TILING AND WEAK TILING IN $(\mathbb{Z}_p)^d$

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Abstract. We discuss the relation of tiling, weak tiling and spectral sets in finite abelian groups. In particular, in elementary p-groups $(\mathbb{Z}_p)^d$, we introduce an averaging procedure that leads to a natural object of study: a 4-tuple of functions which can be regarded as a common generalization of tiles and spectral sets. We characterize such 4-tuples for $d = 1, 2$, and prove some partial results for $d = 3$.

1. Introduction

The concept of weak tiling was introduced recently in [15], in connection with Fuglede's conjecture for convex bodies in $\mathbb{R}^d$. In this note we will study the relation of tiling, weak tiling and spectral sets in finite abelian groups, with particular attention to elementary p-groups $(\mathbb{Z}_p)^d$. Our motivation is to give a new tool to prove the "spectral $\to$ tile" direction of Fuglede's conjecture in finite abelian groups.

We begin by recalling the necessary notions and fixing our notation.

The cardinality of any finite set A will be denoted by $|A|$. We use the standard notation $A \pm B = \{a \pm b : a \in A,\ b \in B\}$, and $kA = \{ka : \ a \in A\}$, for any positive integer k. For any n, let $\mathbb{Z}_n$ denote the cyclic group of order n.

Let G be a finite abelian group. A *character* is a homomorphism from G to the complex unit circle $\mathbb{T}$. The dual group, denoted by $\widehat{G}$, is the collection of all characters of G. For a function $f : G \to \mathbb{C}$, the Fourier transform $\widehat{f} : \widehat{G} \to \mathbb{C}$ is defined as

$$\widehat{f}(\gamma) = \sum_{x \in G} f(x)\gamma(x).$$

A set $A \subset G$ is called *spectral* if the function space $L^2(A)$ admits an orthogonal basis consisting of characters restricted to A. Such an orthogonal basis of characters is called a *spectrum* of A. (The spectrum, if it exists, is not necessarily unique.)

We say that a set $A \subset G$ *tiles* G if there exists another set $B \subset G$ such that each $g \in G$ can be written uniquely as $g = a + b$, where $a \in A, b \in B$. We usually express

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this relation as $A \oplus B = G$ or, in the functional notation, $1_A * 1_B = 1_G$, where $*$ denotes convolution. The convolution of any two functions $f, g : G \to \mathbb{C}$ is defined in the standard way as $(f * g)(x) = \sum_{y \in G} f(x - y)g(y)$.

For any function $f : G \to \mathbb{C}$ the function f_- is defined as $f_-(x) = f(-x)$. Some basic properties of the Fourier transform read as follows: $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$, $\widehat{f_-}(\gamma) = \overline{\widehat{f}(\gamma)}$, $\widehat{f * f_-} = |\widehat{f}|^2$, and for real-valued even functions $\widehat{\widehat{f}} = |G|f$.

Fuglede's conjecture [7] stated that a set $A \subset G$ is spectral if and only if it tiles G. The conjecture was formulated explicitly in $\mathbb{R}^d$, but Fuglede already mentioned that the notions make sense in any locally compact abelian group G. In fact, the first counterexample by Tao [20] in $\mathbb{R}^5$ was based on a counterexample in the finite group $(\mathbb{Z}_3)^5$. Since then, further counterexamples [5, 12] have been constructed (all based on examples in finite groups) to both directions of the conjecture in $\mathbb{R}^d$, $d \geq 3$. The conjecture (both directions of it) remains open for $\mathbb{R}$ and $\mathbb{R}^2$. Some further connections between the discrete and the continuous settings have been revealed by Dutkay and Lai in [2].

In this paper we will restrict our attention to finite abelian groups, and mostly to the case $G = (\mathbb{Z}_p)^d$ with p being a prime. For finite abelian groups, several results have been discovered in recent years [3, 4, 6, 8, 9, 10, 11, 13, 14, 16, 17, 18, 19, 21]. For the purposes of this note, we single out the results of [8] and [6, 17]. In [8] the authors prove both directions of Fuglede's conjecture in $(\mathbb{Z}_p)^2$. To the contrary, in [6] and [17] the authors prove (independently of each other) that for odd primes p there exist spectral sets in $(\mathbb{Z}_p)^d$, $d \geq 4$, which do not tile the group, thus exhibiting a counterexample to the "spectral $\to$ tile" direction of Fuglede's conjecture in these groups.

In connection with Fuglede's conjecture, the notion of *weak tiling* was introduced in [15]. For us, in the case of finite abelian groups, weak tiling can be formulated as follows. A set $A \subset G$ tiles another set $E \subset G$ *weakly*, if there exists a nonnegative function $h : G \to \mathbb{R}$ such that $1_A * h = 1_E$. We will soon see that it makes sense to introduce the further restrictions that $h(0) = 1$, the function h is positive definite (i.e. $\widehat{h} \geq 0$), and restrict our attention to the case $E = G$.

