \begin{cases} x \vee y, & \text{if } (x, y) \text{ is above } \mathcal{P} \cup \mathcal{Q} \\ x \wedge y, & \text{if } (x, y) \text{ is below } \mathcal{P} \cup \mathcal{Q} \\ \text{Proj}_x \text{ or } \text{Proj}_y, & \text{if } (x, y) \in [a, b]^2, \end{cases}$$

and for every $(x, y) \in \mathcal{P} \cup \mathcal{Q}$

$$F(x, y) = \begin{cases} x \wedge y & \text{if } (x, y) = p_i \text{ or } q_i \text{ and } \overline{p_i, p_{i+1}} \text{ is horizontal,} \\ x \vee y, & \text{if } (x, y) = p_i \text{ or } q_i \text{ and } \overline{p_i, p_{i+1}} \text{ is vertical,} \\ x \text{ or } y, & \text{if } (x, y) = p_i \text{ and } \overline{p_i, p_{i+1}} \text{ is diagonal,} \\ x \text{ or } y, & \text{if } (x, y) = q_i \text{ and } \overline{q_i, q_{i+1}} \text{ is diagonal.} \end{cases}$$

If a is the lower half-neutral element f and b is the upper half-neutral element e , then F is Proj_x on $[a, b]^2$, otherwise it is Proj_y .

Moreover F is symmetric exact on $[a, b]^2$ and in the points $p_i \in \mathcal{P}$ and $q_i \in \mathcal{Q}$ where $\overline{p_i, p_{i+1}}$ is diagonal ($i \in \{1, \dots, l-1\}$).

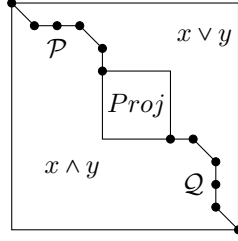


FIGURE 16. Characterization of associative quasitrivial nondecreasing functions on finite chains

Proof. The statement clearly follows from the Algorithm and the definition of paths \mathcal{P} and \mathcal{Q} . \square

The converse statement can be formalized as follows. This statement plays the role of theorem of Martin-Mayor-Torrens [20, Theorem 4.] for finite chains.

Theorem 4.13. *Let $\mathcal{P} = (p_j)_{j=1}^l$ be a downward-right path in $T_1 \subset L_k^2$ from $(1, k)$ to (a, b) ($a \leq b$) and let $\mathcal{Q} = (q_j)_{j=1}^l$ be its reflection to the diagonal Δ_{L_k} . Let $F : L_k^2 \rightarrow L_k$ be defined for every $(x, y) \notin \mathcal{P} \cup \mathcal{Q}$ as*

$$F(x, y) = \begin{cases} x \vee y, & \text{if } (x, y) \text{ is above } \mathcal{P} \cup \mathcal{Q}, \\ x \wedge y, & \text{if } (x, y) \text{ is below } \mathcal{P} \cup \mathcal{Q}, \\ \text{Proj}_x \text{ or } \text{Proj}_y \text{ (uniformly),} & \text{for every } (x, y) \in [a, b]^2. \end{cases}$$

and for every $(x, y) \in \mathcal{P} \cup \mathcal{Q}$

$$F(x, y) = \begin{cases} x \wedge y & \text{if } (x, y) = p_i \text{ or } q_i \text{ and } \overline{p_i, p_{i+1}} \text{ is horizontal,} \\ x \vee y, & \text{if } (x, y) = p_i \text{ or } q_i \text{ and } \overline{p_i, p_{i+1}} \text{ is vertical,} \\ x \text{ or } y \text{ (arbitrarily) ,} & \text{if } (x, y) = p_i \text{ and } \overline{p_i, p_{i+1}} \text{ is diagonal.} \end{cases}$$

If $(x, y) = q_i$ and $\overline{q_i, q_{i+1}}$ (or equivalently $\overline{p_i, p_{i+1}}$) is diagonal, then $F(x, y) \in \{x, y\}$ and $F(x, y) \neq F(y, x)$ uniquely define $F(x, y)$. Then F is associative quasitrivial and nondecreasing.

