Optimal embedded and enclosing isosceles triangles

Áron Ambrus, Mónika Csikós[*], Gergely Kiss[†], János Pach[‡], Gábor Somlai[§]

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Abstract

Given a triangle Δ, we study the problem of determining the smallest enclosing and largest embedded isosceles triangles of Δ with respect to area and perimeter. This problem was initially posed by Nandakumar [16, 18] and was first studied by Kiss, Pach, and Somlai [12], who showed that if Δ' is the smallest area isosceles triangle containing Δ, then Δ' and Δ share a side and an angle. In the present paper, we prove that for any triangle Δ, every maximum area isosceles triangle embedded in Δ and every maximum perimeter isosceles triangle embedded in Δ shares a side and an angle with Δ. Somewhat surprisingly, the case of minimum perimeter enclosing triangles is different: there are infinite families of triangles Δ whose minimum perimeter isosceles containers do not share a side and an angle with Δ.

1 Introduction

The following classical problem is the starting point of our investigation. Given two convex bodies, C and C' in $\mathbb{R}^d$, decide whether C can be moved into a position where it covers C'. One can easily list some necessary conditions, for instance, the volume, the surface area and the diameter of C has to be at least as large as the one of C'. However, solving the decision problem can be rather challenging, even in $\mathbb{R}^2$, or for special cases that might seem friendly at first sight.

For instance, consider the setup, where C' is the 'shadow' of C, that is, C is embedded into $\mathbb{R}^3$ and C' is the orthogonal projection of C onto a 2-dimensional affine subspace. The necessary conditions are clearly satisfied and it looks plausible that there is always a congruent copy of C which covers C'. However, the proof of this fact is far from straightforward [4, 13], and rather surprisingly, the result does not generalize to higher dimensions: for $d \geq 3$, no convex d-polytope embedded in $\mathbb{R}^{d+1}$ can cover all of its shadows [4].

Another special case is where both convex bodies are triangles in $\mathbb{R}^2$: given two triangles Δ and Δ', find an efficient way to decide whether Δ can be brought into a position where it covers Δ'. This is a classical problem posed by Steinhaus [25] in 1964 and an algorithmic solution was proposed only 29 years later by Post [21], who described a set of 18 polynomial

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[†]Rényi Institute, Budapest. Research partially supported by Premium Postdoctoral Fellowship of the Hungarian Academy of Sciences and by National Research, Development and Innovation Office (NKFIH) grant K-124749.

[‡]Rényi Institute, Budapest and IST Austria. Research partially supported by National Research, Development and Innovation Office (NKFIH) grant K-131529 and ERC Advanced Grant "GeoScape."

[§]Eötvös Loránd University, Rényi Institute, partially supported by the János Bolyai Research Fellowship and by the Thematic Excellence Programme TKP2020-NKA-06.

inequalities of degree 4 such that a copy of Δ can cover Δ' if and only if at least one of these inequalities are satisfied. The key geometric component of Post's solution is the following.

Lemma 1.1 (Post [21])**.** *If a triangle Δ can be moved to a position where it covers another triangle Δ', then one can also find a covering position of Δ with a side that contains one side of Δ'.*

Results of this kind help us to reduce the number of configurations to consider, and are of both theoretical and practical interest.

1.1 Optimal covers from a class

In the present paper, we study a variant of the covering problem where the body C (or C') is not fixed, but can be chosen from a family of possible objects and we want to find a solution which is in some sense optimal, for example, has minimum area or perimeter.

Several classical problems in geometry can be viewed as covering problems of this kind: finding an optimal enclosing triangle, polygon, or ellipse (Löwner-John ellipse) for a given input set [2, 3, 5–7, 9, 10, 22, 23] as well as their higher dimensional analogues (that is, simplices, polytopes, ellipsoids, [11, 19, 27]). Apart from their theoretical interest, these problems have found applications in various areas of computer science and mathematics (optimization, packing and covering, approximation algorithms, convexity, computational geometry), see [8, 14, 24]. In the past decade, several explicit algorithms were proposed for the case of triangles [2, 15, 20, 26].

Nandakumar [16–18] raised the following two special instances of the above question: *given a triangle Δ, determine the minimum area and the minimum perimeter isosceles triangles that contain Δ.* In what follows, we answer these questions, together with their 'dual' versions: *given a triangle Δ, determine the maximum area and the maximum perimeter isosceles triangles embedded (i.e., contained) in Δ.*

The case of minimum area isosceles containers has been recently studied by Kiss, Pach, and Somlai [12]: they described all isosceles containers of a given triangle Δ for which the minimum is attained. Here, we complete the picture: we characterize the optimal solutions of the other three problems stated above. We will conclude that for three of the above problems, the optimum is attained for a 'trivial' configuration, where the two triangles share a side and an angle at one end of this side.