Definition 1.1. We say that a set $A \subset G$ *pd-tiles* G *weakly* if there exists a nonnegative function $h : G \to \mathbb{R}$ such that $h(0) = 1$, $1_A * h = 1_G$, and $\widehat{h} \geq 0$.

In the terminology, "pd" stands for positive definite. Note that, due to positive definiteness, h is necessarily an even function, $h(x) = h(-x)$. The assumption $h(0) = 1$ is essential, otherwise the constant function $h = \frac{|G|}{|A|}$ would provide a weak pd-tiling for any set $A \subset G$. Note also, as a comparison with the terminology in [15], that if A pd-tiles G weakly then A tiles its complement weakly.

A simple but important observation, also essentially contained in [15], is that if A pd-tiles G weakly with a function h, then

$$A - A \cap \operatorname{supp} h = \{0\}. \tag{1}$$

The reason is that $x = a - a'$ and $h(x) > 0$ would imply $1_A * h(a) \geq h(0)1_A(a) + h(x)1_A(a') = 1 + h(x) > 1$.

Next, we show that a proper tiling always induces a weak pd-tiling of G.

Lemma 1.2. *If $A \oplus B = G$ is a tiling, then A pd-tiles G weakly with $h_1 = \frac{1}{|B|} 1_B * 1_{-B}$.*

Proof. $A \oplus B = G$ implies $\widehat{1}_A \cdot \widehat{1}_B = |G|\delta_0$, so that the supports of the nonnegative functions $\widehat{1}_A$ and $\widehat{1}_B$ are essentially disjoint (only intersect at 0). This implies that $\widehat{1}_A \cdot \frac{1}{|B|} |\widehat{1}_B|^2 = |G|\delta_0$, which, in turn, implies $1_A * \left(\frac{1}{|B|} 1_B * 1_{-B} \right) = 1_G$. □

We remark that the notation h_1 above (instead of using simply h) is for later convenience.

The essential observation in [15] is that if a set A is spectral, then it tiles its complement weakly. However, slightly more is true.

Lemma 1.3. *Let G be a finite abelian group. If $A \subset G$ is spectral, then A pd-tiles G weakly.*

Proof. Let $S \subset \widehat{G}$ be a spectrum of A. Then $|S| = |A|$ because the space $L^2(A)$ has dimension $|A|$, and S is a basis. Let $h_1 = \frac{1}{|A|^2} |\widehat{1}_S|^2$. Then $h_1 \geq 0$, $h_1(0) = 1$, $\widehat{h}_1 = \frac{|G|}{|A|^2} 1_S * 1_{-S} \geq 0$, so h_1 is nonnegative, normalized and positive definite, as required. Also, the weak tiling condition $1_A * h_1 = 1_G$ is most easily seen by taking Fourier transforms. $\widehat{1}_A \cdot \left(\frac{|G|}{|A|^2} 1_S * 1_{-S} \right) = |G|\delta(0)$ is true because $\widehat{1}_A(0) = |A|$, $1_S * 1_{-S}(0) = |S| = |A|$, and the support of $1_S * 1_{-S}$ is $S - S$, which is a subset of $\{\widehat{1}_A = 0\} \cup \{0\}$ by the orthogonality of S in $L^2(A)$. □

By this lemma, we can establish the "spectral → tile" direction of Fuglede's conjecture in a finite abelian group G, if we prove that any set that pd-tiles G weakly, actually tiles G properly. This motivates the following definition.

Definition 1.4. Assume that a finite abelian group G has the property that whenever a set A pd-tiles G weakly then A tiles G properly. Then we call the group G *pd-flat.*

In the remainder of this note we will focus our attention on elementary p-groups $G = (\mathbb{Z}_p)^d$. As mentioned above, for odd primes and $d \geq 4$, there exist spectral sets in G which do not tile G. Therefore, $(\mathbb{Z}_p)^d$ is not pd-flat for $d \geq 4$.

In Section 2 we introduce an averaging technique which leads to a natural generalization of spectral sets and tiles in $(\mathbb{Z}_p)^d$. Finally, in Section 3 we show that $(\mathbb{Z}_p)^d$ is pd-flat for $d = 1, 2$, and we give some partial results and conjectures for $d = 3$.

2. Averaging

It is natural to go one step further with the generalization of tiling.

Definition 2.1. Let $f, h : G \to \mathbb{R}$ be nonnegative functions such that $f(0) = h(0) = 1$, $\widehat{f} \geq 0, \widehat{h} \geq 0$. We say that the pair (f, h) is a *functional pd-tiling* of G if $f * h = 1_G$.