Proof. It is clear that F is defined for every $(x, y) \in L_k^2$ and F is quasitrivial and nondecreasing (coordinate-wise). Now we show that F is associative. If it is not the case, then by Lemma 3.5, one of the cases of Figure 2 is realized. Let $u, v, w \in L_k$ ($u < v < w$) denote the elements where it is realized. Clearly $F(u, w) \neq F(w, u)$ and F is not a projection on $[u, w]^2$. Thus, by the definition of F , it follows that $(u, w) \in \mathcal{P}$ and $(w, u) \in \mathcal{Q}$. Hence $p_i = (u, w)$ for some $i = \{1, \dots, l-1\}$ and $\overline{p_i, p_{i+1}}$ is diagonal. Thus we have one of the following situation (Figure 17).

Therefore, since $u < v < w$, it follows that $F(u, v) \neq v, F(v, u) \neq v, F(w, v) \neq v, F(v, w) \neq v$. Hence, none of the cases of Figure 2 can be realized. Thus F is associative. \square

Remark 6. According to Theorems 4.12 and 4.13 it is clear that there is a surjection from the set of associative quasitrivial nondecreasing functions defined on L_k to the downward-right paths defined on T_1 and started at $(1, k)$ (and ended somewhere in T_1). This surjection is a bijection if and only if the path \mathcal{P} does not contain a

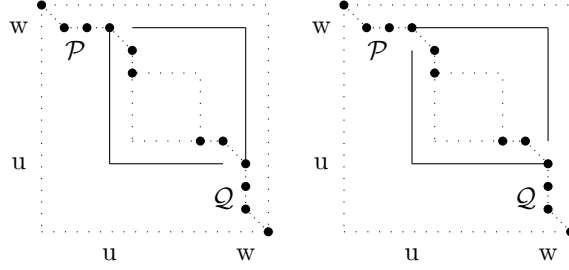


FIGURE 17. Two remaining cases

diagonal move and $a = b$. This condition is equivalent that F is symmetric (and has a neutral element).

Corollary 4.14. *Let $F : L_k^2 \rightarrow L_k$ be an associative quasitrivial nondecreasing function. If F is symmetric, then it is uniquely determined by a downward-right path \mathcal{P} containing only horizontal and vertical line segments and it starts at $(1, k)$ and reaches the diagonal Δ_{L_k} .*

As a consequence of the previous corollary we get the result of [29, Theorem 4.] (see also [7, Theorem 14.]).

Corollary 4.15. *The number of associative quasitrivial nondecreasing symmetric function defined on L_k is 2^{n-1} .*

Proof. Every path from $(1, k)$ to the diagonal Δ_{L_k} using right or downward moves contains k points. According to Corollary 4.14, in each point of the path, except the last one, we have a choice which direction we go further. Thus, in $k-1$ points on the path we have two options. This implies that the number of associative quasitrivial nondecreasing symmetric function defined on L_k is 2^{n-1} . \square

In Theorem 6.1, as an application of the results of this section, we calculate the number of associative quasitrivial nondecreasing functions defined on L_k and also the number of associative quasitrivial nondecreasing functions on L_k that have neutral elements.

Remark 7. (a) We note that from the proof of Lemma 4.5 throughout this section we essentially use that F is defined on a finite chain.
 (b) In the continuous case [5, 20] and also in the symmetric case [7, 29] it is always possible to define a one variable function g , such that the extended graph of g separates the points of the domain of the binary function F into two parts according to if the value of F is a minimum or a maximum. Now the paths \mathcal{P} and \mathcal{Q} play the role of the extended graph of g . Because of the diagonal moves of the path \mathcal{P} , it does not seem so clear how such a 'separating' function can be defined in the non-symmetric discrete case.

5. BISYMMETRIC FUNCTIONS

In this section we show a characterization of bisymmetric quasitrivial nondecreasing functions based on the previous section.

5.1. **Binary case.** The following statement was proved as [7, Lemma 22.].

Lemma 5.1. *Let X be an arbitrary set and $F : X^2 \rightarrow X$ be an operation. Then the following assertions hold.*

- (a) *If F is bisymmetric and has a neutral element, then it is associative and symmetric.*
- (b) *If F is bisymmetric and quasitrivial, then F is associative.*
- (c) *If F is associative and symmetric, then it is bisymmetric.*

Using also the results of Section 4 we get the following statement.

