Decomposition of balls in $\mathbb{R}^d$

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Abstract

We investigate the decomposition problem of balls into finitely many congruent pieces in dimension $d = 2k$. In addition, we prove that the d dimensional unit ball B_d can be divided into finitely many congruent pieces if $d = 4$ or $d \geq 6$. We show that the minimal number of required pieces is less than $20d$ if $d \geq 10$.

1 Introduction

The history of this problem goes back to 1949, when Van der Waerden posed an exercise in Elemente der Mathematatik. The question was whether the disk can be decomposed into 2 disjoint congruent pieces. Different elementary proofs show that it is not possible.

Maybe the simplest one is the following: If there exists such a decomposition, then there exists an isometry connecting the two pieces. We prove that this isometry must be a linear transformation. Let $A \cup B = D$ be a decomposition of the unit ball and ϕ be an isometry with $\phi(A) = B$. The 1 dimensional Hausdorff measure of the boundary of the disk, $\mathcal{H}_1(\partial D)$ is 2π. The outer Hausdorff measure of the intersection of the boundary of the disc with A or B is at least π. We may assume that this holds for A. On the other hand, there is no arc of radius 1 contained in the interior of the disk which has at least π measure. Therefore $\phi(A \cup \partial D) \subset \partial D$. Then the origin stays in place.

This motivates the question whether the d dimensional ball can be decomposed into finitely many congruent pieces. Clearly, it is enough to decide the question for the unit ball B_d. For a cardinal number m, we say that a set K is m**-divisible** (with respect to G) if K can be decomposed into m congruent (with respect to G) and disjoint pieces. Wagon [7] proved in 1984 that the d dimensional ball is not m-divisible for $2 \leq m \leq d$. This was the only well-known lower bound for the number of pieces. In 2012 the authors showed (in an unpublished paper) that the disk is not 3-divisible.

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In 2007, Richter [5] showed that a typical convex body D is not m-divisible for any finite m. Every decomposition can be described by a set A and a set of isometries $\phi_0 = id, \phi_1, \dots, \phi_n$, where $D = \coprod_{i=0}^{n} \phi_i(A)$. He proved that if $\mathcal{H}_{d-1}(\phi_i^{-1}(\partial D) \cap \phi_j^{-1}(\partial D)) = 0$ for every $i \neq j$, then D cannot be decomposed by these isometries. This guarantees that every element of a residual subset of the space of convex bodies (endowed with Hausdorff metric) is not m-divisible for any $m \geq 2$. However, for every d, the d dimensional ball B_d is not in this class, see [5]. In 2010, Laczkovich and the first author proved [2] that the 3 dimensional ball is m-divisible for any $m \geq 22$.

In this paper we prove that the $d = 2k$ dimensional ball can be decomposed into finitely many congruent pieces:

Theorem 1.1. *The $2s$ dimensional ball (either open or closed) is m-divisible for every $m \geq 4(2s+1)+2$ if $s \geq 2$ and $s \neq 3$.*

The original proof was formulated for the four dimensional unit ball. The construction of the proof is a natural generalization of it for higher dimensional cases. As a special case of Theorem 1.1 we get:

Theorem 1.2. *The 4 dimensional ball (either open or closed) is m-divisible for every $m \geq 22$.*

Using Theorem 1.1 and the fact that the 3 dimensional ball can be decomposed into finitely many pieces (see [2]), we prove the following:

Theorem 1.3. *The d-dimensional ball B_d can be decomposed into finitely many pieces for $d \geq 6$ and $d = 3, 4$.*

Furthermore, we show that the minimal number of pieces in our construction grows linearly with the dimension. According to [7] this is the best in the sense that there is a linear lower bound $d+1$ for the number of pieces which is needed for a decomposition.

Theorem 1.4. *Let $d \geq 10$ and $\tau(B_d)$ denote the minimal number of required pieces for a decomposition of the ball B_d. Then*

$$d < \tau(B_d) < 20 \cdot d.$$

This result improves the upper bound given by the construction for the $3k$-dimensional ball in [2].

Our paper is organized as follows. In Section 2 we introduce the notation that we will use throughout the paper. In Section 3 we collect facts about a subgroup of the 4 dimensional special orthogonal group. In Section 4 we define a rational parametrization of special orthogonal matrices in dimensional d. Section 5 is devoted to the main lemma of the paper giving sufficient properties for the existence of decomposition of infinite graphs. In Section 6 and 7 we apply it for graphs defined by isometries. In Section 6 we complete the proof of Theorem 1.2 and Theorem 1.1. Finally, in Section 7 we handle the odd dimensional cases to prove Theorem 1.3 and we collect all the information given in the paper on the number of required pieces for a decompositions to prove Theorem 1.4. In Section 8 we summarize the results and open questions on the decomposition of balls.

2 Notation

For a possible directed graph Γ we denote by $V(\Gamma)$ and $E(\Gamma)$ the set of vertices and edges, respectively. If there is an edge e from U to V, then we say that U is the tail and V is the head of e and we denote them by $T(e)$ and $H(e)$, respectively. We call a sequence of vertices $V_1, V_2, \dots, V_n$ a path if for every $1 \le i \le n-1$ there is an edge from V_i to V_{i+1} and a path $V_1, V_2, \dots, V_n$ is a cycle if $V_i \ne V_j$ if $1 \le i < j \le n-1$ and $V_1 = V_n$. We denote by (P, Q) a path from P to Q. We will also use this notation for graphs, where there are more than one paths connecting P and Q if it is clear which path we consider.

We denote by e the identity element of a group. Let G be a group generated by the elements of the set $S = \{w_\alpha \mid \alpha \in I\}$, where $S = S^{-1}$. Every element W of the group G can be written (not necessarily uniquely) as a word of the generators so W is of the form $w_1 w_2 \dots w_n$, where $w_i \in S$ for every $1 \le i \le n$. As a later terminology, we say that the word W starts with w_1 and ends with w_n. Moreover the i'th letter w_i of the word W will be denoted by $W[i]$ and we use the notation $W[-1]$ for the last letter of W. If W is the empty word, then let $W[i] = e$. We denote by $\lg(W)$ the length n of the reduced word W. However, we will use linear transformations of $\mathbb{R}^d$ as the letters of a word W and we use the convention that linear transformations acts from the left on the elements of $\mathbb{R}^d$. We also say that a word $W = w_1 w_2, \dots, w_k$ has a shorter conjugate if $W[1]^{-1} = W[-1]$.

The special orthogonal group $SO(n, \mathbb{R})$ will be shortly denoted by $SO(n)$ and we denote by $Iso(n)$ the isometry group of the n dimensional Euclidean space. In this paper, by m-divisibility we mean m-divisibility with respect to $Iso(n)$.

Let $p(x) = a_n x^n + \dots + a_0$ be a polynomial. Let $\deg(p)$ denote the degree n of the polynomial p and we denote by $LC(p)$ the leading coefficient a_n of p.

3 Lemmas on a subgroup of $SO(4)$

In this section, for sake of completeness, we prove more than it would be necessary to prove Theorem 1.2.

Lemma 3.1. *Let A and B be the rotations in $SO(4)$ given by the matrices*

$$A = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix} \text{ and}$$

$$B = \begin{pmatrix} \cos\theta & 0 & 0 & -\sin\theta \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ \sin\theta & 0 & 0 & \cos\theta \end{pmatrix},$$

respectively, where $\cos\theta$ is transcendental. We denote by K the group generated by A and B. Then every element $U \neq 1 \in K$ has exactly one fix point, which is the origin.

Proof. The proof can be found in [8, Theorem 6.3]. □

Observation 3.2. *It is easy to see from Lemma 3.1 that K is a free group so every element of K can be written uniquely as the product of the matrices A, A^{-1}, B, B^{-1}. This gives that the length of $M = A^{m_1}B^{n_1}\cdots A^{m_l}B^{n_l} \in K$ is $\lg(M) = \sum_{i=1}^{l} |m_i| + |n_i|$ if M is defined by a reduced word.*

Definition 3.3.

1. *We define the set*

$$\mathcal{M} = \left\{ \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

It is easy to verify that $\mathcal{M}$ is an algebra over $\mathbb{R}$.

2. *Let*

$$\mathcal{M}_1 = \{ M \in \mathcal{M} : \ det(M) = 1 \}.$$

Clearly, $\mathcal{M}_1$ is a subgroup of the orthogonal group $O(n)$.

3. *Similarly, let $\mathcal{M}(\theta)$ denote the set of matrices of the form*

$$\begin{pmatrix} p(\cos\theta) & -\sin\theta\ q(\cos\theta) & -r(\cos\theta) & -\sin\theta\ s(\cos\theta) \\ \sin\theta\ q(\cos\theta) & p(\cos\theta) & -\sin\theta\ s(\cos\theta) & r(\cos\theta) \\ r(\cos\theta) & \sin\theta\ s(\cos\theta) & p(\cos\theta) & -\sin\theta\ q(\cos\theta) \\ \sin\theta\ s(\cos\theta) & -r(\cos\theta) & \sin\theta\ q(\cos\theta) & p(\cos\theta) \end{pmatrix},$$

where $p, q, r, s \in \mathbb{Q}[x]$. Such an element of $\mathcal{M}(\theta)$ is determined by the polynomials p, q, r, s and will be denoted by $M_\theta(p, q, r, s)$.

4. *Let $\mathcal{M}_1(\theta) = \{M \in \mathcal{M}(\theta) : \det M = 1\}$.*

Observation 3.4. *Let U be an element of K, where K is defined in Lemma 3.1. Then $U \in \mathcal{M}_1(\theta)$.*

For further results we need to describe the degree and leading coefficient of the polynomials p, q, r, s for $M_\theta(p,q,r,s) \in \mathcal{M}(\theta)$.

Definition 3.5. *1. For a pair of polynomials p_1, p_2 we write $p_1(\cos(\theta)) \doteq p_2(\cos(\theta))$ and $\sin(\theta)p_1(\cos(\theta)) \doteq \sin(\theta)p_2(\cos(\theta))$ if $\deg(p_1) = \deg(p_2)$ and $LC(p_1) = LC(p_2)$.*

2. For a pair of matrices $M_1, M_2 \in \mathcal{M}(\theta)$ we write $M_1 \doteq M_2$ if and only if $(M_1)_{i,j} \doteq (M_2)_{i,j}$ for every $i, j \in \{1, \dots, 4\}$.

We define the degree of a matrix in $M(\theta)$.

Definition 3.6. *Let $M = M_\theta(p,q,r,s)$. We denote by $\deg(M)$ the maximum of $\deg(p), \deg(q)+1, \deg(r), \deg(s)+1$.*

It is easy to see that if $M, N \in \mathcal{M}(\theta)$ and $M \doteq N$, then $\deg(M) = \deg(N)$.

Observation 3.7. *(a) It is easy to see that for $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$ we have $p(\cos\theta) \doteq a_n(\cos\theta)^n$. We also have $\sin\theta p(\cos\theta) \doteq a_n \sin\theta(\cos\theta)^n$.*

(b) Let p_1, p_2, q_1 and q_2 polynomials in $\mathbb{Z}[x]$. If $p_1 \doteq q_1$ and $p_2 \doteq q_2$, then $p_1p_2 \doteq q_1q_2$.

(c) Let us assume again that $p_1 \doteq q_1$ and $p_2 \doteq q_2$. If $\max\{\deg(p_1), \deg(p_2)\} = \deg(p_1+p_2)$, then $\max\{\deg(q_1), \deg(q_2)\} = \deg(q_1+q_2)$ and $p_1+p_2 \doteq q_1+q_2$.

(d) If $\deg(p_1) > \deg(p_2)$, then $p_1 + p_2 \doteq p_1$.