Theorem 1.2. *Let Δ be a triangle in $\mathbb{R}^2$ and*

- *(i) let $\Delta'' \supseteq \Delta$ be a* minimum area isosceles container *of Δ. Then Δ'' and Δ have a side in common and at one endpoint of this side they also have the same angle [12];*
- *(ii) let $\Delta' \subseteq \Delta$ be a* maximum area embedded isosceles triangle *in Δ. Then Δ' and Δ have a side in common and at one endpoint of this side they also have the same angle;*
- *(iii) let $\Delta' \subseteq \Delta$ be a* maximum perimeter embedded isosceles triangle *in Δ. Then Δ' and Δ have a side in common and at one endpoint of this side they also have the same angle.*

Somewhat surprisingly, the analogous statement is false for minimum perimeter containers.

Theorem 1.3. *There are infinite families of triangles Δ such that none of their minimum perimeter isosceles containers shares a side with Δ (and an angle at the end of this side).*

We describe 5 *different types of isosceles containers such that any triangle Δ has a minimum perimeter isosceles container Δ' belonging to one of these types. Only* 3 *out of these* *types will have the property that Δ and Δ' share a side and at one of the endpoints of this side they also have the same angle.*

Our paper is organized as follows. In Section 2, we fix the notation and list some easy preliminary statements. In Section 3 and Section 4, we present the proofs of Theorem 1.2(ii) and Theorem 1.2(iii), respectively. Finally, Section 5 is dedicated to the description of the 5 types of isosceles containers mentioned in Theorem 1.3 and to the proof of this result.

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2 Preliminaries and notation

In this section, we introduce the notation used in this note and state three easy lemmas (Lemmas 2.1-2.3 and 2.5). Their straightforward proofs are given in the Appendix.

Lemma 2.1. *Let Δ_1 and Δ_2 be two triangles.*

(i) Any maximum area (resp. perimeter) similar copy $\Delta_1' \subseteq \Delta_2$ of Δ_1 satisfies the following properties:
   - *(a) there is a side of Δ_2 that contains a side of Δ_1';*
   - *(b) every side of Δ_2 contains a vertex of Δ_1';*
   - *(c) Δ_1' and Δ_2 have a common vertex.*

(ii) Any minimum area (resp. perimeter) similar copy $\Delta_1' \supseteq \Delta_2$ of Δ_1 satisfies the following properties:
   - *(a) there is a side of Δ_1' that contains two vertices of Δ_2;*
   - *(b) every side of Δ_1' contains a vertex of Δ_2;*
   - *(c) Δ_1' and Δ_2 have a common vertex.*

Optimal isosceles enclosing and embedded triangles satisfy further properties.

Lemma 2.2.

(i) For every triangle Δ there exists a minimum area (resp. perimeter) isosceles container of Δ, and a maximum area (resp. perimeter) isosceles triangle embedded in Δ.

(ii) If Δ_1 is a maximum area (resp. perimeter) isosceles triangle embedded in Δ, then every vertex of Δ_1 lies on a side of Δ.

(iii) If Δ_2 is a minimum area (resp. perimeter) isosceles container of Δ, then every vertex of Δ lies on a side of Δ_2.

For any two points, A and B, let AB denote the closed segment connecting them, and let $|AB|$ stand for the length of AB. To unify the presentation, in the sequel we fix a triangle ABC with side lengths $a = |BC|$, $b = |AC|$, $c = |AB|$. If two sides are of the same length, then ABC is the unique minimum area and perimeter isosceles container (and also maximum area and perimeter embedded isosceles triangle) of itself. Therefore without loss of generality, we assume that $a < b < c$.

2.1 Special embedded isosceles triangles

Given a triangle ABC, we describe its special embedded isosceles triangles, that is, all those isosceles triangles contained in ABC that have a common side with ABC and share an angle at one of the endpoints of the common side. Recall that these triangles play a distinguished role in Theorem 1.2.

Special embedded triangles of the first kind. Let A' be a point of AC with $|A'C| = |BC|$ and let B' and A'' be two points of AB such that $|AB'| = |AC|$ and $|A''B| = |BC|$ (see Figure 1). We say that $A'BC$, $AB'C$, and $A''BC$ are the *special embedded triangles of the first kind* associated with ABC.

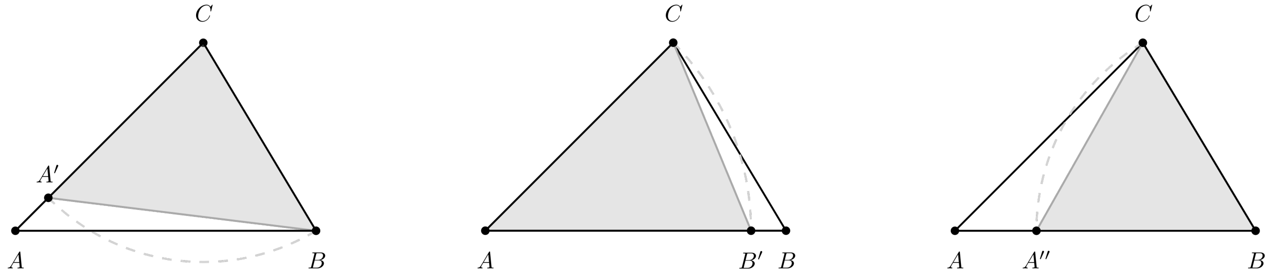


Figure 1: Special embedded triangles of the first kind.