It turns out that this notion is very flexible, and for $G = (\mathbb{Z}_p)^d$ it gives rise to a natural averaging procedure. In connection with this, we introduce a special class of functions.

Definition 2.2. We say that a function $f : (\mathbb{Z}_p)^d \to \mathbb{C}$ is *ray-type*, if for any $\mathbf{x} \in (\mathbb{Z}_p)^d$, and any $k = 1, \dots, p-1$ we have $f(k\mathbf{x}) = f(\mathbf{x})$.

That is, a ray-type function is constant on any punctured line through the origin (but may have a different value at the origin).

We also need some information on the zeroes of the Fourier transform of the indicator function 1_A of a set $A \subset (\mathbb{Z}_p)^d$. We make the identification $G = \{0, 1, \dots, p-1\}^d$, and the same for the dual group $\widehat{G} = \{0, 1, \dots, p-1\}^d$. It is sometimes useful to think of the elements of G as *column* vectors of length d, and elements of $\widehat{G}$ as *row* vectors of length d. Also, in notation, we will use boldface letters for elements of G and $\widehat{G}$ to indicate that they are vectors. With these identifications in mind, the action of a character $\mathbf{t} \in \widehat{G}$ on an element $\mathbf{x} \in G$ is given by $e^{2i\pi\langle \mathbf{t}, \mathbf{x}\rangle/p}$.

For a function $f : G \to \mathbb{C}$, and $\mathbf{t} \in \widehat{G}$, we have $\widehat{f}(\mathbf{t}) = \sum_{\mathbf{x} \in G} f(\mathbf{x}) e^{2i\pi\langle \mathbf{t}, \mathbf{x}\rangle/p}$. In particular, for $A \subset G$, the Fourier transform of 1_A takes the form $\widehat{1}_A(\mathbf{t}) = \sum_{\mathbf{a} \in A} e^{2i\pi\langle \mathbf{t}, \mathbf{a}\rangle/p}$.

The important point here is that for any $\mathbf{t} \neq 0$ we either have $\widehat{1}_A(k\mathbf{t}) = 0$ for all $k = 1, 2, \dots, p-1$, or $\widehat{1}_A(k\mathbf{t}) \neq 0$ for all $k = 1, 2, \dots, p-1$. This is well-known, and the reason is that $\widehat{1}_A(\mathbf{t}) = 0$ if and only if the sum $\sum_{\mathbf{a} \in A} e^{2i\pi\langle \mathbf{t}, \mathbf{a}\rangle/p}$ contains the same number of terms for each pth root of unity, in which case $\sum_{\mathbf{a} \in A} e^{2i\pi\langle k\mathbf{t}, \mathbf{a}\rangle/p}$ also contains the same number of terms of each pth root of unity.

Therefore, the zeros of the Fourier transform $\widehat{1}_A$ consist of punctured lines through the origin (for $\mathbf{t} \neq 0$ a punctured line $\dot{L}$ is given by $\dot{L} = \{\mathbf{t}, 2\mathbf{t}, \dots, (p-1)\mathbf{t}\}$; note that $0 \notin \dot{L}$, and we use the notation $L = \dot{L} \cup \{0\}$). The same is true for the zeroes of the Fourier transform of $f = \frac{1}{|A|} 1_A * 1_{-A}$, because $\widehat{f} = \frac{1}{|A|} |\widehat{1}_A|^2$.

We can now perform the first half of the averaging.

Lemma 2.3. *Let $G = (\mathbb{Z}_p)^d$, and $1_A * h_1 = 1_G$ be a weak pd-tiling of G by a set A. Then, for any $k = 1, \dots, p-1$, $1_{kA} * h_1 = 1_G$ is also a weak pd-tiling. Furthermore, with the notation $f_k = \frac{1}{|A|} 1_{kA} * 1_{-kA}$, we have that $f_k * h_1 = 1_G$ is a functional pd-tiling. Finally, for $f = \frac{1}{p-1} \sum_{k=1}^{p-1} f_k$ we have that f is a ray-type function,* $\operatorname{supp} f \cap \operatorname{supp} h_1 = \{0\}$, *and $f * h_1$ is also a functional pd-tiling.*

Proof. Note that $\widehat{1}_{kA}(\mathbf{t}) = \widehat{1}_A(k\mathbf{t})$, so the zeroes of these functions are the same punctured lines through the origin.

The assumption $1_A * h_1 = 1_G$ is equivalent to $\widehat{1}_A \widehat{h}_1 = |G|\delta_0$. Due to the fact that the zeroes of $\widehat{1}_A$ and $\widehat{1}_{kA}$ coincide, and $|A| = |kA|$, we have $\widehat{1}_{kA}\widehat{h}_1 = |G|\delta_0$, and therefore $1_{kA} * h_1 = 1_G$ is a weak pd-tiling.