Lemma 3.8. *Let $U \in K$ be of the form $A^{m_1}B^{n_1}\cdots A^{m_t}B^{n_t}$, where A and B are given in Lemma 3.1. Let σ denote the length of U.*

(a) (Case $U = A^{m_1}B^{n_1}\cdots A^{m_t}B^{n_t}$)
If m_i, n_i are nonzero integers for $1 \le i \le t$, then $U \in \mathcal{M}(\theta)$, where $\deg(p) = \deg(r) = \deg(q)+1 = \deg(s)+1 = \sigma$. We also have $|LC(p)| = |LC(q)| = |LC(r)| = |LC(s)| = 2^{\sigma-t-1}$.

(b) (Case $U = A^{m_1}B^{n_1}\cdots A^{m_t}$)
If m_i, n_i, m_t are nonzero integers for $1 \le i \le t-1$ and $n_t = 0$, then one of the following two cases holds:

i $\deg(p) = \deg(q)+1 = \sigma$ with $|LC(p)| = |LC(q)| = 2^{\sigma-t-2}$ and $\max(\deg(r), \deg(s)+1) < \sigma$.

ii $\deg(r) = \deg(s)+1 = \sigma$ with $|LC(r)| = |LC(s)| = 2^{\sigma-t-2}$ and $\max(\deg(p), \deg(q)+1) < \sigma$.

(c) (Case $U = B^{n_1}\cdots A^{m_t}B^{n_t}$, similarly)
If n_1, m_i, n_i are nonzero integers for every $2 \le i \le t$ and $m_1 = 0$, then one of the following two cases holds:

i $\deg(p) = \deg(s) + 1 = \sigma$ *with* $|LC(P)| = |LC(S)| = 2^{\sigma-t-2}$ *and* $\max(\deg(r), \deg(q) + 1) < \sigma$.

ii $\deg(q) + 1 = \deg(r) = \sigma$ *with* $|LC(q)| = |LC(r)| = 2^{\sigma-t-2}$ *and* $\max(\deg(p), \deg(s) + 1) < \sigma$.

Proof.

(a) We claim that

$$U \doteq 2^{\sigma-t-1}\cos^{\sigma-1}\theta \begin{pmatrix} \xi\cos\theta & -\mu\sin\theta & -\zeta\cos\theta & -\nu\sin\theta \\ \mu\sin\theta & \xi\cos\theta & -\nu\sin\theta & \zeta\cos\theta \\ \zeta\cos\theta & \nu\sin\theta & \xi\cos\theta & -\mu\sin\theta \\ \nu\sin\theta & -\zeta\cos\theta & \mu\sin\theta & \xi\cos\theta \end{pmatrix} \tag{1}$$

for some $\xi, \mu, \zeta, \nu = \pm1$ with $\mu\nu = \zeta\xi$. The proof of this fact can be found in Wagon [8, page 55].

(b) We write $U = U'A^{m_t}$, where $U' = A^{m_1}B^{n_1}\cdots A^{m_{t-1}}B^{n_{t-1}}$. Equation (1) shows that

$$U' \doteq 2^{\sigma'-t-2}\cos^{\sigma'-1}\theta \begin{pmatrix} \xi'\cos\theta & -\mu'\sin\theta & -\zeta'\cos\theta & -\nu'\sin\theta \\ \mu'\sin\theta & \xi'\cos\theta & -\nu'\sin\theta & \zeta'\cos\theta \\ \zeta'\cos\theta & \nu'\sin\theta & \xi'\cos\theta & -\mu'\sin\theta \\ \nu'\sin\theta & -\zeta'\cos\theta & \mu'\sin\theta & \xi'\cos\theta \end{pmatrix},$$

where $\sigma' = |m_1| + |n_1| + \cdots + |m_{t-1}| + |n_{t-1}|$. Using the fact that $\mu'\nu' = \zeta'\xi'$ and $|\mu'| = |\nu'| = |\zeta'| = |\xi'|$ we get that exactly one of the two sums $\xi' + \mu'$ and $\zeta' - \nu'$ is 0 and the absolute value of the other is 2.

It is easy to show that

$$\begin{aligned} A^{m_t} &= \begin{pmatrix} \cos(m_t\theta) & -\sin(m_t\theta) & 0 & 0 \\ \sin(m_t\theta) & \cos(m_t\theta) & 0 & 0 \\ 0 & 0 & \cos(m_t\theta) & -\sin(m_t\theta) \\ 0 & 0 & \sin(m_t\theta) & \cos(m_t\theta) \end{pmatrix} \\ &\doteq 2^{m_t-1}\cos^{m_t-1}\theta \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix}, \end{aligned}$$

using well known facts about Chebyshev polynomials.

We define the matrix M and N by $U' = (2^{\sigma'-t-2}\cos^{\sigma'-1}\theta)\cdot M$ and $A^{m_t} = (2^{m_t-1}\cos^{m_t-1}\theta)\cdot N$. Using the fact $\sin^2\theta = 1 - \cos^2\theta$, we get that the first row of $M \cdot N$, which is denoted by $(M\cdot N)_{1\cdot}$, is the following

$$\begin{aligned} (M\cdot N)_{1\cdot} \doteq \big(&(\xi'+\mu')cos^2\theta, -(\xi'+\mu')sin\theta\cdot cos\theta, \\ &-(\zeta'-\nu')cos^2\theta, (\zeta'-\nu')sin\theta\cdot cos\theta\big). \end{aligned}$$

Thus either the first two or the second two coordinates vanishes. Easy calculation shows that in the other two coordinates of $U'A^{m_t}$ have degree $(\sigma' - 1) + (m_t - 1) + 2 = \sigma$ and $\sigma - 1$, respectively. The absolute value of the leading coefficients are the same $(2^{m_t-1}2^{\sigma'-t-2} \cdot 2 = 2^{\sigma-2})$.

(c) Similar calculation shows the statement.

$\square$

Lemma 3.9. *Let U and σ be as in Lemma 3.8 (b). We claim that $m_1 m_t > 0$ if and only if*

$$\deg(p) = \deg(q) + 1 \text{ and } \max(\deg(r), \deg(s) + 1) < \deg(p).$$

Proof. If $M = M_\theta(p, q, r, s) \in \mathcal{M}(\theta)$, then $tr(M) = 4p(\cos(\theta))$. Conjugating by A^{m_t} we get

$$tr(M) = tr\left(A^{m_1} B^{n_1} \cdots B^{m_{t-1}} A^{m_t}\right) = tr(A^{m_1+m_t} B^{n_1} \cdots B^{m_{t-1}}).$$

Clearly, the sum of the absolute value of the exponents $\sigma' = |m_1 + m_t| + |n_1| + \ldots + |m_{t-1}|$ is smaller than $\sigma - 1$ if $m_1 m_t < 0$ and $\sigma' = \sigma$ if $m_1 m_t > 0$. By Lemma 3.8 (a) we have $\deg(p) = \sigma' < \sigma - 1$ if $m_1 m_t < 0$ and $\deg(p) = \sigma' = \sigma$ if $m_1 m_t > 0$. Finally, one can identify the two cases of Lemma 3.8 (b), finishes the proof. $\square$

Remark 3.10. *1. The analogue statement is true for U and σ in Lemma 3.8 (c) and for n_1, n_t instead of m_1, m_t. Therefore, for a matrix*

$$M \in \mathcal{M}(\theta)$$

$\deg(M)$*, which is defined in Definition 3.6, is taken in the diagonal if and only if M does not have a shorter conjugate.*

2. Now we can easily calculate the degree of the polynomials in the main diagonal of the word U which equals to

$$\min\left\{lg(U') \mid U' \in K,\ U \text{ and } U' \text{are conjugate}\right\}.$$

It is easy to see that every element M of the group generated by A and B we have $\lg(M) = \sigma = \deg(M)$.

Lemma 3.11. *Suppose that $M \in \mathcal{M}$ and $tr(M) = 4p \neq 1$. Then $M + M^T = 2pI$ is a scalar matrix and $(I - M)^{-1} = \frac{1}{2-2p}(I - M^T)$.*

Proof. Clearly, $M + M^T = 2pI$ and

$$(I - M)(I - M^T) = I - M - M^T + MM^T = I - 2pI + I = (2 - 2p)I$$

since M is an orthogonal matrix. $\square$

Technically, we need the following as well:

Lemma 3.12. *Suppose that for $M \in \mathbb{R}^{d\times d}$, the matrix $I - M$ is invertible, then the entries of $(I - M)^{-1}$ are rational functions of the entries of M.*

Proof. Obvious, using Cramer's rule. $\square$

4 Algebraic independence

It was proved in [2, p. 5-6.] that there exists a rational parametrization $\alpha_d : \Omega \to SO(d)$, where Ω is an open subset of $\mathbb{R}^{d' \cdot d}$ and α_d is surjective, where $d' = d$ if d is even and $d' = d - 1$ if d is odd. Indeed, every element of $SO(d)$ can be written as the product of at most d' reflections given by the vectors $v_i = (x_{di+1}, \dots, x_{di+d})$ for $i = 0, 1, \dots, d' - 1$. For every $w \in \mathbb{R}^d$ the matrices

$$R_w = I - \frac{ww^T}{|w|^2}$$

gives a parametrization of the reflection in a hyperplane perpendicular to w. The entries of the matrix R_w are rational functions of the coordinates of w, where the denominator of the functions does not vanish for any $w \neq \mathbf{0}$. Let $v = (v_0, v_1, \dots, v_{d'-1})$, which is the concatenation of the vectors $v_i \in \mathbb{R}^d$. Hence the entries of the matrix $\alpha(v)$ are rational functions of $x_1, x_2, \dots, x_{d'd}$ with integer coefficients. The denominator of $(\alpha_d(v))_{i,j}$ does not vanish on Ω as a rational function.

Now, we fix a rational parametrization α_d of $SO(d)$. If $v \in \Omega \subset \mathbb{R}^{d'd}$, then we shall denote by O_v the image of the parametrization; both as a matrix and as a linear transformation of $\mathbb{R}^d$. Then $v \mapsto O_v$ is a surjection from Ω onto $SO(d)$, and every entry of the matrix of O_v is a rational function with integer coefficients of the coordinates of v.

Definition 4.1. *We say that $M_1, M_2, \dots, M_m \in SO(d)$ are independent, if there exist $v_1, v_2, \dots, v_m \in \Omega$ such that $\alpha_d(v_i) = M_i$ and the coordinates of v_i are algebraically independent over $\mathbb{Q}$. We will also say that a vector $t \in \mathbb{R}^d$ and the matrices $M_1, M_2, \dots, M_m \in SO(d)$ form an independent system if the coordinates of t and the coordinates of $v_1, \dots, v_m$ are algebraically independent over $\mathbb{Q}$.*

Lemma 4.2. *Let p be a polynomial on k variables. Let $M_1, \dots, M_k$ be independent elements of $SO(d)$ with $p(M_1, \dots, M_k) = 0$. Then $p(N_1, \dots, N_k) = 0$ for all $N_1, \dots, N_k \in SO(d)$.*

Proof. Every entry of the matrix equation is a polynomial expression of the parameters. Since they were chosen algebraically independently, the equation holds if and only if it is trivial. This means that it holds for any substitution of the parameters. The fact that α_d is surjective finishes the proof of Lemma 4.2. □

Similar argument shows the following.

Lemma 4.3. *Let $q(x_1, \dots, x_{N+d})$ be a rational function where $N = d^2 \cdot k$ and $M_1, \dots, M_k \in SO(d)$ and $t \in \mathbb{R}^d$ be an independent system. Let us suppose that $q(m_1, \dots, m_N, t_1, \dots, t_d) = 0$ where (m_i) is an enumeration of the entries of the matrices $M_1, \dots, M_k$. Then $q(n_1, \dots, n_N, s_1, \dots, s_d) = 0$ holds for the same enumeration of the entries (n_i) of arbitrary matrices $N_1, \dots, N_k \in SO(d)$ and arbitrary $s = (s_1, \dots, s_d) \in \mathbb{R}^d$, where the denominator of q does not vanish.*

We usually use this fact contrary, we show that there exists a substitution which is non-trivial, therefore it is non-trivial for any algebraically independent substitution.