Special embedded triangles of the second kind. Let C_1 be the intersection of the perpendicular bisector of AB and the segment AC. Analogously, let A_1 be the intersection of the perpendicular bisector of BC and AC, and let B_1 be the intersection of the perpendicular bisector of BC and the line AC (see Figure 2). The triangles A_1BC, AB_1C, and ABC_1 are the *special embedded triangles of the second kind* associated with ABC.

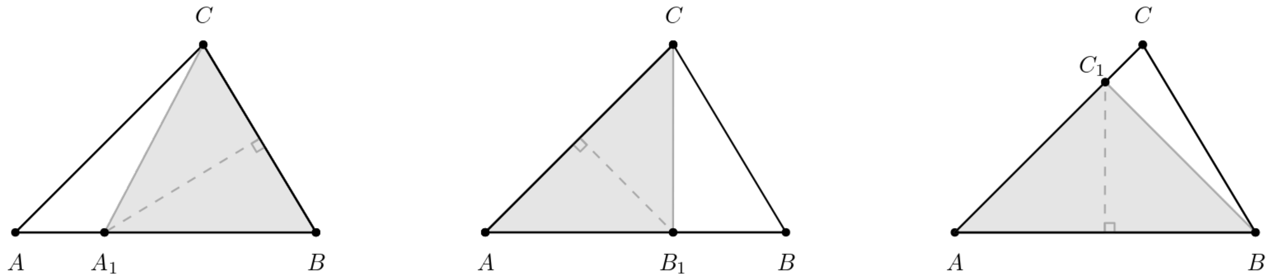


Figure 2: Special embedded triangles of the second kind.

Special embedded triangles of the third kind. Let $\overline{A}$ be a point of AB, where $|\overline{A}C| = |BC|$. Analogously, let $\overline{\overline{A}} \in AC$, and $\overline{B} \in BC$ such that $|\overline{\overline{A}}B| = |BC|$, and $|\overline{B}A| = |AC|$ (see Figure 3). Note that if ABC is non-acute, then $\overline{\overline{A}}BC$ and $A\overline{B}C$ do not exist. $\overline{A}BC$, $\overline{\overline{A}}BC$, and $A\overline{B}C$ are called the *special embedded triangles of the third kind* associated with ABC.

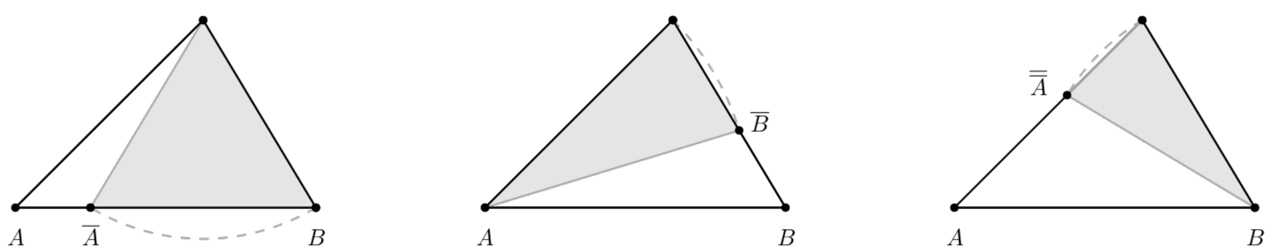


Figure 3: Special embedded triangles of the third kind.

2.1.1 Basic inequalities for special embedded triangles.

We collect a few inequalities on the area and perimeter of special isosceles embedded triangles. For a triangle Δ, let $\mathrm{per}(\Delta)$ and $\mathrm{area}(\Delta)$ denote the perimeter and the area of Δ, respectively.

Lemma 2.3. *If ABC satisfies $a < b < c$, then*

(i) $\mathrm{area}\,(A''BC) < \mathrm{area}\,(A'BC)$*;*

(ii) $\mathrm{area}\,(A_1BC) < \mathrm{area}\,(AB'C)$ *and* $\mathrm{area}\,(AB_1C) < \mathrm{area}\,(ABC_1)$*;*

(iii) $\mathrm{area}\left(\overline{A}BC\right) < \mathrm{area}\,(ABC_1)$*,* $\mathrm{area}\left(\overline{\overline{A}}BC\right) < \mathrm{area}\left(A\overline{B}C\right)$*,* *and* $\mathrm{area}\left(A\overline{B}C\right) < \mathrm{area}\,(AB'C)$*;*

(iv) if ABC is obtuse, then $\mathrm{area}\,(A'BC) < \mathrm{area}\,(ABC_1)$*.*

Lemma 2.3 imply that only 3 of the special embedded triangles of ABC can be optimal.

Corollary 2.4. *If ABC satisfies $a < b < c$, then any maximum area special embedded triangle of ABC is one of the following triangles: $A'BC$, $AB'C$, ABC_1.*

We note that similar results hold for the perimeter function implying that any maximum perimeter special embedded triangle of ABC is $AB'C$, A_1BC, or ABC_1.