The function $f_k = \frac{1}{|A|} 1_{kA} * 1_{-kA}$ is nonnegative, $f_k(0) = 1$, $\widehat{f_k} = \frac{1}{|A|}|\widehat{1}_{kA}|^2 \geq 0$, and the zeroes of $\widehat{f_k}$ and $\widehat{1}_{kA}$ coincide. Also, $\widehat{f_k}(0) = |A| = \widehat{1}_{kA}(0)$, therefore $\widehat{f_k}\widehat{h}_1 = |G|\delta_0$, which implies that $f_k * h_1$ is a functional pd-tiling. Also, supp $f_k = kA - kA$, and hence supp $f_k \ \cap$ supp $h_1 = \{0\}$ by (1).

Finally, for the average $f = \frac{1}{p-1}\sum_{k=1}^{p-1} f_k$, we clearly have $f \geq, 0$, $\widehat{f} \geq 0$, $f(0) = 1$ and $f * h_1 = |G|\delta_0$ is a functional pd-tiling since these properties hold for each f_i. Also, by averaging, it is clear that f is ray-type, and the property supp $f \ \cap$ supp $h_1 = \{0\}$ is inherited. □

We can perform a second step of averaging for the function h_1.

Lemma 2.4. *Let $G = (\mathbb{Z}_p)^d$, and $1_A * h_1 = 1_G$ be a weak pd-tiling of G by a set A, and define the ray-type function f as in Lemma 2.3. For any $k = 1, \dots, p-1$, let $h_k(\mathbf{x}) = h_1(k\mathbf{x})$, and let $h = \frac{1}{p-1}\sum_{k=1}^{p-1} h_k$. Then, for each k, $f * h_k = 1_G$ is a functional pd-tiling, the average h is a ray-type function, and $f * h$ is also a functional pd-tiling such that* supp $f \ \cap$ supp $h = \{0\}$.

Proof. Recall from Lemma 2.3 that f is a ray-type function. We claim that $\widehat{f}$ is also ray-type. To see this, note that f can be written in the form $f = c\delta_0 + \sum_i c_i 1_{L_i}$ for some lines L_i (for convenience, we use proper lines in this decomposition, not punctured lines; the difference is absorbed in the constant c at the origin). The Fourier transform of 1_{L_i} is $p1_{H_i}$ with the hyperplane H_i being orthogonal to L_i, and therefore the $\widehat{f} = c1_{\widehat{G}} + p\sum_i 1_{H_i}$, which is clearly ray-type.

By Lemma 2.3 we have that $f * h_1 = 1_G$, which implies $\widehat{f}\widehat{h}_1 = |G|\delta_0$. This means that supp $\widehat{f} \cap$ supp $\widehat{h}_1 = \{0\}$. As $\widehat{f}$ is ray-type, its support is a union of lines $\dot{R}_i$, and hence $\widehat{h}_1$ must be 0 on the punctured lines $\dot{R}_i$. But $\widehat{h}_k(\mathbf{t}) = \widehat{h}_1(\mathbf{t}/k)$, so $\widehat{h_k}$ is also 0 on $\dot{R}_i$, and hence $\widehat{f}\widehat{h}_k = |G|\delta_0$ holds (for the value at zero we have $\widehat{h}_1(0) = \widehat{h}_k(0) = |G|/|A|$). In turn, this implies that $f * h_k = 1_G$ is indeed a functional pd-tiling.

Also, supp $f \ \cap$ supp $h_1 = \{0\}$ by Lemma 2.3, and f is ray-type, so the support of h_1 is contained in rays where $f = 0$. This clearly implies that the support of h_k is also contained in these rays, and therefore supp $f \ \cap$ supp $h_k = \{0\}$

After averaging, it is clear that h is ray-type, and the functional pd-tiling property $f * h$ is preserved, as well as the condition supp $f \ \cap$ supp $h = \{0\}$. □

We see from Lemma 1.2 and 1.3 that A pd-tiles G weakly in both cases when A tiles G, or A is spectral in G. After applying Lemma 2.3 and 2.4 we arrive at the following corollary.