Lemma 4.4. *Let α_d be a rational parametrization of the d dimensional special orthogonal linear transformations, where d is even. If $O_1, O_2, \ldots, O_k$ are independent orthogonal transformations and U is not an empty word, then $\hat{U} = U(O_1, O_2, \ldots, O_k)$ does not have a nonzero fix point.*

Proof. The characteristic polynomial $p(y) = \det(Iy - U)$ of the orthogonal transformation $\hat{U} = U(O_1, O_2, \ldots, O_k)$ can be considered as a rational function with integer coefficients of the variables $y, x_1, x_2, \ldots, x_{k \cdot d^2}$. Let us assume indirectly that $\hat{U}$ has a nonzero fixpoint, thus p vanishes at $y = 1$. By the algebraic independence of the parameters we get that p vanishes at $y = 1$ for any substitution to the variables $x_1, x_2, \ldots, x_{k \cdot d^2}$. This shows that 1 is the eigenvalue of every element of the form $U(M_1, M_2, \ldots, M_k)$, where M_i are orthogonal transformations, which clearly contradicts Lemma 3.1 if $d = 4$. Moreover, free subgroup of the orthogonal group consisting of fixed point free elements (except the identity) was given in [3, 4] for every d dimensional orthogonal groups where d is even and $d \geq 4$, finishing the proof of Lemma 4.4. □

5 Decomposition in $\mathbb{R}^{2s}$

Let X be a set, and let $f_1, \ldots, f_n$ be maps from subsets of X into X. Our aim is to find a sufficient condition for the existence of a decomposition $X = A_0 \cup A_1 \cup \ldots \cup A_n$ such that $f_i(A_0) = A_i$ for every $i = 1, \ldots, n$.

Suppose that for $i = 1, 2, \ldots, n$ the function f_i is defined on $D_i \subset X$ ($i = 1, \ldots, n$), and put $D = \bigcap_{i=1}^n D_i$. We say that the point x is a core point, if $x \in D$, and the points $x, f_1(x), \ldots, f_n(x)$ are distinct. By the image of a point x we mean the multiset $\mathcal{I}_x = \{f_1(x), \ldots, f_n(x)\}$. The multiset $\mathcal{I}_x$ is a set if x is a core point.

For a set $\mathcal{F} = \{f_1, \ldots, f_n\}$ we define a graph $\Gamma_{\mathcal{F}}$ on the set X as follows. We connect the distinct points $x, y \in X$ by an edge if there is an $i \in \{1, \ldots, n\}$ such that $f_i(x) = y$. Then $\Gamma_{\mathcal{F}}$ will be called the graph generated by the functions $f_1, \ldots, f_n$.

Lemma 5.1. *Let X, $f_1, \ldots, f_n$, D, and $\Gamma_{\mathcal{F}}$ be as above, and suppose that the graph $\Gamma_{\mathcal{F}}$ has the property that*

$$\begin{array}{c}\textit{whenever two cycles } \mathcal{C}_1 \textit{ and } \mathcal{C}_2 \textit{ in } \Gamma_{\mathcal{F}} \textit{ share a common edge,}\\ \textit{then the sets of vertices of } \mathcal{C}_1 \textit{ and } \mathcal{C}_2 \textit{ coincide.}\end{array} \tag{2}$$

Suppose further that there is a point $x_0 \in X$ satisfying the following conditions.

(a) x_0 is in the image of at least one core point;

(b) every $x \in X \setminus \{x_0\}$ is in the image of at least three core points.

Then there is a decomposition $X = A_0 \cup A_1 \cup \ldots \cup A_n$ such that $A_0 \subset D$, and $f_i(A_0) = A_i$ for every $i = 1, \ldots, n$.

Proof. The proof is based on the axiom of choice and can be found in [2]. $\square$

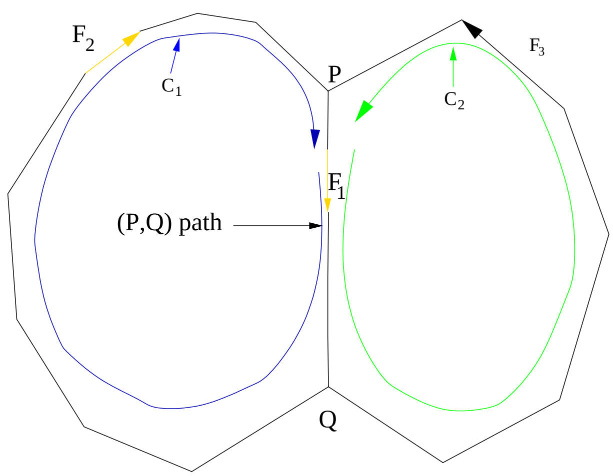
Lemma 5.2. *If a connected component Γ' of Γ contains two different cycles sharing at least a common edge, then Γ' contains two cycles $\mathcal{C}_1 = P_1, P_2, \ldots, P_m$ and $\mathcal{C}_2 = Q_1, Q_2, \ldots, Q_n$ such that for some $1 < k < \min\{n, m\}$ we have $P_i = Q_i$ for $i = 1, \ldots, k$ and $\{P_1, P_2, \ldots, P_m\} \cap \{Q_1, Q_2, \ldots, Q_n\} = \{P_1, P_2, \ldots, P_k\}$.*

Proof. We may assume that $P_1 = Q_1$ is one of the endpoints of a common edge such that $P_2 \neq Q_2$. Then there exists a minimal integer b such that $Q_b = P_a$ for some $1 < a < m$. Since $P_1, P_2, \ldots, P_m$ are different points, the cycles $Q_1, \ldots, Q_b, P_{a-1}, \ldots, P_1$ and $Q_1, \ldots, Q_b, P_{a+1}, \ldots, P_m$ have a common path and share only the points $Q_1, Q_2, \ldots, Q_b$. $\square$

Remark 5.3. *Essentially, this means that we can find two points P and Q such that between these points there are three paths which have no other common points.*

Theorem 5.4. *Let us assume that $t \in \mathbb{R}^d$ and $\alpha_{v_0} = O_0, \alpha_{v_1} = O_1, \ldots, \alpha_{v_m} = O_m$ in $SO(d)$ form an independent system, where $d = 2s \geq 4$ and $d \neq 6$. Let $F(x) = O_0 x + t$. Then $\Gamma_{\mathcal{F}}$ has the property (2), where $\mathcal{F} = \{F, O_1, \ldots, O_m\}$.*

Proof. Let us assume indirectly that there exists a connected component of Γ' which the contains cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and the two cycles share at least one edge. Using Lemma 5.2 we may assume that the two cycles share a common path. Thus Γ' contains a subgraph $\Delta = (V(\Delta), E(\Delta))$:



Let us denote by P and Q the endpoints of the common paths and denote by (P, Q) path the common path as in Figure 1. For each edge of the graph we can naturally assign a letter O_i or F.

Remark 5.5. *O_i are independent orthogonal transformations and F is the only isometry involving translation, therefore by Lemma 4.4 there must be a letter F or F^{-1} in every cycle. Thus we may assume that at least two of the three paths between P and Q contain the letter $F^{\pm 1}$.*

Using the previous remark we may assume that the (P, Q) path contains an F or an F^{-1}. We denote the closest $F^{\pm 1}$ to P on the path (P, Q) by F_1.

We choose a starting point S from which we start going around the cycles $\mathcal{C}_1, \mathcal{C}_2$ (as in Figure 1) and then the two cycles naturally determine two words W_1 and W_2, respectively. According to Remark 5.5, there is another $F^{\pm 1}$ in W_1, which as an edge is not contained in $E(\mathcal{C}_2)$. Let us denote the first $F^{\pm 1}$ in W_1 by F_2. Similarly to F_1 and F_2 one can define F_3 to be the edge corresponding to the last $F^{\pm 1}$ on the cycle $\mathcal{C}_2$. Note that F_3 might be equal to F_1 and it might also happen that $F_1 \neq F_3$ but F_3 is on the (P, Q) path. We consider the edge corresponding to F_2 and F_3 as a directed edge which has the same direction as the cycle $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively.

The starting point S of the two cycles can be identified with an element of $x \in \mathbb{R}^d$ which satisfies

$$W_1(x) = W_2(x) = x. \tag{3}$$

Every direction-preserving isometry of $\mathbb{R}^d$ can be written as $W(x) = U(x) + b$ for some $U \in SO(d)$ and $b \in \mathbb{R}^d$. Using equation (3) we get that there are $U_1, U_2 \in SO(d)$ and $b_1, b_2 \in \mathbb{R}^d$ such that

$$W_1(x) = U_1 x + b_1 = x \text{ and } W_2(x) = U_2 x + b_2 = x. \tag{4}$$

Let H denote the group generated by $O_0, O_1, \ldots, O_m$. Since the edges of Δ are labelled by $F^{\pm 1}$ and $O_i^{\pm 1}$ we have U_1 and U_2 are in H. Thus

$$(I - U_1)x = b_1 \text{ and } (I - U_2)x = b_2.$$

Since $\mathcal{C}_1$ and $\mathcal{C}_2$ are cycles, U_1 and U_2 are nonempty reduced words of the generators $O_0, , \ldots, O_m$. By Lemma 4.4, the orthogonal transformation U_1 and U_2 do not have a fix point thus $I - U_i$ are invertible for $i = 1, 2$ and hence

$$(I - U_1)^{-1} b_1 = (I - U_2)^{-1} b_2 = x. \tag{5}$$

One can easily verify that $(I - U_1)^{-1} b_1 = (I - U_2)^{-1} b_2$ is equivalent to the fact the words W_1 and W_2 have a common fix point, which was formulated in equations (3) and (4).

We write

$$W_i(x) = S_{i,1} F^{\alpha_{i,1}} S_{i,2} F^{\alpha_{i,2}} \ldots S_{i,n_i} F^{\alpha_{i,n_i}} S_i^*(x),$$

where S_i^* and $S_{i,j}$ are elements of the group $H' = \langle O_1, \ldots, O_m \rangle$ and $\alpha_{i,j}$ is 1 or -1 for every $j = 1, \ldots, n$ and $i = 1, 2$. In this case for W_i is of the following form for $i = 1, 2$:

$$\begin{aligned} W_i = {} & S_{i,1} {O_0}^{\alpha_{i,1}} S_{i,2} {O_0}^{\alpha_{i,2}} \cdots S_{i,n_i} {O_0}^{\alpha_{i,n_i}} S_i^* x \\ & + t \left(\sum_{k=1}^{n_i} (-1)^{\beta_{i,k}} \left(\prod_{j=1}^{k-1} S_{i,j} {O_0}^{\alpha_{i,j}} \right) S_{i,k} {O_0}^{\beta_{i,k}} \right), \end{aligned} \tag{6}$$

where $\beta_{i,j} = 0$ if $\alpha_{i,j} = 1$ and $\beta_{i,j} = -1$ if $\alpha_{i,j} = -1$.