Theorem 5.2. *Let $F : L_k^2 \rightarrow L_k$ be a bisymmetric quasitrivial nondecreasing function. Then there exists the upper half-neutral element e and the lower half-neutral element f and F is symmetric on $(L_k \setminus [e \wedge f, e \vee f])^2$.*

Proof. According to Lemma 5.1(b), every quasitrivial bisymmetric functions are associative. Thus, by Proposition 4.4 it has an upper and lower half-neutral element (e and f , respectively).

Let us assume that $e \leq f$ (the case when $f \leq e$ can be handled similarly).

If there exists $u, v \in L_k$ such that $u < v$, $F(u, v) \neq F(v, u)$, then by Corollary 4.7, either $u, v \in [e, f]$ (then we do not need to prove anything) or $u, v \notin [e, f]$. Moreover, if $u, v \notin [e, f]$, then Lemma 4.5(a) or Remark 5(a) holds. The existence of e implies that $v - u \geq 2$.

If

$$u = F(u, v) \neq F(v, u) = v$$

is satisfied, then Lemma 4.5 (a) holds (i.e, $F(x, v) = v$ if $x \in [u+1, v]$ and $F(y, u) = u$ if $y \in [u, v-1]$). Since $u-v \geq 2$, $u+1 \leq v-1$, hence $F(u+1, u) = u$. On the other hand, F is monotone and idempotent, thus by Lemma 3.2, $F(v, t) = v$ and $F(u, t) = u$ for all $t \in [u, v]$. Using bisymmetric equation we get the following

$$u = F(u, v) = F(F(u+1, u), F(v, v-1)) = F(F(u+1, v), F(u, v-1)) = F(v, u) = v,$$

which is a contradiction.

Similarly, if

$$v = F(u, v) \neq F(v, u) = u$$

is satisfied, then Remark 5 (a) holds (i.e, $F(v, x) = v$ if $x \in [u+1, v]$ and $F(u, y) = u$ if $y \in [u, v-1]$). Since $u-v \geq 2$, $u+1 \leq v-1$, hence $F(v-1, v) = v$. Applying Lemma 3.2 again, we have $F(t, v) = v$ and $F(t, u) = u$ for all $t \in [u, v]$. Using bisymmetric equation we get a contradiction as

$$u = F(v, u) = F(F(v-1, v), F(u, u+1)) = F(F(v-1, u), F(v, u+1)) = F(u, v) = v.$$

□

Applying Theorem 5.2 we get the following characterization.

Theorem 5.3. *Let $F : L_k^2 \rightarrow L_k$ be a quasitrivial nondecreasing function. Then F is bisymmetric if and only if there exists $a, b \in L_k$ ($a \leq b$) and a downward-right path $\mathcal{P} = (p_j)_{j=1}^l$ (for some $l \in \mathbb{N}$) from $(1, k)$ to (a, b) containing only horizontal and vertical line segments such that for every $(x, y) \notin \mathcal{P} \cup \mathcal{Q}$*

$$(4) \quad F(x, y) = \begin{cases} x \vee y, & \text{if } (x, y) \text{ is above } \mathcal{P} \cup \mathcal{Q}, \\ x \wedge y, & \text{if } (x, y) \text{ is below } \mathcal{P} \cup \mathcal{Q}, \\ \text{Proj}_x \text{ or Proj}_y \text{ (uniformly),} & \text{for every } (x, y) \in [a, b]^2. \end{cases}$$

and for every $(x, y) \in \mathcal{P} \cup \mathcal{Q}$

$$(5) \quad F(x, y) = \begin{cases} x \wedge y & \text{if } (x, y) = p_i \text{ or } q_i \text{ and } \overline{p_i, p_{i+1}} \text{ is horizontal,} \\ x \vee y, & \text{if } (x, y) = p_i \text{ or } q_i \text{ and } \overline{p_i, p_{i+1}} \text{ is vertical,} \end{cases}$$

where $\mathcal{Q} = (q_j)_{j=1}^l$ is the reflection of \mathcal{P} to the diagonal Δ_{L_k} .