Corollary 2.5. *Let $G = (\mathbb{Z}_p)^d$, and assume A pd-tiles G weakly with $1_A * h_1 = 1_G$ (in particular, this is the case if A tiles G, or A is spectral in G). For $k = 1, \dots, p-1$, let $f_k = \frac{1}{|A|} 1_{kA} * 1_{-kA}$, $h_k(\mathbf{x}) = h_1(k\mathbf{x})$, and $f = \frac{1}{p-1}\sum_{k=1}^{p-1} f_k$, $h = \frac{1}{p-1}\sum_{k=1}^{p-1} h_k$. Then*

$$|A| = \sum_{x \in G} f(x), \tag{2}$$

and the 4-tuple of functions $(f, h, \widehat{f}, \widehat{h})$ satisfy the following properties:

(i) $f, h, \widehat{f}, \widehat{h}$ *are all ray-type functions,*
(ii) $f \geq 0$, $h \geq 0$,
(iii) $f(0) = 1$, $h(0) = 1$,
(iv) $f * h = 1_G$,
(v) $\mathrm{supp}\, f \cap \mathrm{supp}\, h = \{0\}$,
(vi) $\widehat{f} \geq 0$, $\widehat{h} \geq 0$,
(vii) $\widehat{f}(0) = |A|$, $\widehat{h}(0) = |G|/|A|$,
(viii) $\widehat{f} * \widehat{h} = |G| 1_{\widehat{G}}$,
(ix) $\mathrm{supp}\, \widehat{f} \cap \mathrm{supp}\, \widehat{h} = \{0\}$.

Proof. Only (viii) needs further explanation, and it follows by taking Fourier transform of the equation, noting that $\widehat{\widehat{f}} = |G| f$ (and the same for h), and applying (v) and (iii). □

This corollary motivates the following definition.

Definition 2.6. A 4-tuple of functions $(f, h, \widehat{f}, \widehat{h})$ is called a *complementary 4-tuple* if they satisfy conditions (i)-(ix) of Corollary 2.5. (We do not necessarily require that f be constructed from a set A.)

The notion of complementary 4-tuples is very appealing, because it puts the functions f, h and their Fourier transform $\widehat{f}, \widehat{h}$ on equal footing. When studying spectral sets and tiles in $(\mathbb{Z}_p)^d$ it is natural to try to characterize all complementary 4-tuples. We will do this for $d = 1, 2$ in the next section, and present some partial results for $d = 3$.

We also remark that an analogous averaging procedure can be carried out in cyclic groups $\mathbb{Z}_n$, leading to a similar notion of complementary 4-tuples in those groups. Studying these 4-tuples could lead to some insights concerning the Coven-Meyerowitz conjecture. This is subject to future research.

3. Complementary 4-tuples in $(\mathbb{Z}_p)^d$

In this section we analyze complementary 4-tuples in $(\mathbb{Z}_p)^d$, and prove that $(\mathbb{Z}_p)^d$ is pd-flat for $d = 1, 2$, and also give some partial results for $d = 3$.

For convenience, we recall here what is known about Fuglede's conjecture in these groups. Both directions of the conjecture are true for $d = 1, 2$, with $d = 1$ being fairly trivial, and $d = 2$ being treated in [8]. A simpler proof of the case $d = 2$ can be found in [9]. For $d = 3$ it is known that all tiles are spectral [1], but it is not known whether all spectral sets are tiles. For $d \geq 4$ (and p being odd) there are examples of spectral sets which do not tile the group [17, 6] but the tile-spectral direction of the conjecture is open.

We now turn to the main results of this section. We emphasize here that a group G being pd-flat is *formally* stronger than the "spectral $\to$ tile" direction of Fuglede's conjecture in G.

Proposition 3.1. *$G = (\mathbb{Z}_p)^d$ is pd-flat for $d = 1, 2$, that is, every set A which pd-tiles G weakly, tiles G properly. Also, G is not pd-flat for $d \geq 4$.*

Proof. Assume A pd-tiles G weakly. Then there exists a complementary 4-tuple $(f, h, \widehat{f}, \widehat{h})$ with the properties listed in Corollary 2.5, and f is constructed from 1_A as in Corollary 2.5.

For $d = 1$ the support of any ray-type function is either $\{0\}$ or G. If supp $f = \{0\}$ then $|A| = 1$, and A tiles G trivially. If supp $f = G$ then necessarily supp $h = \{0\}$, and $h(0) = 1$ implies $h = \delta_0$, and hence f must be the constant function 1. Therefore, $|A| = p$ by (2), and hence $A = G$.

For $d = 2$ we must perform a case-by-case analysis of the function f. Let $c = \inf_{x \in G} f(x) \geq 0$, and let L_i be the lines in the support of f. Then f can be decomposed uniquely as $f = c1_G + \sum_i d_i 1_{L_i} + m\delta_0$ with some $d_i \geq 0$ and $m \in \mathbb{R}$ (note that L_i denotes the full line here, not the punctured line).

If $c > 0$ then supp $f = G$, and hence supp $h = \{0\}$, $h = \delta_0$, so $f = 1_G$. By (2) we obtain that $|A| = p^2$ and $A = G$.

If $c = 0$ and $m > 0$ then $\widehat{f} > 0$ everywhere on $\widehat{G}$, so supp $\widehat{h} = \{0\}$. Thus h is constant, and $h(0) = 1$ implies that h is constant 1. This implies that $f = \delta_0$, and $|A| = 1$.