For every $k \in \{1, \dots, n_i\}$ we define

$$U_{i,k} = \left(\prod_{j=1}^{k-1} S_{i,j} {O_0}^{\alpha_{i,j}} \right) S_{i,k}$$

and let

$$\hat{U}_i = (-1)^{\beta_{i,n_i}} U_{i,n_i} {O_0}^{\beta_{i,n_i}} \text{ and } \mathring{U}_i = (-1)^{\beta_{i,1}} S_{i,1} O_0^{\beta_{i,1}}.$$

Using the previous notation one can see from equation (6) that

$$U_i = S_{i,1} {O_0}^{\alpha_{i,1}} S_{i,2} {O_0}^{\alpha_{i,2}} \cdots S_{i,n_i} {O_0}^{\alpha_{i,n_i}} S_i^* \tag{7}$$

and we can also write

$$U_i = U_{i,n} \cdot {O_0}^{\alpha_{i,n_i}} S_i^*.$$

The vectors b_1 and b_2 can be written as $V_i t$, where

$$V_i = \sum_{k=1}^{n_i} (-1)^{\beta_{i,k}} U_{i,k} {O_0}^{\beta_{i,k}}. \tag{8}$$

Equation (5) can be reformulated as follows

$$(I - U_1)^{-1} V_1 t = (I - U_2)^{-1} V_2 t.$$

By Lemma 3.12, every entry of $(I - U_i)^{-1}$ is a rational function of the entries of U_i, which is generated by $O_0, \dots, O_m$. Using Lemma 4.3 and the algebraic independence assumption on the coordinates of t and v_i it is clear that the previous equation holds for every vector $s \in \mathbb{R}^d$ and $O'_1, \dots, O'_m \in SO(d)$. Thus we can eliminate t from the previous equation and we get

$$(I - U_1)^{-1} V_1 = (I - U_2)^{-1} V_2. \tag{9}$$

First, we prove that it is enough to deal with the four dimensional case.

Remark 5.6. Let us assume that $2s \neq 4$. From now on, we substitute block matrices into O_i for $i = 0, 1, \dots, m$, of the form

$$M = \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix},$$

where $N_1 \in M(\theta) \subset SO(4)$ and $N_2 \in SO(2s-4)$. Since multiplying and adding these matrices we can count with the blocks separately. Clearly, a block matrix is invertible if and only if every block is invertible.

We need to guarantee that after the substitution, $I - U_1$ and $I - U_2$ are invertible. Since $2s - 4 \geq 4$, Lemma 4.4 shows that the group $SO(2s-4)$ contains a free subgroup (freely generated by m elements) consisting of fix point free elements. In order to prove that equation (9) does not hold for some substitution, it is enough to prove it for four dimensional matrices as in the following proposition.

Proposition 5.7. *We can substitute elements of the group K, defined in Lemma 3.1, into O_i for $i = 0, 1, \dots, m$ such that equation* (9) *does not hold.*

Proof.

Substituting words of A and B we may assume that U_1 and U_2 are in $\mathcal{M}_1(\theta) \subset \mathcal{M}_1$. Clearly, $U_i \in \mathcal{M}_1$ is of the following form for $i = 1, 2$:

$$\begin{pmatrix} p_i & -q_i & -r_i & -s_i \\ q_i & p_i & -s_i & r_i \\ r_i & s_i & p_i & -q_i \\ si & -r_i & q_i & p_i \end{pmatrix}.$$

Using Lemma 3.11 and the fact that W_1 and W_2 are non-empty words, we get

$$\begin{aligned} (I - U_i)^{-1} &= \frac{1}{2 - 2p_i} \begin{pmatrix} 1 - p_i & q_i & r_i & s_i \\ -q_i & 1 - p_i & s_i & -r_i \\ -r_i & -s_i & 1 - p_i & q_i \\ -s_i & r_i & -q_i & 1 - p_i \end{pmatrix} \\ &= \frac{1}{2 - 2p_i}(I - U_i)^T \end{aligned}$$

since $U_i \in SO(4)$ and $(I - U_i) + (I - U_i)^T$ is a scalar matrix and $p_1, p_2 \neq 1$. Equation (5) can be reformulated as

$$\frac{1}{2 - 2p_1}(I - U_1)^T V_1 = \frac{1}{2 - 2p_2}(I - U_2)^T V_2.$$

This is equivalent to

$$(1 - p_2)(I - U_1)^T V_1 = (1 - p_1)(I - U_2)^T V_2. \tag{10}$$

Using equation (8) we get

$$\begin{aligned} &(1 - p_2)(I - U_1^T)(\mathring{U}_1 + \sum_{k=2}^{n-1} (-1)^{\beta_{1,k}} U_{1,k} {O_0}^{\beta_{1,k}} + \hat{U}_1) = \\ &(1 - p_1)(I - U_2^T)(\mathring{U}_2 + \sum_{k=2}^{n-1} (-1)^{\beta_{2,k}} U_{2,k} {O_0}^{\beta_{2,k}} + \hat{U}_2). \end{aligned} \tag{11}$$

Let

$$\begin{gathered} M_1 = (I - U_1^T)(\mathring{U}_1 + \sum_{k=2}^{n-1} (-1)^{\beta_{1,k}} U_{1,k} {O_0}^{\beta_{1,k}} + \hat{U}_1) \\ \text{and similarly} \\ M_2 = (I - U_2^T)(\mathring{U}_2 + \sum_{k=2}^{n-1} (-1)^{\beta_{2,k}} U_{2,k} {O_0}^{\beta_{2,k}} + \hat{U}_2). \end{gathered} \tag{12}$$

Remark 5.8. Equation (11) depends only on the matrices $O_0, O_1, \ldots, O_m$. For an element of $O \in H = \langle O_0, \ldots, O_m \rangle$ we denote by $\overline{O}$ the element of K what we get after the substitution. Since $\overline{M}_1$ and $\overline{M}_2$ are generated by A and B defined in Lemma 3.1 we can write $\overline{M}_1 = M_\theta(p_1, q_1, r_1, s_1)$ and $\overline{M}_2 = M_\theta(p_2, q_2, r_2, s_2)$. By expanding the brackets in equation (12) we get a sum where every summand is a subword or the inverse of a subword of U_1 and U_2 endowed with a sign. It is easy to see from Observation 3.7 (d) that in order to determine the degree of the matrix in equation (12) we have to find the longest summands after the substitution. Basically the longest subword and the longest inverse of a subword occurring in M_i are $\hat{U}_i$ and $-U_i^T \mathring{U}_i = -U_i^{-1} \mathring{U}_i$, respectively.

From now on we distinguish five major cases:

(a) $F_1 = F^{-1}$

(b) $F_1 = F$, and there is no more $F^{\pm 1}$ on the paths (P, Q).

(c) $F_1 = F$ and $F_2 = F$

(d) $F_1 = F$ and $F_2 = F^{-1}$.

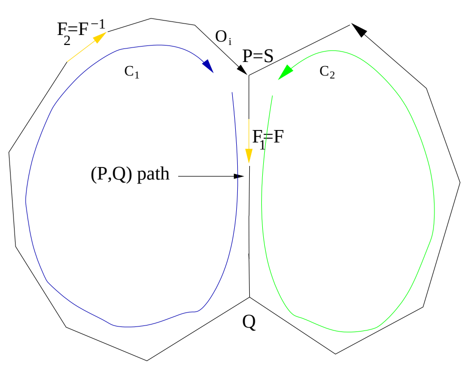
Some of these cases originate in case (a).

Case (b) $\Rightarrow$ Case (a): If there is only one $F^{\pm 1}$ on the path (P, Q), then we just change the role of P and Q and we get case (a).

Case (c) $\Rightarrow$ Case (a): If $F_1 = F$ and $F_2 = F$, then we can change the role of the paths such that the common path of W_1 and W_2 contains F_2 instead of F_1. This is again case (a).

However, case (d) does not originate in case (a), we can modify it to get a simpler form. In this case the role of F_1 and F_2 is symmetric hence we may assume that F_1 is not further from P than F_2. This implies that there are some $O_i^{\pm 1}$'s on the path from the head of F_2 to the tail of F_1 (see figure Case (e)) which are not on the path (P, Q) since the letters F and F^{-1} cannot succeed each other on a cycle.

Thus, instead of to case (d) it is enough to investigate the following case:



Case e

(e) $F_1 = F$ and $F_2 = F^{-1}$ and there are some O_i's in the path $(H(F_2), T(F_1))$ which are not in (P,Q). Moreover we may assume that F_1 is not the only F or F^{-1} on its way since we assume that this case does not originate in case (a). Using the same argument, the last $F^{\pm 1}$ on the (P,Q) path has to be F^{-1}, otherwise we change the role of P and Q. By symmetry again, we may assume that F_1 is the closest F or F^{-1} to P on Δ.

There are two major cases left and in both cases (and in every subcase) the starting point will be $S = P$. Clearly, $S_1^* = S_2^*$ in this case so we denote it by S_*.

Now we substitute $O_i = A^{i_1} B^{i_3} A^{i_2}$ for $i = 1, \dots, m$, where the absolute value of the exponents are pairwise different integers and $|i_3| \geq |i_1|, |i_2| > 1$. Further, according to the case we investigate we substitute $O_0 = \overline{S}_* A^{\varepsilon_1} \cdot B^D \cdot A^{\varepsilon_2}$ or $O_0 = A^{\varepsilon_1} \cdot B^D \cdot A^{\varepsilon_2} \overline{S}_*^{-1}$, where D, ε_1 and ε_2 will be chosen later.

We claim that if D is large enough, then this substitution is monotone. More precisely we have the following.

Lemma 5.9. *(a) Let $O_i = A^{i_1} B^{i_3} A^{i_2}$ for $i = 0, \dots, m$, where the absolute value of the exponents are pairwise different integers and $|i_3| \geq |i_1|, |i_2|$. Let V be a reduced word on the letters $O_0, \dots, O_m$ and let $V = UO_i^{\pm 1}$ or $V = O_i^{\pm 1} U$ and let U be a subword of V. Then $\lg(\overline{U}) < \lg(\overline{V})$.*

(b) Let $O_i = A^{i_1} B^{i_3} A^{i_2}$ for $i = 1, \dots, m$, where the absolute value of the exponents are pairwise different integers and $|i_3| \geq |i_1|, |i_2| > 1$. Let σ_i^1 and σ_i^2 denote the sum of the absolute value of the exponent of O_i's occurring in U_1 and U_2, respectively. Let us assume that $|\varepsilon_1| = |\varepsilon_2| = 1$ for every $1 \leq i \leq m$ and let

$$D = \sum_{i=1}^{m} (\sigma_i^1 + \sigma_i^2)(|i_1| + |i_3| + |i_2|).$$

Then for every pair of subwords V_1 and V_2 of $W_j^{\pm 1}$ $(j = 1,2)$ we have $\lg(\overline{V_1}) > \lg(\overline{V_2})$ if V_1 contains more $O_0^{\pm 1}$ than V_2.

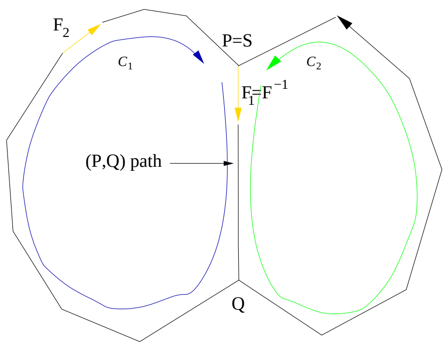
Proof.

1. It is enough to prove it, when V is of the form $UO_i^{\pm 1}$ or $O_i^{\pm 1} U$. Since the absolute value of the exponents are different B cannot be eliminated after the substitution so it is easy to see that $\lg(\overline{U}) < \lg(\overline{UO_i^{\pm 1}})$ and $\lg(\overline{U}) < \lg(\overline{O_i^{\pm 1} U})$.

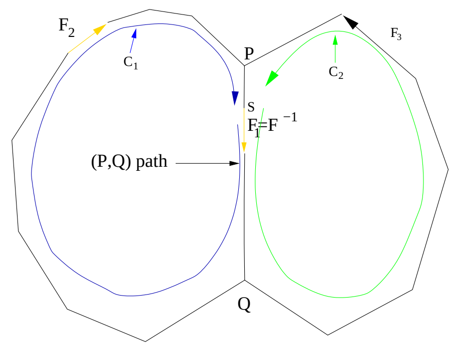
2. It is easy to show that $B^{\pm D}$ cannot be eliminated from $\overline{O_0}^{\pm 1}$ since $|i_1|, |i_2| > 1$ are different numbers.