In particular, F is symmetric on $L_k^2 \setminus [a, b]^2$ and one of the projections on $[a, b]^2$.

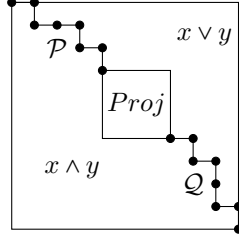


FIGURE 18. Characterization of bisymmetric quasitrivial nondecreasing functions on finite chains

Proof. (Necessity) Since F is bisymmetric and quasitrivial, by Lemma 5.1(b), F is associative. By Theorem 4.12, there exist half-neutral elements $a, b \in L_k$ ($a < b$) and a downward-right path \mathcal{P} from $(1, k)$ to (a, b) . By Theorem 5.3, F is symmetric on $L_k^2 \setminus [a, b]^2$. Thus \mathcal{P} does not contain a diagonal line segment. Hence, applying again Theorem 4.12 we get that F satisfies (4) and (5).

(Sufficiency) The function F defined by (4) and (5) satisfies the conditions of Theorem 4.13, thus F is quasitrivial nondecreasing and associative. Now we show that F is bisymmetric (i.e., $\forall u, v, w, z \in L_k$

$$(6) \quad F(F(u, v), F(w, z)) = F(F(u, w), F(v, z)).$$

Let us assume that $F(x, y) = Proj_x$ on $[a, b]^2$ (for $F(x, y) = Proj_y$ on $[a, b]^2$ the proof is analogue). By Corollary 4.7, this implies that $a = f$ and $b = e$ ($f < e$) and, by Proposition 4.4, it is clear that

$$(7) \quad F(x, y) = x \quad \forall x \in L_k, \forall y \in [a, b].$$

Since F is associative, we have

$$F(F(u, v), F(w, z)) = F(F(F(u, v), w), z) = F(F(u, F(v, w)), z)$$

and

$$F(F(u, w), F(v, z)) = F(F(F(u, w), v), z) = F(F(u, F(w, v)), z).$$

If $F(v, w) = F(w, v)$, then (6) follows and we are done.

If $F(v, w) \neq F(w, v)$, then $v, w \in [a, b]^2$ and, since $F(x, y) = Proj_x$ on $[a, b]^2$, $F(v, w) = v$ and $F(w, v) = w$. Then, by (7),

$$\begin{aligned} F(F(u, F(v, w)), z) &= F(F(u, v), z) = F(u, z), \\ F(F(u, F(w, v)), z) &= F(F(u, w), z) = F(u, z). \end{aligned}$$

Thus F is bisymmetric. \square

- Remark 8.* (a) There is a one-to-one correspondence between downward-right paths containing only vertical and horizontal line segments and the quasitrivial nondecreasing bisymmetric functions where we fix that the function is $Proj_x$ on $[a, b]^2$ (a and b are the half neutral-elements of the function). The same is true, if the function is $Proj_y$ on $[a, b]^2$.
- (b) The nondecreasing assumption can be substituted by monotonicity. Indeed, Corollary 3.3, monotonicity is equivalent with nondecreasingness for quasitrivial functions.

5.2. n -ary bisymmetric functions ($n \geq 3$).

5.2.1. *Negative result.* In this section we show that the analogue of Lemma 5.1(b) does not hold which implies that we cannot necessarily reduce this case to a binary one. The following example also answers the question whether a quasitrivial, bisymmetric operation is associative (see [10, Remark 10.(b)]).

Example 5.4. Let l be a natural number such that $1 < l < n$ and $F : L_2^n \rightarrow L_2$ be defined as

$$(8) \quad F(x_1, \dots, x_n) = \begin{cases} 1 & \text{if } x_1 = \dots = x_l = 1, \\ 2 & \text{otherwise.} \end{cases}$$

It is clear that F is quasitrivial and also easy to show that it is nondecreasing.