2.2 Special enclosing isosceles triangles

Given a triangle ABC, now we describe its special enclosing isosceles triangles, that is, all those isosceles triangles containing ABC that have a common side with ABC and share an angle at one of the endpoints of the common side. Recall that these triangles play a distinguished role in Theorems 1.2 and 1.3.

Special containers of the first kind. Let B' denote the point on the ray $\vec{CB}$, for which $|B'C| = |AC|$. Analogously, let C' (and C'') denote the points on $\vec{AC}$ (resp., $\vec{BC}$) such that $|AC'| = |AB|$ (resp., $|BC''| = |AB|$), see Figure 4. We call the triangles $AB'C$, ABC', and ABC'' *special containers of the first kind* associated with ABC.

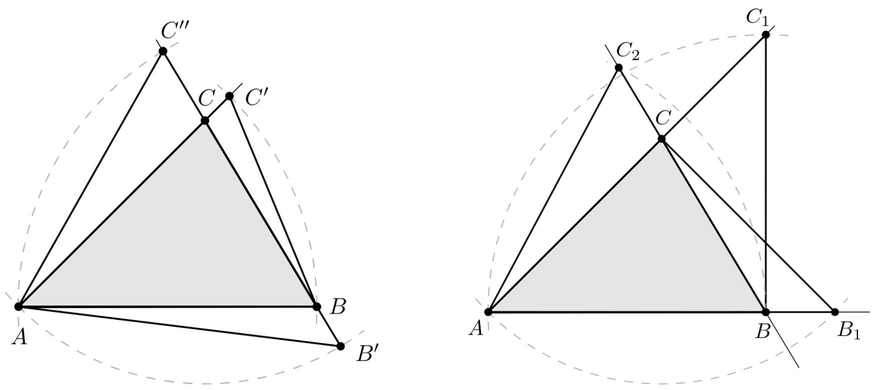


Figure 4: Special containers of the first kind ($AB'C, ABC'$, and ABC'') and second kind (AB_1C, ABC_1, and ABC_2).

Special containers of the second kind. Let B_1 denote the point on the ray $\vec{AB}$, different from A, for which $|B_1C| = |AC|$. Analogously, let C_1 (resp., C_2) denote the point on $\vec{AC}$ (resp., $\vec{BC}$) for which $|BC_1| = |AB|$ and $C_1 \neq A$ (resp., $|AC_2| = |AB|$ and $C_2 \neq B$), see Figure 4. The triangles AB_1C, ABC_1, and ABC_2 are called the *special containers of the second kind* associated with ABC.

Special containers of the third kind. Let $\overline{A}$ be the intersection of the perpendicular bisector of BC and the line AC. Since we have $b = |AC| < |AB| = c$, the point $\overline{A}$ lies outside of ABC. Analogously, denote by $\overline{B}$ (resp., $\overline{C}$) the intersection of the perpendicular bisector of AC (resp. AB) and the line BC. (If ABC is non-acute $\overline{A}BC$ and $A\overline{B}C$ do not contain ABC (Figure 5).) The triangles $\overline{A}BC$, $A\overline{B}C$, and $AB\overline{C}$ are called the *special containers of the third kind* associated with ABC, provided that they contain ABC.

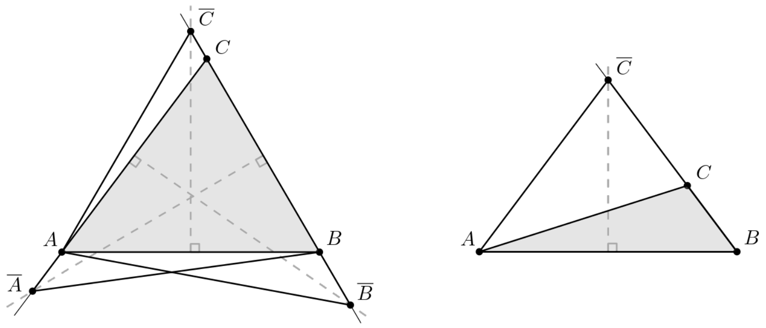


Figure 5: Special containers of the third kind ($\overline{A}BC, A\overline{B}C, AB\overline{C}$) in the acute and in the non-acute cases.

2.2.1 Basic inequalities for special containers

Similarly to the case of maximum area embedded triangles, we can show that not all special containers can be of minimum perimeter.

Lemma 2.5. *If ABC satisfies $a < b < c$, then*

(i) $\operatorname{per}(ABC') < \operatorname{per}(ABC'')$ *and* $\operatorname{per}(AB'C) < \operatorname{per}(AB_1C)$*;*

(ii) $\operatorname{per}(ABC') < \operatorname{per}(ABC_2) < \operatorname{per}(ABC_1)$*;*

(iii) $\operatorname{per}(ABC') < \operatorname{per}\left(\overline{A}BC\right) < \operatorname{per}\left(A\overline{B}C\right)$.