$\square$

Case (a):



Case a/1



Case a/2

1. Let us assume first that the tail of F_1 equals to P.

   Since the orthogonal transformations acts from the left, $W_1[1] \neq W_2[1]$ and neither of these letters are F since both W_1 and W_2 represent a cycle. Now we substitute $O_i = A^{i_1} B^{i_3} A^{i_2}$ for $i = 0, \dots, m$, where the absolute value of all of these exponents are pairwise different integers and $|i_3| \geq |i_1|, |i_2|$.

   One can see from Lemma 5.9 (a) that the longest summands in $\overline{M}_1$ and $\overline{M}_2$ are $-\overline{U}_1$ and $-\overline{U}_2$, respectively since $\hat{U}_1 = -U_1$ and $\hat{U}_2 = -U_2$, while the first letter of W_i is not F so $\mathring{U}_1, \mathring{U}_2 \neq 1$ and every other summand in equation (11) is a subword of one of them. By Observation 3.7 (d) we have $\deg(\overline{U}_i) = \deg(\overline{M}_i)$ for $i = 1, 2$.

   Both $-\overline{U}_1$ and $-\overline{U}_2$ starts and ends with A or A^{-1}. Since $W_1[-1] = W_2[-1]$ we have $\overline{U}_1[-1] = \overline{U}_2[-1]$ and since $W_1[1] \neq W_2[1]$ we can choose i_1, i_2 $(0 \leq i \leq m)$ such that $\overline{U}_1[1] \neq \overline{U}_2[1]$. Thus by the symmetry of U_1 and U_2 we may assume $\overline{U}_1[1] \neq \overline{U}_1[-1]$ and $\overline{U}_2[1] = \overline{U}_2[-1]$. Lemma 3.9 shows that $\deg(p_1) < \deg(\overline{M}_1)$ and $\deg(p_2) = \deg(\overline{M}_2)$. This gives that $(1 - p_2)\overline{M}_1 \doteq (1 - p_1)\overline{M}_2$ does not hold so $(1 - p_2)\overline{M}_1 \neq (1 - p_1)\overline{M}_2$.

   It is important to note that what we proved here is that both $\overline{M}_1$ and $\overline{M}_2$ have a unique longest summand and exactly one of these summands has a shorter conjugate. These facts guarantee that $(1 - p_2)\overline{M}_1 \neq (1 - p_1)\overline{M}_2$.

2. Let us assume that the tail of F_1 is not P.

   We use Lemma 5.9 to calculate

$$\deg\left((I - U_i^T)(\hat{U}_i + \sum_{k=1}^{n_i - 1} (-1)^{\beta_{i,k}} U_{i,k} {O_0}^{\beta_{i,k}})\right). \tag{13}$$

   Now we substitute $\overline{O}_0 = \overline{S}_* A^{\varepsilon_1} B^D A^{\varepsilon_2}$. Since $F_1 = F^{-1}$, we have

$$\begin{aligned}
\overline{\hat{U}}_i &= -\overline{S}_{i,1} \overline{O}_0^{\alpha_{i,1}} \overline{S}_{i,2} \dots \overline{S}_{i,n_i} O_0^{-1} \\
&= -\overline{S}_{i,1} (\overline{S}_* A^{\varepsilon_1} B^D A^{\varepsilon_2})^{\alpha_{i,1}} \overline{S}_{i,2} \dots \overline{S}_{i,n_i} A^{-\varepsilon_2} B^{-D} A^{-\varepsilon_1} {\overline{S}_*}^{-1}.
\end{aligned}$$

If $\alpha_{i,1} = -1$, then $\overline{U_i^{-1}\mathring{U}_i}$ contains less $O_0^{\pm 1}$ than $\overline{\hat{U}}_i$.

If $\alpha_{1,1} = 1$ (i.e. $F_2 = F$), then

$$\begin{aligned}\overline{U_i^{-1}\mathring{U}_i} &= \overline{S}_*{}^{-1}\overline{S}_* A^{\varepsilon_1} B^D A^{\varepsilon_2} \overline{S}_{i,n_i}{}^{-1} \dots \overline{S}_{i,2}^{-1} A^{-\varepsilon_2} B^{-D} A^{-\varepsilon_1} \overline{S}_*{}^{-1} \overline{S}_{i,1}^{-1} \overline{S}_{i,1} \\ &= A^{\varepsilon_1} B^D A^{\varepsilon_2} \overline{S}_{i,n_i}{}^{-1} \dots \overline{S}_{i,2}^{-1} A^{-\varepsilon_2} B^{-D} A^{-\varepsilon_1} \overline{S}_*{}^{-1}\end{aligned}$$

and
$$\overline{\hat{U}}_i = -\overline{S}_{i,1}\overline{S}_* A^{\varepsilon_1} B^D A^{\varepsilon_2} \overline{S}_{i,2} \dots \overline{S}_{i,n_i} A^{-\varepsilon_2} B^{-D} A^{-\varepsilon_1} \overline{S}_*^{-1}.$$

It is easy to see that $S_* S_{i,1} \neq e$ since the path corresponding to $S_* S_{i,1}$ on Δ is non-trivial. It follows that $\overline{S}_*\overline{S}_{i,1} \neq e$, which is equivalent to $\overline{S}_{i,1}\overline{S}_* \neq e$. This shows using Lemma 5.9 (b) as well that $\overline{\hat{U}}_i$ is the longest summand of $\overline{M}_i$ again.

By the symmetry of paths between P and Q we may assume that if $\alpha_{i,1} = 1$ for $i = 1$ or 2, then $\lg(S_{i,1}) \geq \lg(S_*)$. This gives that $S_{i,1}S_*[1] = S_{i,1}[1] \neq e$ if $\alpha_{i,1} = 1$.

We may assume that $(P, F^{-1}(P)) \notin E(\Delta)$ so if $S_{i,1} = e$ for $i = 1$ or 2, then $\alpha_{i,1} = -1$ since otherwise this case goes back to case (a). This also implies that $S_{1,1} = e$ and $S_{2,1} = e$ cannot hold at the same time.

Let us assume that neither $S_{1,1}$ nor $S_{2,1}$ is e. We also have $S_{1,1}[1] \neq S_{2,1}[1]$ since the corresponding paths end in P. Therefore for suitable choice of the sign of the exponents i_1 and i_2 we may assume that $\overline{S}_{1,1}[1] = A^{e_1} = \overline{\hat{U}}_1[1]$ and $\overline{S}_{2,1}[1] = A^{e_2} = \overline{\hat{U}}_2[1]$ with $e_1 e_2 < 0$. It is easy to see that $\overline{\hat{U}}_1[-1] = \overline{\hat{U}}_2[-1]$ since $S_* \neq e$ so exactly one of $\overline{\hat{U}}_1$ and $\overline{\hat{U}}_2$ has shorter conjugate.

Let us assume that $S_{1,1} = e$. We have already proved that $S_{2,1} \neq e$ and $\alpha_{1,1} = -1$ in this case. Then $\overline{\hat{U}}_1[1] = A^{-\varepsilon_2}$. Since $\hat{U}_1[-1], \hat{U}_2[1]$ and $\hat{U}_2[-1]$ are in H' we have that for any choice of i_1, i_2 for $i = 1, \dots, m$ we may choose ε_2 such that exactly one of $\overline{\hat{U}}_1$ and $\overline{\hat{U}}_2$ has shorter conjugate.

Similar result can be proved if $S_{2,1} = e$ so for suitable substitution we have $(1 - p_2)\overline{M}_1 \doteq (1 - p_1)\overline{M}_2$.

Case (e): Let us assume that this case does not originate in case (a). It implies that $\alpha_{1,1} = \alpha_{2,1} = -1$ if F_3 is not on the (P, Q) path. We have already assumed that F_1 is not the only $F^{\pm 1}$ on the (P, Q) path and the last $F^{\pm 1}$ is F^{-1} so if F_3 is on (P, Q) path, then $F_3 = F^{-1}$ again.

Therefore
$$U_i = S_{i,1} O_0{}^{-1} S_{i,2} O_0{}^{\alpha_{i,2}} \cdots S_{i,n_i} O_O S_i^*.$$

As in the previous cases we write $S_* = S_1^* = S_2^*$. Again we substitute $O_i = A^{i_1} B^{i_3} A^{i_2}$ for $i = 1, \dots, m$, where the absolute value of the exponents are pairwise different integers and $|i_3| \geq |i_1|, |i_2|$ and $O_0 = A^{\varepsilon_1} B^D A^{\varepsilon_2} \overline{S}_*^{-1}$, where D is as large as in Lemma 5.9 (b). By Lemma 5.9 (b) there are two possible choices for the longest term in $\overline{M}_1$. One of them is

$$\overline{U}_1^{-1}\overline{S}_{1,1}\overline{O}_0^{-1} = \overline{S}_*^{-1}\overline{O}_0^{-1}\overline{S}_{1,n_1}^{-1}\dots\overline{S}_{1,2}^{-1}\overline{O}_0\overline{S}_{1,1}^{-1}\overline{S}_{1,1}\overline{O}_0^{-1},$$

which equals to

$$\overline{S}_*^{-1}\overline{S}_*A^{-\varepsilon_2}B^{-D}A^{-\varepsilon_1}\overline{S}_{1,n_1}^{-1}\dots\overline{S}_{1,2}^{-1} = A^{-\varepsilon_2}B^{-D}A^{-\varepsilon_1}\overline{S}_{1,n_1}^{-1}\dots\overline{S}_{1,2}^{-1}.$$

The other one is

$$\overline{S}_{1,1}\overline{O}_0^{-1}\overline{S}_{1,2}\dots\overline{S}_{1,n_1} = \overline{S}_{1,1}\overline{S}_*A^{-\varepsilon_2}B^{-D}A^{-\varepsilon_1}\overline{S}_{1,2}\dots\overline{S}_{1,n_1}.$$

Since both $S_{1,1}^{-1}$ and S_* starts at P we have $S_{1,1}^{-1} \neq S_*$ hence

$$L_1 = \overline{S}_{1,1}\overline{S}_*A^{-\varepsilon_2}B^{-D}A^{-\varepsilon_1}\overline{S}_{1,2}\dots\overline{S}_{1,n_1}$$

is the longest term of $\overline{M}_1$. Similarly, the longest term of $\overline{M}_2$ is

$$L_2 = \overline{S}_{2,1}\overline{S}_*A^{-\varepsilon_2}B^{-D}A^{-\varepsilon_1}\overline{S}_{2,2}\dots\overline{S}_{2,n_2}.$$

We have already also assumed that F_1 is not the only F or F^{-1} on the (P,Q)-path so $\overline{S}_{2,n_2} = \overline{S}_{1,n_1}$. Further, $S_{1,1}[1]$ and $S_{2,1}[1]$ are different since their tail is P and. The assumption that F_1 is the closest to P among F_1, F_2, F_3 shows that $L_i[1] = \overline{S_{i,1}S_*}[1] = \overline{S}_{i,1}[1]$ for $i = 1,2$. Therefore we may choose the exponents i_1 and i_2 such that exactly one of L_1 and L_2 has shorter conjugate in K. This gives that for exactly one of $\overline{M}_1$ and $\overline{M}_2$ takes its degree in the main diagonal, which gives again $(1-p_2)(\overline{M}_1) \neq (1-p_1)(\overline{M}_2)$, finishing the proof of Proposition 5.7.

□

6 Construction of the congruent pieces in high dimension

6.1 A set of symmetries of the regular simplex

In this section we select isometries satisfying the conditions of Lemma 5.1. Therefore, this set of isometries gives a decomposition of the balls (either open or closed) in $\mathbb{R}^d$, where $d = 4$ or $d = 2s$ with $s \geq 4$.