Let X_1 and X_2 denote the preimage of 1 and 2, respectively. Formally,

$$\begin{aligned} X_1 &= \{(x_1, \dots, x_n) \in L_2^n : F(x_1, \dots, x_n) = 1\}, \\ X_2 &= \{(x_1, \dots, x_n) \in L_2^n : F(x_1, \dots, x_n) = 2\}. \end{aligned}$$

We show that the sets X_1 and X_2 are invariant under the bisymmetric equation which implies that F is bisymmetric. We introduce the notation $'\star'$ which means that its value is either 1 or 2 (we do not want to specify) and we set $m \cdot \star = \star, \dots, \star$ (m times). Then

$$1 = F(l \cdot 1, (n-l) \cdot \star) = F(l \cdot F(l \cdot 1, (n-l) \cdot \star), (n-l)F(n \cdot \star)).$$

The main observation is that the last expression does not change by the bisymmetric equation. Since the value of that expression is 1 for any substitution of $'\star'$, this immediately implies that X_1 and hence $L_n^k \setminus X_1 = X_2$ are invariant for the bisymmetric equation. Thus F is bisymmetric.

Now we show that F is not associative. We proceed by contradiction. Suppose that F is associative. Then

$$1 = F(l \cdot 1, (n-l) \cdot 2) = F(l \cdot 1, F(n \cdot 2), (n-l-1) \cdot 2)$$

using the associativity of F we get

$$= F(F(l \cdot 1, (n-l) \cdot 2), (n-1) \cdot 2) = F(1, (n-1) \cdot 2) = 2.$$

This contradiction shows that F is not associative.

Remark 9. (a) Since F is not associative, it cannot be derived from a binary function G as in Theorem 3.1.

- (b) It is an open question how can we characterize bisymmetric quasitrivial n -ary operations for $n \geq 3$ either with or without the assumption of nondecreasingness.

5.2.2. *Positive results.* For some special classes of n -ary bisymmetric functions ($n \geq 3$) we can find a characterization. These results based on the following lemma of Jimmy Devillet [9].

Lemma 5.5. *Let X be a nonempty set and let $F : X^n \rightarrow X$ be a bisymmetric function which is derived from a binary function G which satisfies*

$$(9) \quad G(a, b) = F(a, (n-1) \cdot b) = F((n-1) \cdot a, b)$$

for every $a, b \in X$. Then $G : X^2 \rightarrow X$ is also bisymmetric.

Proof. Let $x, y, u, v \in X$ and let $\mathbf{a} = ((n-1) \cdot x, y)$, $\mathbf{b} = ((n-1) \cdot u, v)$, $\mathbf{c} = ((n-1) \cdot x, u)$, $\mathbf{d} = ((n-1) \cdot y, v) \in X^n$. Using (9), we have

$$G(G(x, y), G(u, v)) = F((n-1) \cdot F(\mathbf{a}), F(\mathbf{b}))$$

and, by the bisymmetry of F , we get

$$= F((n-1) \cdot F(\mathbf{c}), F(\mathbf{d})) = G(G(x, u), G(y, v)).$$

□

Remark 10. The statement automatically implies that F and G is associative, since F is derived from G and by definition, this means that F is (n) -associative and G is associative.

As an easy consequence of Theorem 3.1 and Lemma 5.5 we get the following.

Proposition 5.6. *Let X be a nonempty chain and $F : X^n \rightarrow X$ be bisymmetric associative idempotent nondecreasing. Then F is derived from a bisymmetric associative idempotent binary nondecreasing function G .*

Proof. By Theorem 3.1 we get that F is derived from a binary function G . Moreover, G is associative idempotent nondecreasing and can be defined by (9). According to Lemma 5.5, G is bisymmetric. □

As a consequence of [10, Corollary 4.9 and Corollary 4.12] we get the following.

Proposition 5.7. *Let X be a nonempty set and $F : X^n \rightarrow X$ be a bisymmetric quasitrivial function which satisfies any of the following condition*

- (a) F is symmetric,
- (b) F has a neutral element,

Then F is associative.

Combining the previous results we deduce a characterization for some special classes of functions.

Corollary 5.8. *Let $F : L_k^n \rightarrow L_k$ be a bisymmetric quasitrivial nondecreasing function which satisfies any of the following conditions*

- (a) F is symmetric,
- (b) F has a neutral element,
- (c) F is associative,

then F is derived from a bisymmetric (associative) quasitrivial nondecreasing binary operation G which satisfies (9) and can be characterized by Theorem 5.2.