Lemma 2.5 immediately gives the following corollary.

Corollary 2.6. *If ABC satisfies $a < b < c$, then any minimum perimeter special container of ABC is one of the following triangles: $AB'C$, ABC', $AB\overline{C}$.*

Again, we note that similar results hold for the area function implying that a minimum area special container of ABC is $AB'C$, ABC', or AB_1C.

3 Maximum area embedded isosceles triangles – Proof of Theorem 1.2(ii)

Let ABC be a triangle and let XYZ denote one of its maximum area isosceles embedded triangles. In this section, we prove that XYZ has to be a special embedded triangle. We use the notation $a = |BC|$, $b = |AC|$, $c = |AB|$, $x = |YZ|$, $y = |XZ|$, $z = |XY|$, and assume (with no loss of generality) that $a < b < c$. By Lemmas 2.1 and 2.2, we have the following statements on maximum area embedded isosceles triangles.

Lemma 3.1. *Let XYZ be any maximum area isosceles triangle embedded in ABC. Then*

(i) a side of ABC contains a side of XYZ;
(ii) every side of ABC contains a vertex of XYZ;
(iii) ABC and XYZ have a common vertex;
(iv) no vertex of XYZ lies in the interior of ABC.

If XYZ has at least two common vertices with ABC, then by Lemma 3.1(iv), XYZ and ABC have a common side and a common angle. Therefore, we can assume that ABC and XYZ have exactly one common vertex.

Denote the midpoints of the sides BC, AC, and AB by m_A, m_B, and m_C, respectively. We divide the boundary of ABC into 3 polylines defined as

$$\widehat{m_A m_B} = m_A C \cup C m_B, \quad \widehat{m_B m_C} = m_B A \cup A m_C, \quad \widehat{m_C m_A} = m_C B \cup B m_A.$$

We get the following constraint on the position of X, Y, and Z:

Lemma 3.2. *Let XYZ be a maximum area embedded isosceles triangle of the triangle ABC. Then each of $\widehat{m_A m_B}, \widehat{m_B m_C}$, and $\widehat{m_C m_A}$ contains exactly one vertex of XYZ.*

Proof. By Lemma 3.1, X, Y, Z lies on the boundary of ABC. Assume, without loss of generality, that $\widehat{m_A m_C}$ contains X and Z (see Figure 6).

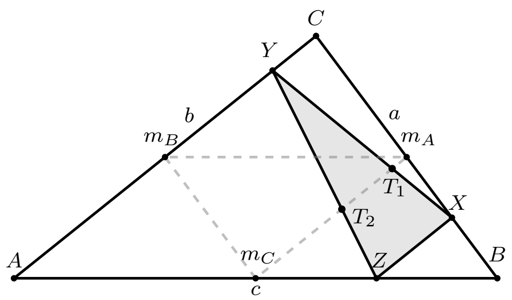


Figure 6: Proof of Lemma 3.2.

Let $T_1 = m_A m_C \cap XY$ and $T_2 = m_A m_C \cap YZ$. Then area $(XT_1T_2Z) \leq$ area (Bm_Am_C) and by $|T_1T_2| \leq |m_Am_C|$ we obtain that area $(T_2T_1Y) \leq$ area $(m_Am_Bm_C)$. Thus we have

$$\text{area}\,(XYZ) \leq \text{area}\,(Bm_Am_C) + \text{area}\,(m_Am_Bm_B) = \frac{\text{area}\,(ABC)}{2}.$$

On the other hand, since $c \leq a + b \leq 2b$, the special embedded triangle $AB'C$ satisfies

$$\text{area}\,(AB'C) = \frac{b^2 \sin(\sphericalangle CAB)}{2} > \frac{bc \sin(\sphericalangle CAB)}{4} = \frac{\text{area}\,(ABC)}{2}.$$

Hence, area $(XYZ) <$ area $(AB'C)$, which contradicts the maximality of the area of XYZ. $\square$

Lemmas 3.1 and 3.2 imply that a maximum area embedded isosceles triangle of ABC is either special or its vertex arrangement corresponds to one of the 9 cases illustrated in Figure 7.