We denote by $|v|$ the standard Euclidean norm of a vector $v \in \mathbb{R}^d$ and we use the induced norm $\|M\| = \sup_{v\neq 0}\frac{|Mv|}{|v|}$ for $M \in SO(d)$.

Remark 6.1. *Suppose that $\phi_i \in SO(d)$ $(i = 1,\dots,k)$ are orthogonal transformations. Then for every $\varepsilon > 0$ and for every $i \in \mathbb{N}$ there exists $O_{i,j} \in SO(d)$ for $j = 1,\dots,l$ such that $\|\phi_i - O_{i,j}\| < \varepsilon$ and the matrices $O_{i,j}$ are independent.*

Proof. It is easy to see that the parametrization $\alpha_d : \Omega \to SO(d)$ is a continuous function of $\omega \in \Omega$, where the topology on $SO(d)$ is defined by the induced norm. There exists an everywhere dense subset of Ω whose elements are algebraically independent over $\mathbb{Q}$, finishing the proof of the Remark 6.1. $\square$

Definition 6.2. *Let $A_1, \ldots, A_{d+1}$ be the vertices of a regular simplex S_d such $A_1, \ldots, A_{d+1}$ are in the boundary of the unit ball B_d. For $k = 1, \ldots, d+1$, let H_k denote the affine hyperplane containing A_i for every $i \neq k$. Let $\mathcal{A}'_k$ denote the set of A_i which is contained by H_k. For instance, $\mathcal{A}'_1 = \{A_2, \ldots, A_{d+1}\}$.*

Let O denote the origin of the unit ball B_d of dimension $d \geq 2$. It is easy to see that H_k is perpendicular to the vector $\overrightarrow{OA_k}$.

Lemma 6.3. *Let H_1 and H_k be two affine hyperplanes as in Definition 6.2. Then there is a $\phi_k \in SO(d)$, $k = 1, \ldots, d+1$ such that $\phi_k(H_1) = H_k$. Furthermore $\phi_k(\mathcal{A}'_1) = \mathcal{A}'_k$.*

Proof. It is enough to show that the statement is true for $k = 2$. It is easy to check that there is a reflection $r \in O(d)$ which fixes the points $A_3, \ldots, A_{d+1}$ and maps A_1 to A_2. Clearly, r is not in $SO(d)$. Therefore, we take another reflection r' which fixes the points $A_1, A_2, \ldots, A_{d-1}$ and maps A_d to A_{d+1}. Then the composition $\phi_2 = r \circ r'$ $SO(d)$ and $\phi_k(\mathcal{A}'_1) = \mathcal{A}'_2$. $\square$

The image $\mathcal{I}_x$ of $x \in \mathbb{R}^d$ was defined in Section 5. For the multiset $\mathcal{I}_x$ we write $\mathcal{I}_x \subset H \subset \mathbb{R}^d$ if and only if every element of $\mathcal{I}_x$ is in H.

Lemma 6.4. *Let $\phi_k \in SO(d)$ $(k = 2, \ldots, d+1)$ as in Lemma 6.3 and we fix $\phi_1 = id$. Let $T_b(x) = x + b$, where $b \in \mathbb{R}^d$ with $t = |b| < \frac{2}{3d+4}$. Then every point $x \in B_d$ has a preimage $y = \phi_j^{-1}(x)$ for some $j = 1, \ldots, d+1$ such that for every $z \in B(y,t) \cap B_d$ the multiset $\mathcal{I}_z \subset B_d$.*

Proof. We write a vector $u \in \mathbb{R}^d$ as $u = (u_1, u_2, \ldots, u_d)$.

We may assume that $A_1 = (0, 0, \ldots, 0, 1)$, where A_1 is a vertex of the simplex given in Definition 6.2 and the vector b and $\overrightarrow{OA_1}$ have the same direction. Since $\phi_1, \phi_2, \ldots, \phi_{d+1}$ are orthogonal transformations, in order to verify for some $z \in B_d$ that $\mathcal{I}_z = \{\phi_1(x), \phi_2(x), \ldots, \phi_{d+1}(x), T_b(x)\} \subset B_d$ it is enough to verify that $T_b(z) = z + b \in B_d$. It is easy to see that if $z \in B_d$ with $z_d < -\frac{t}{2}$, then $z + b \in B_d$.

Every affine hyperplane H_k divides B_d into two parts. Let F_k denote the part containing the simplex S_d and E_k denote the other one.

We denote by $B_d^{1-2t} = \{x \in \mathbb{R}^d : |x| < 1 - 2t\}$, A_k^{1-2t} and H_k^{1-2t} the objects what we get from A_k and H_k by contracting B_d with ratio $1-2t$ from the origin 0, respectively. The affine hyperplane H_k^{1-2t} divides B_d into two parts. We denote by F^{1-2t} and E^{1-2t} the two parts of B_d which contains and which does not contain the contracted simplex, respectively.

If $x \in B_d^{1-2t}$, then we choose $\phi_1 = id$ so $y = x$. It is easy to see that if $z \in B(y,t) \cap B_d = B(x,t) \cap B_d$, then $z + b \in B_d$.

Since the average of the coordinates of the points of the simplex S_d is 0 we have that the last coordinate of $A_2, \ldots, A_{d+1}$ is $-\frac{1}{d}$. And similarly, since the last coordinate of A_1^{1-2t} is $1-2t$, the last coordinate of A_k^{1-2t} $(k=2,\dots,d+1)$ is $-\frac{(1-2t)}{d}$.

If $x \in E_k^{1-2t}$, where k is not necessarily unique, then we choose ϕ_k. Lemma 6.2 gives that $y \in E_1'$. Since the last coordinate of the point in E_1^{1-2t} is smaller than $-\frac{(1-2t)}{d}$ and $t = |b| < \frac{2}{3d+4}$, we have $z_d \leq -\frac{(1-2t)}{d} + t < -\frac{t}{2}$ for every $z \in B_d \cap B(y,t)$. Therefore, $z + b \in B_d$.

Clearly, $B_d \subseteq \bigcup_{k=1}^{d+1} E_k^{1-2t} \cup B_d^{1-2t}$, finishing the proof of Lemma 6.4. $\square$

Remark 6.5. For $x \in B_d$, in order to find a preimage y which is a core point we use Lemma 6.4 and besides, we guarantee that the elements of $\mathcal{I}_y$ are different.

6.2 Proof of Theorem 1.1

Now we can complete the proof of Theorem 1.1.

We construct $4(2s+1)+1$ maps satisfying the conditions of Lemma 5.1, where the bijections are orthogonal transformations of $SO(d)$ with $d = 2s \geq 4$ and $d \neq 6$. In this case the d dimensional unit ball can be decomposed into $4(2s+1)+2$ pieces.

According to Lemma 6.4, there exist orthogonal transformations $\phi_i \in SO(d)$ $(i = 1,\dots,d+1)$ and $|b| = t < \frac{2}{3d+4}$ which have the property that every point has a preimage y such that for every $z \in B(y,t)$ we have $\mathcal{I}_z \subset B_d$.

By Lemma 6.1, for every $1 \leq i \leq 2s+1$ there exist independent orthogonal transformations $O_{i,1}, O_{i,2}, O_{i,3}$ and $O_{i,4}$ such that $\|O_{i,j} - \phi_i\| < t$ and $\|O_0 - id\| < t$. Furthermore, we assume that $O_{i,j}$, O_0 and b form an independent system with respect to the standard basis of $\mathbb{R}^d$.

By choosing a suitable orthonormal basis in $\mathbb{R}^d$, we may use Lemma 6.4. If $x \neq x_0 = 0$, then there exists an i such that for every $j = 1,\dots,4$ we have $\mathcal{I}_y \subset B_d$ for $y = O_{i,j}^{-1}(x)$. By Remark 6.5, if we can guarantee that $\mathcal{I}_y$ consists of different points, then y is a core point.

Lemma 4.4 shows that $O_{i_1,j_1}(y) \neq O_{i_2,j_2}(y)$ if $y \neq 0$, which is the case since $x \neq 0$. The only case which remains is that

$$O_{k,l}(y) = O_0(y) + b \tag{14}$$

for some y and $O_{k,l}$, where $k \in \{1,\dots,d+1\}, l \in \{1,\dots,4\}$.

Lemma 6.6. *Let $O_{m,n}$ $(m = 1,\dots,2s+1,\ n = 1,\dots,4)$, O_0 and b an independent system in $\mathbb{R}^d$, where $d = 2s$. For every $x \neq 0$, there exists at most one pair of linear transformations $O_{i,j}$ and $O_{k,l}$ such that for the point $y = O_{i,j}^{-1}(x)$ the equation $O_{k,l}(y) = O_0(y) + b$ might be satisfied.*

Proof. Let us assume that for some $0 \neq x \in B_d$ we have

$$O_{k_1,l_1}(y_1) = O_0(y_1) + b \quad \text{and} \quad O_{k_2,l_2}(y_2) = O_0(y_2) + b,$$

where $y_1 = O_{i_1,j_1}^{-1}(x)$ and $y_2 = O_{i_2,j_2}^{-1}(x)$. Thus, we get

$$(O_{k_1,l_1} - O_0)(O_{i_1,j_1}^{-1}(x)) = b \text{ and } (O_{k_2,l_2} - O_0)(O_{i_2,j_2}^{-1}(x)) = b. \tag{15}$$

Using again Lemma 4.4 we get that $O_0^{-1}O_{i,k} - I$ is invertible. Therefore equation (15) can be written in the form

$$x = O_{i_1,j_1}(O_0^{-1}O_{k_1,l_1} - I)^{-1}O_0^{-1}b \text{ and } x = O_{i_2,j_2}(O_0^{-1}O_{k_2,l_2} - I)^{-1}O_0^{-1}b.$$

Hence

$$O_{i_1,j_2}(O_0^{-1}O_{k_1,l_1} - I)^{-1}O_0^{-1}b = O_{i_2,j_2}(O_0^{-1}O_{k_2,l_2} - I)^{-1}O_0^{-1}b.$$

Since $O_{m,n}, O_0$ and b form an independent system, we can eliminate b from the previous equation, and we get the following:

$$O_{i_1,j_1}(O_0^{-1}O_{k_1,l_1} - I)^{-1}O_0^{-1} = O_{i_2,j_2}(O_0^{-1}O_{k_2,l_2} - I)^{-1}O_0^{-1}. \tag{16}$$

Using Lemma 4.2, we may substituting $O_0 = id$ and we get

$$O_{i_1,j_1}(O_{k_1,l_1} - I)^{-1} = O_{i_2,j_2}(O_{k_2,l_2} - I)^{-1}. \tag{17}$$

If $O_{i_1,j_1} = O_{i_2,j_2}$ or $O_{k_1,l_1} = O_{k_2,l_2}$, then it is clear from equation (17) that $O_{i_1,j_1} = O_{i_2,j_2}$ and $O_{k_1,l_1} = O_{k_2,l_2}$.

Thus we can assume that $O_{i_1,j_1} \neq O_{i_2,j_2}$ and $O_{k_1,l_1} \neq O_{k_2,l_2}$. Then we substitute such that $\overline{O}_{i_1,j_1} = \overline{O}_{i_2,j_2}$. This implies $\overline{O}_{k_1,l_1} = \overline{O}_{k_2,l_2}$. Hence either $O_{i_1,j_1} = O_{k_1,l_1}$ and $O_{i_2,j_2} = O_{k_2,l_2}$, or $O_{i_1,j_1} = O_{k_2,l_2}$ and $O_{i_2,j_2} = O_{k_1,l_1}$.

1. Let us assume first that $O_{i_1,j_1} = O_{k_1,l_1}$ and $O_{i_2,j_2} = O_{k_2,l_2}$. We shortly denote O_{i_1,j_1} by U and we substitute $O_{i_2,j_2} = U^2$. From equation (17) we get
$$U(U - I)^{-1} = U^2(U^2 - I)^{-1}.$$
This gives
$$U^2 - I = (U - I)(U + I) = (U - I)U,$$
which is a contradiction since $U - I$ is invertible by Lemma 4.4.