6. THE NUMBER OF FUNCTIONS OF GIVEN CLASS

This section is devoted to calculate the number of associative quasitrivial nondecreasing functions. Byproduct of the following argument we also consider the number of associative quasitrivial nondecreasing functions having neutral elements. With the same technique one can easily deduce the number of bisymmetric quasitrivial nondecreasing binary functions (see Proposition 6.5).

Theorem 6.1. *Let A_k denote the number of associative quasitrivial nondecreasing functions defined on L_k and B_k denote the number of associative quasitrivial nondecreasing functions defined on L_k and having neutral elements. Then*

$$A_k = \frac{1}{6}((2 + \sqrt{3})(1 + \sqrt{3})^k + (2 - \sqrt{3})(1 - \sqrt{3})^k - 4),$$

$$B_k = \frac{1}{2 \cdot \sqrt{3}}((1 + \sqrt{3})^k - (1 - \sqrt{3})^k).$$

The following observations show that these numbers are related to the downward-right path $\mathcal{P} = (p_j)_{j=1}^l$ (for some $l \leq k$) in T_1 starting from $(1, k)$. Let $m_{\mathcal{P}}$ be the number of diagonal line segments $\overline{p_i, p_{i+1}} \in \mathcal{P}$ ($i \in \{1, \dots, l-1\}$). We say that the downward-right path \mathcal{P} is *weighted* with weight $2^{m_{\mathcal{P}}}$.

Lemma 6.2. (a) B_k is the sum of the weights of weighted paths that starts at $(1, k)$ and reaches Δ_{L_k} .
 (b) $A_k + B_k$ is twice the sum of the weights of weighted paths in T_1 that starts at $(1, k)$ and ends at any point of T_1 .

Proof. (a) Applying Theorem 4.12, it is clear that if an associative quasitrivial nondecreasing binary function F has a neutral element, then the downward-right path \mathcal{P} defined for F reaches the diagonal Δ_{L_k} . By Theorem 4.13, there can be defined $2^{m_{\mathcal{P}}}$ different functions for a given path \mathcal{P} that reaches the diagonal, since we have a choice in each case when the path contains a diagonal line segment. This show the first part of the statement.

(b) This statement follows from the fact that for any associative quasitrivial nondecreasing function F one can define a downward-right path which starts at $(1, k)$ and ends somewhere in T_1 . If its end in (a, b) where $a < b$ (not on Δ_{L_k}), then F is one of the projections in $[a, b]^2$, and a and b are the half-neutral elements of F . This makes the extra 2 factor in the statement.

Let Π_1 denote set of the weighted paths in T_1 that starts at $(1, k)$ and ends at (a, b) where $a < b$. Similary, Π_2 denote the set of weighted paths that starts at $(1, k)$ and reaches Δ_{L_k} . Hence,

$$A_k = 2 \cdot \sum_{\mathcal{P} \in \Pi_1} 2^{m_{\mathcal{P}}} + \sum_{\mathcal{P} \in \Pi_2} 2^{m_{\mathcal{P}}}$$

According to the (a) part

$$B_k = \sum_{\mathcal{P} \in \Pi_2} 2^{m_{\mathcal{P}}}.$$

Adding these equations, we get the statement for $A_k + B_k$. □

Now we present a recursive formula for A_k and B_k .

Lemma 6.3. (a) $B_1 = 1$, $B_2 = 2$ and $B_k = 2 \cdot B_{k-1} + 2 \cdot B_{k-2}$ for every $k \geq 3$.

(b) $A_k = 2 \sum_{i=1}^k B_i - B_k$ for every $k \in \mathbb{N}$.