If $c = 0$ and $m = 0$, then the total mass of f is exactly p times the mass at 0, because this is true for all 1_{L_i}. The mass at zero is $f(0) = 1$, and hence we have $|A| = p$, by equation (2). Also, $c = 0$ implies that f must be zero on some punctured line $\dot{R}$, so A can have at most one element in each coset of the line R since $A - A \subset$ supp (f). But $|A| = p$, so A must have exactly one element in each coset of R, and therefore $A \oplus R = G$ is a tiling.

Finally, we claim that $c = 0$ and $m < 0$ is not possible. Indeed, $c = 0$ means that f must be zero on some punctured line $\dot{R}$, and therefore $\widehat{f} = m < 0$ on the orthogonal punctured line $\dot{R}^{\perp}$, contradicting the nonnegativity of $\widehat{f}$.

For $d \geq 4$ we know that there exist spectral sets in $(\mathbb{Z}_p)^d$ which do not tile the group [17, 6]. Therefore, $(\mathbb{Z}_p)^d$ cannot be pd-flat. □

For $d = 3$, we conjecture that $(\mathbb{Z}_p)^3$ is pd-flat, but unfortunately we cannot give a complete characterization of complementary 4-tuples, we can only prove some partial results in this direction. Therefore, the "spectral $\to$ tile" direction of Fuglede's conjecture remains open in this case. We have the following partial result.

Proposition 3.2. *Let $G = (\mathbb{Z}_p)^3$, and assume $A \subset G$ pd-tiles G weakly with $1_A * h_1 = 1_G$. Let $(f, h, \widehat{f}, \widehat{h})$ be the complementary 4-tuple constructed in Corollary 2.5. Then either A tiles G properly, or the functions $f, h, \widehat{f}, \widehat{h}$ all have the following "dispersive" property: for any 2-dimensional subspaces $S \subset G$, $S' \subset \widehat{G}$ the intersections* $\mathrm{supp}\, f \cap S$, $\mathrm{supp}\, h \cap S$, $\mathrm{supp}\, \widehat{f} \cap S'$ *and* $\mathrm{supp}\, \widehat{h} \cap S'$ *are all non-trivial (i.e. not equal to $\{0\}$ or the whole plane).*

Proof. Exploiting the fact that all appearing functions are ray-type, we can represent each function in the following normalized form. Let us label the planes and lines through the origin as $S_1, S_2, \dots, S_{p^2+p+1}$ and $L_1, L_2, \dots, L_{p^2+p+1}$. Then f can be represented uniquely as

$$f = w 1_G + \sum_i (c_i 1_{S_i} + d_i 1_{L_i}) + m\delta_0, \tag{3}$$

where the coefficients $w, c_i, d_i \geq 0$, $m \in \mathbb{R}$ are defined by the following greedy algorithm: w is the maximal value such that $f - w1_G \geq 0$, c_1 is the maximal value such that $f - w1_G - c_1 1_{S_1} \geq 0$, c_2 is the maximal value such that $f - w1_G - c_1 1_{S_1} - c_2 1_{S_2} \geq 0$, etc, and subsequently d_1 is the maximal value such that $f - w1_G - (\sum_{i=1}^{p^2+p+1} c_i 1_{S_i}) - d_1 1_{L_1} \geq 0$, etc. We can represent $f, h, \widehat{f}, \widehat{h}$ in this manner uniquely, after fixing the order of planes and lines in G and $\widehat{G}$.

We will show that if any of $f, h, \widehat{f}, \widehat{h}$ does not have the dispersive property (stated in the proposition), then A tiles G. This is done by a case-by-case analysis, where we will use the properties of complementary 4-tuples stated in Corollary 2.5.

Case I. Let us first consider the trivial case when the support of some of the appearing functions is the whole underlying group.

(a) If $\mathrm{supp}\, f = G$, then $\mathrm{supp}\, h = \{0\}$ by (v), and hence $h = \delta_0$ by (iii), and $f = 1_G$ by (iv), and $|A| = p^3$ by (2). Therefore, $A = G$, and A tiles the group trivially.

(b) Similarly, if $\mathrm{supp}\, h = G$ then $\mathrm{supp}\, f = \{0\}$, and $|A| = 1$, and A tiles G trivially.

(c) If $\mathrm{supp}\, \widehat{f} = \widehat{G}$, then $\mathrm{supp}\, \widehat{h} = \{0\}$ by (ix), and hence h is a constant function, $h = 1_G$ by (iii), and therefore $f = \delta_0$ by (iv), and again $|A| = 1$.

(d) Similarly, if $\mathrm{supp}\, \widehat{h} = \widehat{G}$, then $\widehat{f} = \{0\}$, and hence f is constant 1, and $|A| = p^3$ by (2).