2. Let us assume that $O_{i_1,j_1} = O_{k_2,l_2}$ and $O_{i_2,j_2} = O_{k_1,l_1}$. Then we denote $U = O_{i_1,j_1}$ and we substitute $O_{i_1,j_1} = U^2$ again. Similar calculation gives
$$I = (U + I)U.$$
This gives $U^2 + U - I$, which is a polynomial expression, contradicting Lemma 4.2.

This shows that equation (16) holds if and only if $\{i_1, j_1\} = \{i_2, j_2\}$ and $\{k_1, l_1\} = \{k_2, l_2\}$, finishing the proof of Lemma 6.6. □

For every $x \neq 0$ we have already found $O_{i,1}$, $O_{i,2}$, $O_{i,3}$ and $O_{i,4}$ such that $\mathcal{I}_{y_j} \subset B_d$, where $y_j = O_{i,j}^{-1}(x)$ for every $j = 1, \dots, 4$. By Lemma 6.6 at least three of y_j is a core point satisfying Lemma 5.1 (b).

If x_0 is the origin it has to satisfy condition Lemma 5.1 (a). Therefore we need to guarantee that $F^{-1}(x_0) = F^{-1}(0)$ is a core point. Indeed, if $y = F^{-1}(0) \neq$

0, then $O_{i_1,k_1}(y) \neq O_{i_2,k_2}(y)$ holds again by Lemma 4.4. Clearly, $F(y) = 0$, thus $O_{k,l}(y) = F(y)$ cannot hold for any k and l. Due to the choice of b we have $\mathcal{I}_y \subset B_d$.

Since the matrices O_0, $O_{i,j}$ and b is an independent system and the dimension $d = 2s \geq 4$ and $s \neq 3$, we can use Theorem 5.4. Thus the graph $\Gamma_{\mathcal{F}}$ has the property (2) for $\mathcal{F} = \{F, O_{m,n}\}$ $(m = 1, \dots, 2s+1, n = 1, \dots, 4)$, where $F = T_b O_0$.

We conclude that $\mathcal{F}$ satisfies the conditions of Lemma 5.1, finishing the proof of Theorem 1.1.

□

Remark 6.7.

1. The proof above gives a construction for $m = 4\cdot(2s+1)+1 = 4d+5$ pieces in dimension $d = 2s$. We can easily obtain a construction for $m > 4d+5$, since we can add any finite number of orthogonal transformations with algebraically independent parameters to the already defined ones, which satisfy the conditions of Lemma 5.1.
2. Most probably, this bound $4d+5$ is practically not the best but this construction of Section 6 cannot be modified without difficulties.

7 Decomposition in higher dimension

In this section we prove Theorem 1.3.
In [2] the authors proves the following theorem

Theorem 7.1. *The $3s$ dimensional ball can be decomposed into finitely many pieces for every $s \in \mathbb{Z}^+$.*

This shows that there is a decomposition for $d = 6$ and $d = 9$. In order to prove Theorem 1.3, by Theorem 1.1 it is enough to prove it when $d \geq 7$ is odd and $d \neq 9$. Such an integer can be written in the form $d = d' + 3$. Then we write the elements x of $\mathbb{R}^d$ as $x = (y, z)$, where $y \in \mathbb{R}^{d'}$ and $z \in \mathbb{R}^3$, where $d' \geq 4$ is even and $d' \neq 6$.

We shall recall some of the results of [2] for the 3 dimensional case using our notation.

The following lemma is essentially the same as [2, Lemma 3.5].

Lemma 7.2. *Suppose that $O'_0, O'_1, \dots, O'_m \in SO(3)$ and $b \in \mathbb{R}^3$ form an independent system. Let $F = T_b O'_0$ and $\mathcal{F} = \{F, O'_1, \dots, O'_m\}$. If $\mathcal{C}$ is a cycle in $\Gamma_{\mathcal{F}}$, then the corresponding word does not contain the letter F or F^{-1}.*

We remind that a cycle has distinct points aside from the first and the last vertices of it which coincide. We refer to [2, Lemma 4.1] which states the following.

Lemma 7.3. *Suppose that $O'_0, O'_1, \dots, O'_m \in SO(3)$ and $b \in \mathbb{R}^3$ form an independent system. Let $\mathcal{F} = \{T_b O'_0, O'_1, \dots, O'_m\}$. Then $\Gamma_{\mathcal{F}}$ has property (2).*

Finally, a version of Lemma 4.2. in [2] states following

Lemma 7.4. *Suppose that $O'_1, \dots, O'_m \in SO(3)$ are independent orthogonal transformations. Then for every $0 \neq x \in \mathbb{R}^3$ there are at most two elements of the form O'_i such that $I_{O'^{-1}_i(x)}$ does not consist of different points.*

Our aim is to construct d dimensional special orthogonal transformations satisfying the conditions of Lemma 5.1.

Let b_1 be a vector in $\mathbb{R}^{d'}$. Let $\phi_1, \phi_2, \dots, \phi_{d'+1} \in SO(d')$ as in Lemma 6.4 with the additional assumption that $\overrightarrow{OA_1}$ and b_1 have the same direction. For every $1 \leq i \leq d'+1$ we choose 20 orthogonal transformations $O_{i,j}$ $(j = 1, \dots, 20)$ such that $\|O_{i,j} - \phi_i\| \leq \varepsilon'$ for some $\varepsilon' > 0$. Let $O_0 \in SO(d')$ satisfies $\|O_0 - I_{d'}\| \leq \varepsilon'$, where I_n denotes the n dimensional identity matrix and let $F = T_{b_1} O_0$. We assume that $O_0, O_{i,j}$ and b_1 form an independent system.

Let $\phi'_1, \phi'_2, \phi'_3, \phi'_4 \in SO(3)$ as in Lemma 6.4 and $b_2 \in \mathbb{R}^3$. We assume again that for one of the points A'_1 of the 3 dimensional simplex, the vector $\overrightarrow{OA'_1}$ and b_2 have the same direction. We denote by $1 \leq j' \leq 4$ the integer such that $j \equiv j' \mod 4$. For every $1 \leq j \leq 20$ we choose $O'_{i,j} \in SO(3)$ for $(1 \leq i \leq d'+1)$ such that $\|O'_{i,j} - \phi'_{j'}\| \leq \varepsilon'$ and let $\|O'_0 - I_3\| < \varepsilon'$ and let $F' = T_{b_2} O'_0$. We assume again that $O'_0, O'_{i,j}$ and b_2 form an independent system, where $b_2 \in \mathbb{R}^3$.

We define the orthogonal transformations $\hat{O}_0$ and $\hat{O}_{i,j} \in SO(d)$ by

$$\hat{O}_0(x, y) = (O_0(y), O'_0(z)) \text{ and } \hat{O}_{i,j}(y, z) = (O_{i,j}(y), O'_{i,j}(z))$$

and for $b = (b_1, b_2) \in \mathbb{R}^d$ let

$$\hat{F} = T_b \hat{O}_0(x) = (T_{b_1} O_0(y), T_{b_2} O'_0(z)).$$

One can see that there exists some ε depending only on ε' and d such that for $\Phi_{i,j} = (\phi_i, \phi'_j) \in SO(d)$ $(i = 1, \dots, d+1,\ j = 1, \dots, 4)$ there are at least 5 orthogonal transformations of the form $\hat{O}_{i,j}$ such that $\|\hat{O}_{i,j} - \Phi_{i,j}\| < \varepsilon$ and if ε' tends to 0 then ε tends to 0 as well.

Lemma 7.5. *The graph $\Gamma_{\hat{\mathcal{F}}}$ has property* (2)*, where* $\hat{\mathcal{F}} = \left\{\hat{F}, \hat{O}_{1,1}, \dots, \hat{O}_{d'+1,20}\right\}$.

Proof. We claim that if $\mathcal{C}$ is a cycle in $\Gamma_{\hat{F}}$, then for every vertex $(y, z) \in \mathbb{R}^d$ on the cycle $\mathcal{C}$ we have $y = 0$. One can easily assign to $\mathcal{C}$ a word W by identifying the vertices by the letters $\hat{O}_{i,j}$ and $\hat{F}$. Clearly, W is a reduced word.

Let W' be the restriction of W to the last three coordinates. Lemma 7.2 shows that if W' contains $F^{\pm 1}$, then W' does not have a fixed point. Hence W does not contain the letter $\hat{F}$ or $\hat{F}^{-1}$. In this case W'' which is the restriction of W to the first d' coordinates can be identified by an element of the group K which is not the identity element. Lemma 4.4 shows that the only fixed point of W'' is 0 so $y = 0$ for each vertex (y, z) of $\mathcal{C}$.

Thus if $\Gamma_{\hat{F}}$ contains two cycles $\mathcal{C}_1$ and $\mathcal{C}_2$ sharing an edge, then $\mathcal{C}_1$ and $\mathcal{C}_2$ can be considered as cycles in $\mathbb{R}^3$ in the graph $\Gamma_{\mathcal{G}}$, where $\mathcal{G} = \{F', O'_1, \dots, O'_m\}$. In this case [2, Lemma 2.1] shows that $\mathcal{C}_1$ and $\mathcal{C}_2$ coincide, finishing the proof of Lemma 7.5. $\square$

We prove that $x_0 = 0$ satisfies the conditions given in Lemma 5.1 with respect to the graph $\Gamma_{\mathcal{F}'}$.

It is easy to see that for $\hat{F}^{-1}(x_0) = \hat{F}^{-1}(0) = (y_0, z_0)$ we have $0 \neq y_0 \in \mathbb{R}^{d'}$ which shows that $\hat{O}_{i,j}((y_0, z_0)) \neq \hat{O}_{k,l}((y_0, z_0))$ if $(i,j) \neq (k,l)$ using Lemma 4.4. Moreover these points differ from $\hat{F}(\hat{F}^{-1})(0) = 0$ hence $\hat{F}^{-1}(0)$ is a core point if $|b| < 1$.

Lemma 7.6. *For every $0 \neq x = (y,z) \in \mathbb{R}^d$ there are at most two $\hat{O}_{i,j}$ such that $\mathcal{I}_{\hat{O}^{-1}_{i,j}(x)}$ does not consist of different elements.*

Proof. If $y \neq 0$, then Lemma 6.6 shows that there is at most one $\hat{O}_{i,j}$ such that the elements of $\mathcal{I}_{\hat{O}^{-1}_{i,j}(x)}$ are not different. Lemma 7.2 shows that $O'_{i,j}(v) \neq T_{b_2}O'_0(v)$ for every $v \in \mathbb{R}^3$ so it can only happen that for $(0,v) = \hat{O}^{-1}_{i,j}(x) \in \mathbb{R}^d$ we have $\hat{O}_{k_1,l_1}(0,v) = \hat{O}_{k_2,l_2}(0,v)$ for some $1 \leq k_1, k_2 \leq d'$ and $1 \leq l_1, l_2 \leq 20$. Lemma 7.4 gives that there at most two $\hat{O}_{k,l}$ with this property. □

One can see from Lemma 7.6 that in order to verify that every $0 \neq x = (y,z) \in B_d$ is the image of at least three core points we only have to show that for at least five (i,j) pair $\mathcal{I}_{\hat{O}^{-1}_{i,j}(x)} \subset B_d$.

We will compare the length of vectors of different dimension so we denote by $|v|_k$ the length of the vector $v \in \mathbb{R}^k$. Now we assume that $|b_1|_{d'} = |b_2|_3 = t$. Choosing a suitable basis again, we assume that $b = (b_1, b_2)$, where $b_1 = (0, \ldots, 0, 0, t) \in \mathbb{R}^{d'}$ and $b_2 = (0,0,t) \in \mathbb{R}^3$ and $A_1 \in \mathbb{R}^{d'}$, which is a vertex of the simplex defined in Lemma 6.2, is just $\underbrace{(0, \ldots, 0, 1)}_{d'}$.