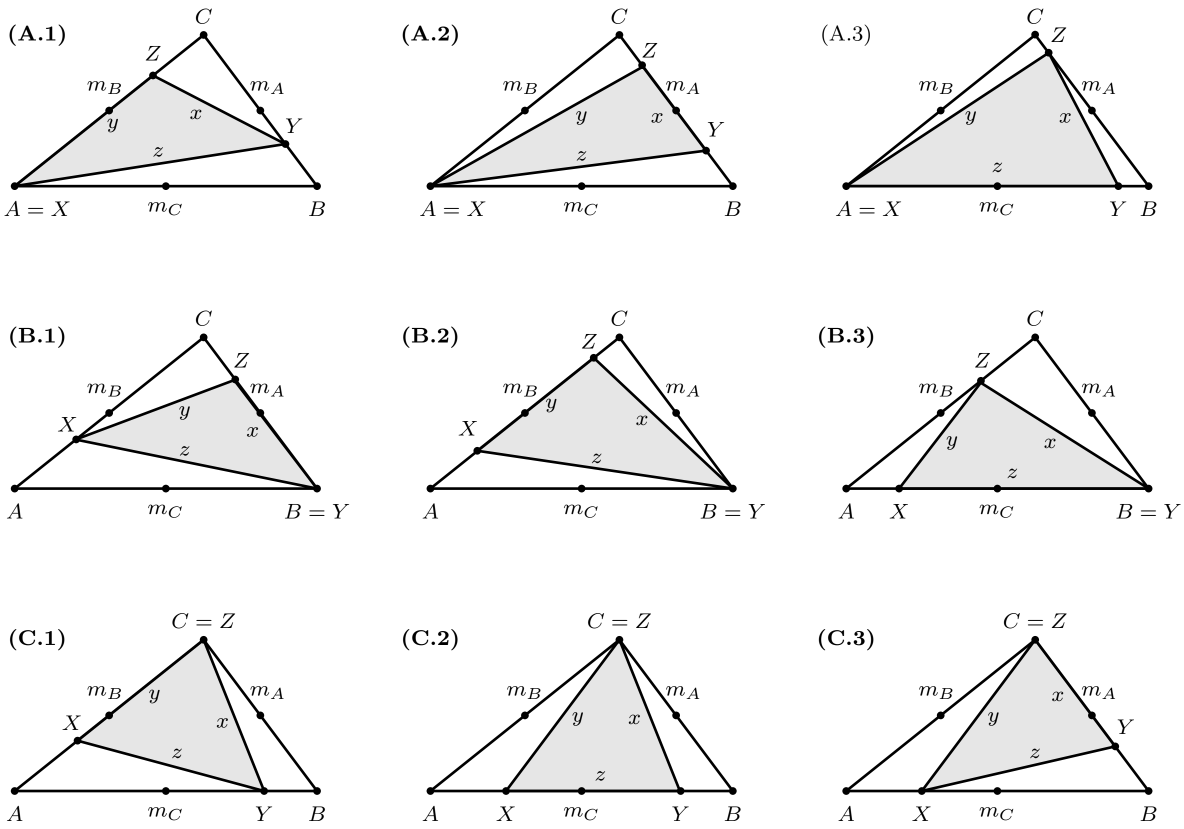


Figure 7: The 9 possible arrangements of the points X, Y, Z in a given triangle ABC.

To complete the proof of Theorem 1.2(ii), it remains to prove that none of the arrangements depicted on Figure 7 can be optimal. We prove this for each of the 9 cases, separately. Note that in some instances, we will refer to special embedded triangles using their specific labeling introduced in Section 2.1.

Case A: *The common vertex of ABC and XYZ is $A = X$.*

Subcase A.1: *$Y \in BC$ and $Z \in AC$.*

Observe that since $b < c$, the orthogonal projection of A onto CB is contained in Cm_A, which implies that $\sphericalangle AYB$ is obtuse. Thus, we can rotate XYZ about X such that two of its vertices get to the interior of ABC and so, by Lemma 3.1, XYZ cannot be of maximum area.

Subcase A.2: *Both Y and Z are in BC.*

If $y = z$, then we can increase area(XYZ) by moving Z towards C and Y towards B while maintaining $|XZ| = |XY|$, since $\alpha = \sphericalangle CAB < 90°$. If ABC is acute, then we can do this until the vertices Z and C will coincide, and triangle XYZ will be the same as the special embedded triangle $A\overline{B}C$. If ABC is non-acute, then $y \neq z$. Clearly, $|AZ| = y > |ZB| > |YZ| = x$. Hence, $x \neq y$. A similar argument shows that $x \neq z$.

Subcase A.3: *$Y \in AB$ and $Z \in BC$.*

Since $a < b$, the orthogonal projection $\widehat{Z}$ of Z to the line segment AB lies in $m_C B$.

If $x = y$, then $|AY| = 2|A\hat{Z}| > 2|Am_C| = |AB|$, a contradiction to $Y \in AB$.

If $x = z$, then the altitude with base z in XYZ is smaller than the altitude with base c in ABC. On the other hand, $x = z < a$, if $\sphericalangle ZYB \geq 90°$. In this case, the special embedded triangle $A''BC$ satisfies $\text{area}\,(A''BC) > \text{area}\,(XYZ)$. Otherwise, $x = z < y$ (as $\sphericalangle AYZ > 90°$) and $y < c$. Let Y' be the point in AB that is defined by the equality $|AY'| = |AZ|$. (The existence of $Y' \in AB$ is a consequence of $y < c$.) Then, $\text{area}\,(XY'Z) > \text{area}\,(XYZ)$. In both cases it follows that the area of XYZ cannot be optimal.