Proof. (a) $B_1 = 1$, $B_2 = 2$ are clear. The recursive formula follows from the Algorithm presented in Section 4 and the definition of downward-right path $\mathcal{P} = (p_j)_{j=1}^l$. Now we assume that $k \geq 3$. If $\overline{p_1, p_2}$ is horizontal or vertical, then Case I. (a) or (b) of the Algorithm holds (see also Figure 11). Thus we reduce the square Q_1 of size k to a square Q_2 of size $k-1$. If $\overline{p_1, p_2}$ is diagonal, then Case III (a) or (b) holds (see also Figure 13). Thus we reduce the square Q_1 of size k to a square Q_2 of size $k-2$. By definition, the number of associative quasitrivial nondecreasing functions having neutral elements defined on a square of size k is B_k . Thus we get that $B_k = 2 \cdot B_{k-1} + 2 \cdot B_{k-2}$.
 (b) This follows from Lemma 6.2 (b) and the fact that 'sum of the weights of weighted paths from $(1, k)$ to any point of T_1 ' is exactly $\sum_{i=1}^k B_i$. Indeed, let $s \in \{1, \dots, k\}$ be fixed. Then B_s is equal to the sum of the weights of weighted paths \mathcal{P} that starts at $(1, k)$ and ends at (a, b) where $b - a = s$. \square

Proof of Theorem 6.1. We use a standard method of second-order linear recurrence equations for the formula of Lemma 6.3 (a). Therefore,

$$B_k = c_1 \cdot (\alpha_1)^k + c_2 (\alpha_2)^k,$$

where α_1, α_2 ($\alpha_1 < \alpha_2$) are the solutions of the equation $x^2 - 2x - 2 = 0$. Thus, $\alpha_1 = 1 + \sqrt{3}, \alpha_2 = 1 - \sqrt{3}$. By the initial condition $B_1 = 1$ and $B_2 = 2$, we get that $c_1 = -c_2 = \frac{1}{2\sqrt{3}}$. Thus,

$$B_k = \frac{1}{2\sqrt{3}} \left((1 + \sqrt{3})^k - (1 - \sqrt{3})^k \right).$$

According to Lemma 6.3 (b), A_k can be calculated as $2 \cdot \sum_{i=1}^k B_i - B_k$. This gives that

$$\begin{aligned} A_k &= \frac{1}{3} \left((1 + \sqrt{3})^{k+1} + (1 - \sqrt{3})^{k+1} \right) - \frac{1}{2\sqrt{3}} \left((1 + \sqrt{3})^k - (1 - \sqrt{3})^k \right) \\ &= \frac{1}{6} \left(2 \cdot (1 + \sqrt{3})^{k+1} + 2 \cdot (1 - \sqrt{3})^{k+1} - \sqrt{3} \cdot (1 + \sqrt{3})^k + \sqrt{3} \cdot (1 - \sqrt{3})^k \right) \\ &= \frac{1}{6} \left((2 + \sqrt{3})(1 + \sqrt{3})^k + (2 - \sqrt{3})(1 - \sqrt{3})^k - 4 \right) \end{aligned}$$

\square

Here we present a list of the first 10 value of A_k . $A_1 = 1, A_2 = 4, A_3 = 12, A_4 = 34, A_5 = 94, A_6 = 258, A_7 = 706, A_8 = 1930, A_9 = 5274, A_{10} = 14410$.

By Theorem 3.1, we get the similar results for the n -ary case.

Corollary 6.4. (a) *The number of associative quasitrivial nondecreasing functions $F : L_k^n \rightarrow L_k$ ($k \in \mathbb{N}$) having neutral elements is*

$$\frac{1}{2\sqrt{3}} \left((1 + \sqrt{3})^k - (1 - \sqrt{3})^k \right),$$

(b) *The number of associative quasitrivial nondecreasing functions $F : L_k^n \rightarrow L_k$ ($k \in \mathbb{N}$) is*

$$\frac{1}{6} \left((2 + \sqrt{3})(1 + \sqrt{3})^k + (2 - \sqrt{3})(1 - \sqrt{3})^k - 4 \right).$$

Proposition 6.5. *Let C_k denote number of bisymmetric quasitrivial nondecreasing binary functions defined in L_k and D_k denote the number of bisymmetric quasitrivial nondecreasing binary functions having neutral elements. Then*

$$\begin{aligned} D_k &= 2^{k-1}, \\ C_k &= 3 \cdot 2^{k-1} - 2. \end{aligned}$$

Proof. (a) By Lemma 5.1 (a) every bisymmetric binary function having a neutral element is symmetric and associative. By Lemma 5.1 (b), a symmetric and associative function is bisymmetric. According to Proposition 3.8, a quasitrivial symmetric nondecreasing function defined on a finite chain has a neutral element. Thus, bisymmetric quasitrivial nondecreasing binary functions having neutral elements are exactly the associative quasitrivial symmetric nondecreasing binary functions. Hence, the statement follows from Corollary 4.15.