Lemma 7.7. *(a) Let $x \in \mathbb{R}^k$ with $|x|_k > \frac{1}{3}$. Then there exists $\phi_i \in SO(k)$ as in Lemma 6.2 such that for $\phi_i(x) = (y_1, \ldots, y_k)$ we have $y_k \leq -\frac{1}{3k}$. Moreover, if $|c|_k = |(0, \ldots, 0, c_k)|_k \leq \frac{1}{6k}$, then for every $u \in B(\phi_i(x), |c|_k)$ with $|u|_k = |x|_k$ we have $u_k \leq -\frac{1}{6k}$ and $|u + c|_k \leq |x|_k$.*

(b) For every $\varepsilon > 0$ there exists $r > 0$ such that if $|c| < r$, then for every $x \in B_k \setminus B_k^{\varepsilon}$ there exists $\phi_i \in SO(k)$ such that $|u + c|_k \leq |x|_k$ for every $u \in B(\phi_i(x), |c|_k)$ with $|u|_k = |x|_k = |\phi_i(x)|_k$.

Proof.

(a) Clearly, $B_k \setminus B_k^{\frac{1}{3}} \subset \cup_{i=1}^{k+1} E_i^{\frac{1}{3}}$, where $E_i^{\frac{1}{3}}$ denotes the intersection of a half-plane with B_k as in Lemma 6.4. Hence x is contained in $E_i^{\frac{1}{3}}$ for some $1 \leq i \leq k+1$. Since $\phi_i(E_i^{\frac{1}{3}}) = E_1^{\frac{1}{3}}$, the last coordinate of $\phi_i(x)$ is smaller than or equal to $-\frac{1}{3k}$. Finally, one can easily verify that the last coordinate of z is smaller than $-\frac{|c|_k}{2}$, which gives that $|u + c|_k \leq |u|_k = |x|_k$.

(b) Using the same argument again, we may assume that $\phi_i(x) \in E_1^{\varepsilon}$. This shows that the last coordinate of $u = (u_1, \ldots, u_k)$ is smaller than or equal to $-\varepsilon\frac{1}{k} + |c|_k$. If $|c|_k$ is small enough, then $u_k < -\frac{|c|_k}{2}$ which guarantees that $|u + c|_k \leq |u|_k$. □

Lemma 7.8. *If $\sqrt[d]{2t^2} = |b|_d = |(0,\dots,0,t,0,0,t)|$ is small enough, then for every $x = (y,z) \in \mathbb{R}^d$ there exists $\Psi = (\phi_i, \phi'_j)$ such that $u + b \in B_d$ for every $u = (u_1, u_2) \in B(\Psi(x), |b|)$ with $|u_1|_{d'} = |y|_{d'}$ and $|u_2|_3 = |z|_3$.*

Proof. It is easy to see that if $|x|_d < 1 - |b|_d$, then $u + b \in B_d$. Therefore, if $|b|_d < \frac{1}{3}$, then we may assume that either $|y|_{d'}$ or $|z|_3$ is greater than $\frac{1}{3}$ since $\frac{\sqrt{2}}{3} + \frac{1}{3} < 1$.

Let us assume first that $|y|_{d'} \geq \frac{1}{3}$. Let $\theta_{d'}$ be a negative number what we will define later and let $\varepsilon = -\frac{\theta_{d'}}{6}$. By Lemma 7.7 (b) there exists $r > 0$ such that if $|b_2|_3 < r$, then for every fixed $w = (w_1, w_2) \in \mathbb{R}^{d'+3}$ with $|w_2|_3 > \varepsilon$ there exists ϕ'_j such that for every $u_2 \in B(\phi'_j(w_2), |b_2|_3)$ with $|u_2|_3 = |w_2|_3$ we have $|\phi'_j(u_2) + b_2|_3 \leq |w_2|_3$. Therefore, if $|y|_{d'} \geq \frac{1}{3}$ and $|z|_3 \geq \varepsilon$, then there exists a d dimensional orthogonal transformation of the form $\Psi = (\phi_i, \phi'_j)$ such that for every if $u = (u_1, u_2) \in \mathbb{R}^d$ with $|u_1|_{d'} = |y|_{d'}$, $|u_2|_3 = |z|_3$ and $u \in B(\Psi(x), |b|)$, then $u + b \in B_d$. Thus we may assume $|z|_3 < \varepsilon$.

We show that $|u_1 + b_1|^2_{d'} + |u_2 + b_2|^2_3 \leq |y|^2_{d'} + |z|^2_3$. This is equivalent to $|b_1|^2_{d'} + |b_2|^2_3 \leq -2u_1 b_1 - 2u_2 b_2$, where the product of two vectors is the standard inner product. Using $|b_1| = |b_2| = t$ we get

$$t \leq -|u_1|_{d'} \cos\tau_1 - |u_2|_3 \cos\tau_2, \tag{18}$$

where τ_1 and τ_2 denotes the angle between b_1 and u_1 and between b_2 and u_2, respectively. Lemma 7.7 (a) gives that the last coordinate u_1 is smaller than $-\frac{1}{6d'}$ so $\cos\tau_1$ can be estimated from above by a number $\theta_{d'} = -\frac{1}{3}\frac{1}{6d'}$ which only depends on d. Thus

$$-|u_1|_{d'} \cos\tau_1 - |u_2|_3 \cos\tau_2 \geq -\frac{1}{3}\theta_{d'} - |u_2|_3 \geq -\frac{1}{3}\theta_{d'} + \frac{1}{6}\theta_{d'} = -\frac{1}{6}\theta_d. \tag{19}$$

It is easy to see that this last term in equation (19) is a positive number which only depends on d so for suitable choice of t combining with the previous conditions for $|b| = \sqrt[d]{2t^2}$ we have that equation (18) holds, finishing the proof of Lemma 7.8. $\square$

Lemma 7.5, Lemma 7.6 and Lemma 7.8 imply that for every $x \in B_d$ there are at least five $\overline{O}_{i,j}$ such that $\mathcal{I}^{-1}_{\overline{O}_{i,j}}(x) \subset B_d$ and clearly, three of them are core points. We conclude that all the conditions of Lemma 5.1 are satisfied, finishing the proof of Theorem 1.3. $\square$

Now we collect the results on the number of pieces required for the decomposition in different dimensions to prove Theorem 1.4.

Proof of Theorem 1.4:
We distinguish 3 major cases.

1. If $d = 3, 6$ or 9 then B_d can be decomposed into finitely pieces by [2, Theorem 1].

2. If $d = 2s \geq 4$, where $s \geq 2$ and $s \neq 3$, then the number of orthogonal transformations is $4(2s+1)+1 = 4d+5$ and hence B_d can be decomposed into $(4d+5)+1 = 4d+6$ pieces, by Theorem 1.2.

3. If $d = 2s + 3$, where $s \geq 2$ and $s \neq 3$, then the number of orthogonal transformations is $5(4(2s+1)) + 1 = 20(d-2) + 1$ and hence B_d can be decomposed into $(20(d-2)+1)+1 = 20d - 38$ pieces, by Theorem 1.3.

Therefore the number of pieces is asymptotically, less than or equal to $20d$ if d.

8 Problems and results in dimension d

In dimension $d = 2$ the transformation group O_2 does not contain noncommutative free subgroups, thus the methods worked out in [2] and in this article and cannot say anything about the divisibility of the discs. C. Richter posed a question about decomposition of the disc using affine transformations instead of orthogonal transformations. A celebrated result of von Neumann shows that the group of affine transformations contains noncommutative free subgroups. In this case, the main difficulty is to satisfy the conditions (b) of Lemma 5.1. We do not know whether or not Richter's problem can be solved along these lines.

By Theorem 1.2, the minimal number of pieces τ_d which is needed to decompose B_d, is less than $20d$ for $d \geq 10$. The main result of [7], which was reproved in [6], shows that $\tau_d > d$. Thus we get that $\tau_d = \Theta(d)$, which is best possible in some sense. This widely improves the upper bound of τ_d for $d = 3s$ given in [2], where it was shown that $\tau_d \leq \exp(c_1 d \log d)$ for a positive constant c_1.

As for $d = 3$, the question whether or not B_3 is m-divisible for $4 \leq m \leq 21$ is open, for $d \geq 10$, the question whether or not B_d is m-divisible for $d+1 \leq m \leq 20d$ also remains open. There are several obstacles in the way of improving these bounds. One of them is the condition of Lemma 5.1 which requires that every point $x \neq x_0$ has to be the image of at least three core points. In [2, Example 6.1] was shown that this condition of Lemma 5.1 is sharp.

The most related question is whether or not B_d is divisible for $d = 5$. It is very likely that the answer is affirmative. However, our proof does not seem to work in this case. The crucial step in the proof of Theorems 1.3 is to check that the conditions of Lemma 5.1 is satisfied on the graph generated by the isometries. Our proof in even dimension $d = 2s$ is based on the fact that if $O \in SO(2s)$ is a 'generic' rotation then O has no fixed point other than the origin. Thus $T_b O$ has a fixed point for *every* vector $b \in \mathbb{R}^{2s}$, since $I - O$ is invertible, and $(I-O)^{-1}(b)$ is a fixed point of $T_b O$. This statement does not hold for dimension $d = 2s+1$. Furthermore it can be easily shown that 1 is an eigenvalue of a 'generic' rotation $O \in SO(2s+1)$ with multiplicity at least 1.

However, it is also not clear if the method applied in [2] works for $d = 5$. The result of [2] is based on the fact that if $O_0, \ldots, O_N \in SO(3)$ are 'generic' rotations, $b \in \mathbb{R}^3$ is a 'generic' vector and $F = T_b O_0$, then a nonempty reduced word on the alphabet $O_1^{\pm 1}, \ldots, O_N^{\pm 1}$ and $F^{\pm 1}$ has a fixed point only if the word is a conjugate of a word on the alphabet $O_1^{\pm 1}, \ldots, O_N^{\pm 1}$. (See [2, Lemma 3.5] for the precise statement.) Unfortunately, this statement does not generalize for higher dimensions. The generalization has a difficulty. The authors use the fact that the axis of a 'generic' rotation O can be expressed by the entries of the matrix O. On the other hand, Borel [1] proved that for every odd $d \geq 3$ there is a dense subset of pairs (A, B) in $(SO(d))^2$ such that each pair generates a locally commutative group. Thus if two 'generic' rotations have a common axis, then they commute. (This property is called the locally commutativity.) Still, we conjecture that the corresponding graph has property (2) in every dimension $d \geq 3$.

References

[1] A. Borel, On free subgroups of semisimple groups, *Enseign. Math.* **29** (1983) 151-164.

[2] G. Kiss and M. Laczkovich, Decomposition of balls into congruent pieces *Mathematika* **57** (2011), no. 1, 89-107.

[3] T. J. Dekker, Decomposition of sets and spaces I,II,III, Indag. Math. 18 (1956), 581-595, 19 (1957), 104-107.

[4] P. Deligne and D. Sullivan, Division algebras and the Hausdorff-Banach-Tarski paradox *Enseign. Math.* (2) **29** (1983), no. 1-2, 145-150.

[5] C. Richter, Affine divisibility of convex sets, *Bull. Lond. Math. Soc.* **41** (2009), no. 4, 757-768.

[6] E. van Douwen, Indivisibility of balls in Euclidean n-space, *Topology Appl.* **51** (1993), no. 2, 183-185.

[7] S. Wagon, Partitioning intervals, spheres and balls into congruent pieces, *Canad. Math. Bull.* **26** (1983), no. 3, 337-340.

[8] S. Wagon, *The Banach-Tarski paradox.* Second edition, Cambridge Univ. Press, 1986. First paperback edition, 1993.

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