Having dealt with Case I, we can assume for the rest of the argument that $w = 0$ in equation (3). Moreover, this also implies that $m \leq 0$ in (3) because $m > 0$ would imply $\mathrm{supp}\, \widehat{f} = \widehat{G}$. Altogether, we can conclude that the representation of f takes the form

$$f = \sum_i (c_i 1_{S_i} + d_i 1_{L_i}) + m\delta_0, \tag{4}$$

where $c_i, d_i \geq 0$, and $m \leq 0$. The same is true for the representations of $h, \widehat{f}, \widehat{h}$.

Case II. Next, assume that the support of one of the appearing functions intersects a plane only at 0.

(a) Let supp $f \ \cap \ S = \{0\}$. Let $\dot{L} \subset \widehat{G}$ denote the punctured line orthogonal to S. Then $\widehat{f} = m \leq 0$ on $\dot{L}$, and the nonnegativity of $\widehat{f}$ implies $m = 0$. Also, the "empty" plane S intersects every other plane S_i, so every S_i has at least one "empty" line $S \cap S_i$, and hence the weight c_i must be 0 in equation (4) for every i. Hence, f can be represented in the form $f = \sum_i d_i 1_{L_i}$, and therefore the total mass of f is p times the mass at 0 (as this is the case for every L_i), and hence $|A| = p$ is implied by (2) and $f(0) = 1$. Furthermore, the fact that S is empty means that $A - A \ \cap \ S = \{0\}$, so there is exactly one point of A on each coset of S. Therefore, $A \oplus S = G$ is a tiling.

(b) If supp $h \ \cap \ S = \{0\}$, then the same argument as in (a) shows that h has a decomposition $h = \sum_i \tilde{d}_i L_i$, and therefore $\frac{\sum_{\mathbf{x} \in G} h(\mathbf{x})}{h(0)} = p$. Considering $h(0) = 1$, this implies $\sum_{\mathbf{x} \in G} h(x) = p$, and hence $|A| = p^2$ by (iv). Also, supp $f \neq G$, so there exists a line L such that $A - A \ \cap \ L = \{0\}$, and hence $A \oplus L = G$ is a tiling.

(c) If supp $\widehat{f} \ \cap \ S' = \{0\}$, then the same argument as in (a) shows that $\widehat{f}$ has a decomposition $\widehat{f} = \sum_i d_i' L_i'$, and therefore $\frac{\sum_{\mathbf{t} \in \widehat{G}} \widehat{f}(t)}{\widehat{f}(0)} = p$. Considering that $\widehat{f}(0) = \sum_{\mathbf{x} \in G} f(x)$ and $\sum_{\mathbf{t} \in \widehat{G}} \widehat{f}(t) = p^2 f(0)$, we obtain $\sum_{\mathbf{x} \in G} f(\mathbf{x}) = p^2$, that is $|A| = p^2$ by (2). We can finish the argument as in (b): as supp $f \neq G$, there exists a line L such that $A - A \ \cap \ L = \{0\}$, and hence $A \oplus L = G$ is a tiling.

(d) If supp $\widehat{h} \ \cap \ S' = \{0\}$, then the same argument as in (a) shows that $\widehat{h}$ has a decomposition $\widehat{h} = \sum_i \tilde{d}_i' L_i'$, and therefore $\frac{\sum_{\mathbf{t} \in \widehat{G}} \widehat{h}(t)}{\widehat{h}(0)} = p$. This implies that $\sum_{\mathbf{x} \in G} h(\mathbf{x}) = p^2$, and $\sum_{\mathbf{x} \in G} f(\mathbf{x}) = p$. This implies that in the decomposition (4) all coefficients $c_i = 0$ and $m = 0$, otherwise the total mass of f would be greater than p times the mass at 0 (recall that $c_i \geq$ and $m \leq 0$). Therefore, $f = \sum_i d_i 1_{L_i}$. Also, as supp $\widehat{f} \neq \widehat{G}$, we have a line $L' \subset \widehat{G}$ such that $\widehat{f} = 0$ on $\dot{L}'$, which implies that $f = 0$ on the punctured plane $S = L'^{\perp}$. Finally, as $|A| = p$ and $A - A \ \cap \ S = \{0\}$ we have that $A \oplus S = G$ is a tiling.