If $y = z$, consider the special embedded triangle $AB'C$, define l to be the line parallel to $B'C$ going through Y and let $Z' = l \cap BC$, see Figure 8. Since $Z' \in CZ$, we have

$$\text{area}\,(XYZ) < \text{area}\,(XYZ') = \text{area}\,(AB'C) \cdot \frac{b + |B'Y|}{b} \cdot \frac{c - b - |B'Y|}{c - b}.$$

The inequality follows from the fact that $Z' \in CZ$. Therefore, the altitude of XYZ with base z is greater than the altitude of XYZ' with base z. Thus, it is enough to show that

$$\frac{b + |B'Y|}{b} \cdot \frac{c - b - |B'Y|}{c - b} < 1.$$

As $b > 0$ and $c - b > 0$, this is equivalent to $|B'Y|(2b - c + |B'Y|) > 0$, which follows from the triangle inequality $c < a + b < 2b$.

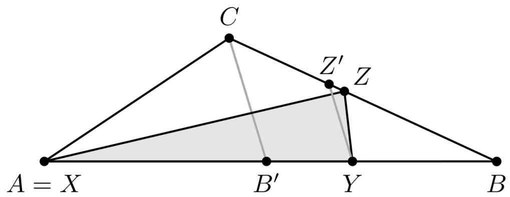


Figure 8: Illustration for Subcase A.3.

Case B: *The common vertex of ABC and XYZ is $B = Y$.*

Subcase B.1: *$X \in AC$ and $Z \in BC$.*
Since $a < c$, we have that $\sphericalangle AXY > 90°$, and hence, we can rotate the triangle XYZ around Y so that the image of the vertices X, Z will be inside of ABC. As in Subcase A.1, this implies that the area of XYZ is not optimal.

Subcase B.2: *Both X and Z are in AC.*
Observe that $b < c$ implies that A and C are on the same side of the perpendicular bisector of BC. This implies that $|XY| = z > |XC| > |XZ| = y$. If $x = z$, we can 'open' $\sphericalangle XYZ$ as in Subcase A.2 and get that $\text{area}\,(XYZ) < \text{area}\left(\bar{\bar{A}}BC\right)$. Hence, we can assume that $x = y$.

If the triangle ABC is non-acute, then consider the special embedded triangle ABC_1. Since the altitudes of ABC_1 and XYZ from vertex $B = Y$ are equal, and $x = y < |BC_1| = |AC_1|$ (as $\sphericalangle BCA \geq 90°$), we have that $\text{area}\,(XYZ) < \text{area}\,(ABC_1)$.

If ABC is acute, let $\hat{B}$ denote the orthogonal projection of B onto AC. If $Z \in A\hat{B}$, then we can slightly rotate XYZ around Y (as $\sphericalangle YXA > \sphericalangle YZA > 90°$). Thus, by Lemma 3.1(iv), the area of XYZ is not maximal. Thus, we can assume that $Z \in C\hat{B}$, that is, $\sphericalangle YZA \leq 90°$. Similarly as above, this implies that $x = |YZ| < a = |BC|$ and thus the special embedded triangle $A'BC$ satisfies $\text{area}\,(XYZ) < \text{area}\,(A'BC)$.

Subcase B.3: *$X \in AB$ and $Z \in AC$.*
If $y = z$, then, since $\sphericalangle CAB < \min(\sphericalangle AXZ, \sphericalangle ZXY)$, we get that $y = |XZ| < |AZ| <$

$b = |AC|$, which immediately implies that the special embedded triangle $AB'C$ satisfies area $(XYZ) <$ area $(AB'C)$.

Now we assume that $x = z$. If A and Z lie on the same side of the perpendicular bisector of AB, then we can reflect XYZ to this perpendicular bisector. We denote this reflection by $X'Y'Z'$. Clearly, $X', Y' \in AB$, and Z' is inside of ABC, which implies that area (XYZ) is not maximal. If Z is on the perpendicular bisector of AB, then XYZ is strictly contained in the special embedded triangle ABC_1, so area $(XYZ) <$ area (ABC_1). If Z and C are on the same side of the perpendicular bisector of AB, then $x = z < |AZ| < |AC| = b$, and hence area $(XYZ) <$ area $(AB'C)$.

It remains to handle the case $x = y$. We show that area $(XYZ) <$ area (ABC_1). The condition $x = y$ implies that $Z \in C_1C$. Plainly, $z = c - |AX|$. Denote the lengths of the altitudes from C_1 in ABC_1 and from Z in XYZ by h_{C_1} and h_Z, respectively. Clearly, we get $h_Z = h_{C_1}\frac{c+|AX|}{c}$, and hence

$$\text{area}\,(XYZ) = \text{area}\,(ABC_1)\,\frac{c+|AX|}{c}\cdot\frac{c-|AX|}{c} = \text{area}\,(ABC_1)\,\frac{c^2-|AX|^2}{c^2} < \text{area}\,(ABC_1)\,.$$

Case C: *The common vertex of ABC and XYZ is $C = Z$.*

Subcase C.1: *$X \in AC$ and $Y \in AB$.*
If Y and B are on the same side of the altitude from C, then we can rotate XYZ about Z so that X and Y get to the interior of ABC which by Lemma 3.1(iv) implies that XYZ is not optimal. If Y and B are on different sides of the altitude from C, then XYZ is strictly contained in the special embedded triangle $AB'C$.