(b) Same argument as in Lemma 6.3(b) shows that $C_k = 2 \sum_{i=1}^k D_i - D_k$. Using this we get that $C_k = 2 \cdot (2^k - 1) - 2^{k-1} = 3 \cdot 2^{k-1} - 2$. \square

Remark 11. During the finalization of this paper the author have been informed by [9] that Jimmy Devillet, Jean-Luc Marichal and Miguel Couceiro found an alternative and independent approach for similar estimations. This will be presented in their upcoming paper.

APPENDIX

This section is devoted to prove the analogue of Corollary 3.3. As it was already mentioned in Remark 2, the proof is just a slight modification of the proof of [18, Theorem 3.2]. The difference is based on the following easy lemma.

Lemma 6.6. *Let X be a chain and $F : X^n \rightarrow X$ be an associative monotone function. Then F is non-decreasing in the first and the last variable.*

Proof. The argument for the first and for the last variable is similar. We just consider it for the first variable. From the definition of associativity it is clear that an associative function $F : X^n \rightarrow X$ satisfies

$$(10) \quad \begin{aligned} F(F(x_1, \dots, x_n), x_{n+1}, \dots, x_{2n-1}) &= \\ F(x_1, F(x_2, \dots, x_{n+1}), x_{n+2}, \dots, x_{2n-1}). \end{aligned}$$

for every $x_1, \dots, x_{2n-1} \in X$. Now let us fix $x_2, \dots, x_{2n-1} \in X$ and define

$$h(x) = F(F(x, x_2, \dots, x_n), x_{n+1}, \dots, x_{2n-1}).$$

The function F is monotonic in the first variable thus it is clear that $h(x)$ is nondecreasing, since we apply F twice when x is in the first variable. Then using (10) we get that F must be nondecreasing in the first variable. \square

As it was also mentioned in [18] the following condition is an easy application of [1, Theorem 1.4] using the statement therein for $A_2 = \emptyset$.

Theorem 6.7. *Let X be an arbitrary set. Suppose $F : X^n \rightarrow X$ be a quasitrivial, (n -)associative function. Then F is not derived from a binary function G if n is odd and there exist b_1, b_2 ($b_1 \neq b_2$) such that for any $a_1, \dots, a_n \in \{b_1, b_2\}$*

$$(11) \quad F(a_1, \dots, a_n) = b_i \quad (i = \{1, 2\}),$$

where b_i occurs odd number of times.

Proposition 6.8. *Let X be a totally ordered set and let $F: X^n \rightarrow X$ be an associative, quasitrivial, monotone function. Then F is reducible.*

Proof. According to Theorem 6.7, if F is not reducible, then n is odd. Hence $n \geq 3$ and there exist b_1, b_2 satisfying equation (11). Since $b_1 \neq b_2$, we may assume that $b_1 < b_2$ (the case $b_2 < b_1$ can be handled similarly). By the assumption (11) for b_1 and b_2 we have

$$(12) \quad F(n \cdot b_1) = b_1, \quad F(b_2, (n-1) \cdot b_1) = b_2, \quad F(b_2, (n-2) \cdot b_1, b_2) = b_1.$$

By Lemma 6.6, F is nondecreasing in the first and the last variable. Thus we have

$$F(n \cdot b_1) \leq F(b_2, (n-1) \cdot b_1) \leq F(b_2, (n-2) \cdot b_1, b_2).$$

This implies $b_1 = b_2$, a contradiction. \square

The following was proved as [18, Corollary 4.9].

Corollary 6.9. *Let X be a nonempty chain and $n \geq 2$ be an integer. An associative, idempotent, monotone function $F: X^n \rightarrow X$ is reducible if and only if F is nondecreasing.*

Using Proposition 6.8 and Corollary 6.9 we get the statement.

Corollary 6.10. *Let $n \geq 2 \in \mathbb{N}$ be given, X be a nonempty chain and $F: X^n \rightarrow X$ be an associative quasitrivial function.*

$$F \text{ is monotone} \iff F \text{ is nondecreasing.}$$

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