Case III. Finally, assume that the support of one of the appearing functions contains a whole plane. In this case, we can invoke (v) and (ix) to conclude that the support of some other function intersects the same plane at 0 only, and we are done by Case II above. □

It is worth summarizing here what this result means for spectral sets. For a spectral set $A \subset (\mathbb{Z}_p)^3$, perform the avearaging procedure leading to the complementary 4-tuple in Corollary 2.5. If any of the functions $f, h, \widehat{f}, \widehat{h}$ does not have the dispersive property described in Proposition 3.2 then A necessarily tiles $(\mathbb{Z}_p)^3$. If all appearing functions have the dispersive property, they must have a representation

$$\sum_i d_i 1_{L_i} + m\delta_0, \tag{5}$$

where the lines L_i are situated such that they do not cover any plane fully, and $d_i > 0$, $m < 0$ (the latter is true because $m \geq 0$ would imply the Fourier transform being positive on the planes $L_i^{\perp}$).

It would be tempting to conjecture that this situation cannot occur, that is, complementary 4-tuples $(f, h, \widehat{f}, \widehat{h})$ with all functions having the dispersive property do not exist. However, unfortunately, the following example shows that this is not the case.

Example 3.3. Let S_1, S_2, S_3 be three different planes through the origin in $(\mathbb{Z}_p)^3$, and let $L_1 = S_2 \cap S_3$, $L_2 = S_1 \cap S_3$, $L_3 = S_1 \cap S_2$ be the lines of intersection. For the sake of easier calculations we will use a representation for f and h different from (5). (It is not difficult to re-write the representations below according to equation (5), and we invite the reader to do so.)

Let $f_0 = 2 \cdot 1_G - 2(1_{S_1} + 1_{S_2} + 1_{S_3}) + p(1_{L_1} + 1_{L_2} + 1_{L_3})$, and $h_0 = (1_{S_1} + 1_{S_2} + 1_{S_3}) - 2(1_{L_1} + 1_{L_2} + 1_{L_3}) + 2p\delta_0$.

Then, $\widehat{f_0} = p^2(1_{L_1^{\perp}} + 1_{L_2^{\perp}} + 1_{L_3^{\perp}}) - 2p^2(1_{S_1^{\perp}} + 1_{S_2^{\perp}} + 1_{S_3^{\perp}}) + 2p^3\delta_0$, and $\widehat{h}_0 = 2p1_{\widehat{G}} - 2p(1_{L_1^{\perp}} + 1_{L_2^{\perp}} + 1_{L_3^{\perp}}) + p^2(1_{S_1^{\perp}} + 1_{S_2^{\perp}} + 1_{S_3^{\perp}})$.

Based on these formulae, it is easy (but somewhat tedious) to check that the normalized functions $f = \frac{1}{3p-4} f_0$, $h = \frac{1}{2p-3} h_0$, and their Fourier transforms $\widehat{f}, \widehat{h}$ form a complementary 4-tuplet $(f, h, \widehat{f}, \widehat{h})$, such that none of the appearing functions have the dispersive property of Proposition 3.2.

It may be easier to visualize this example if we identify the punctured lines $\dot{L}_i$ of $(\mathbb{Z}_p)^3$ with points of the projective plane $PG(2, \mathbb{Z}_p)$. In this picture, S_i become lines on the projective plane, and the functions f and h can be described in terms of a triangle in the projective plane. For example, the function h is positive on the lines of a triangle (with the exception of the vertices, where h is 0), and zero everywhere outside. We leave the details to the reader.

Notice, however, that in this example neither f nor h can come from the averaging of an indicator function of a set A as in Lemma 2.3. The reason is that by equation (2) we would have $|A| = \frac{p^2(2p-3)}{3p-4}$ or $|A| = \frac{p(3p-4)}{2p-3}$, neither of which is an integer for $p \neq 2, 3$, and for $p = 2, 3$ it is easy to check (with a finite case-by-case analysis) that f and h cannot be a result of averaging an indicator function of any set A.

Example 3.3 can be generalized: one can take k points on the projective plane, with $k - 1$ lying on a line, and one point being outside. Then, a complementary 4-tuple $(f, h, \widehat{f}, \widehat{h})$ can be constructed, such that the function h is positive on all the lines connecting these points (with exception of the points themselves, where h is 0), and zero everywhere outside. The function f is then supported on the complement supp h. The exact formulae are somewhat cumbersome, so we choose to omit them.

At this point, we do not have a reasonable conjecture as to whether $(\mathbb{Z}_p)^3$ is pd-flat or not. Therefore, we cannot decide whether the "spectral $\to$ tile" direction of Fuglede's conjecture holds in these groups.

As a final remark we mention here Rédei's conjecture which concerns the structure of tilings in $(\mathbb{Z}_p)^3$. It states that in any normalized tiling $A \oplus B = (\mathbb{Z}_p)^3$ (normalized meaning that $0 \in A, B$ is assumed) we must have that either A or B is contained in a proper subgroup.

A complete characterization of complementary 4-tuples in $(\mathbb{Z}_p)^3$ could, in principle, lead to the solution of both Fuglede's and Rédei's conjecture in these groups.

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