Subcase C.2: *Both X and Y are contained in AB.*
If $x = y$, then we can open $\sphericalangle YZX$, which increases the area of XYZ, so area (XYZ) is not maximal. Suppose that $x = z$. If Y and A are on the same side of the altitude from C, then XYZ is strictly contained in the special embedded triangle $AB'C$. If Y and A lie on different sides of the altitude, then the special embedded triangle $A''BC$ satisfies area $(XYZ) <$ area $(A''BC)$. Indeed, their altitudes from C are the same, and for their bases we have $x = z < a$. Thus XYZ is not maximal. Analogously, for $y = z$ a similar argument shows that area $(XYZ) <$ area $(AB'C)$.

Subcase C.3: *$X \in AB$ and $Y \in BC$.*
We can rotate XYZ about Z such that the images of X and Y lie in the interior of ABC, and so, by Lemma 3.1(iv), we get that area (XYZ) is not maximal.

We have shown that none of the triangles XYZ of the 9 cases in Figure 7 is a maximum area embedded isosceles triangle of ABC, which completes the proof of Theorem 1.2(ii). □

4 Maximum perimeter embedded isosceles triangles – Proof of Theorem 1.2(iii)

In this section, we prove that for any triangle ABC, any maximum perimeter isosceles triangle XYZ embedded in ABC shares a vertex and the angle at that vertex with ABC. First we collect the observations in Lemmas 2.1 and 2.2 concerning maximum perimeter embedded isosceles triangles.

Lemma 4.1. *Let XYZ be a maximum perimeter isosceles triangle embedded in ABC. Then*

(i) each side of ABC contains a vertex of XYZ;

(ii) no vertex of the triangle XYZ lies in the interior of the triangle ABC;

(iii) there is a side of ABC which contains a side of XYZ;

(iv) ABC and XYZ share a vertex.

We will show that an isosceles triangle embedded in ABC which does not share an angle with ABC cannot be of minimum perimeter. Notice that if ABC and XYZ share at least two vertices, then, by Lemma 4.1(ii), they also share an angle, so we are done. Thus, it is enough to consider those cases where the triangles XYZ and ABC share exactly one vertex, without loss of generality the common vertex is A. Note that in this section, we do not assume a special labeling of ABC, in particular, we do not necessarily have $|BC| < |AC| < |AB|$. On the other hand, we assume that XYZ is labeled so that $|XY| = |YZ|$. We consider the following cases, separately:

Case A: *X and Z lie on the same side of ABC.*
We can always rotate X or Z (for simplicity, assume it is X) about Y so that the rotated point X' lies in the interior of ABC and $\sphericalangle XYZ < \sphericalangle X'YZ$, see Figure 9. By the Hinge theorem[1], $\mathrm{per}\,(XYZ) < \mathrm{per}\,(X'YZ)$.

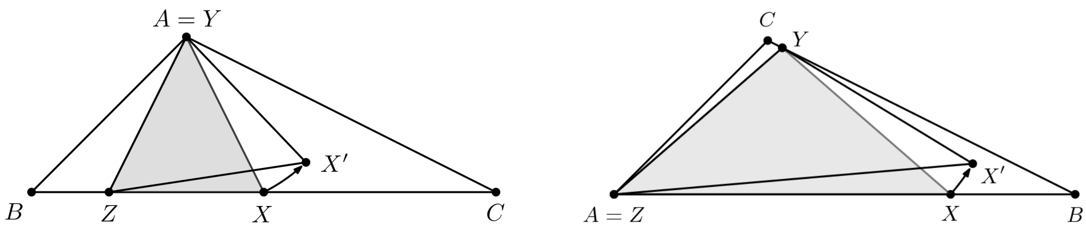


Figure 9: Illustration for Case A

Case B: *X and Z lie on different sides of ABC.*
We will make use of the following classical lemma on the perimeter of the Minkowski sum of convex bodies.

Lemma 4.2 (see e.g. [28, exercise 4-7])**.** *Let K_1 and K_2 be two convex bodies in the plane and let $K = \frac{K_1+K_2}{2}$ be the Minkowski mean of K_1 and K_2. Then the perimeter of K is equal to the arithmetic mean of the perimeters of K_1 and K_2. If K_1 and K_2 are not homothetic triangles, then K is a convex polygon with at least four sides.*

Subcase B.1: *The common vertex of ABC and XYZ is $A = Y$.*
If none of X and Z is on the side opposite to Y, then XYZ and ABC have a common angle at Y. Thus, we can assume that either X or Z is on the side opposite to Y, say it is X.

The idea is to show that the triangle XYZ is strictly contained in the Minkowski mean M of two other non-homothetic isosceles triangles embedded in ABC, thus, by Lemma 4.2, one of these two must have a strictly larger perimeter (by the fact that if $\mathcal{C}_1, \mathcal{C}_2$ are two convex planar sets such that $\mathcal{C}_1 \subseteq \mathcal{C}_2$, then $\mathrm{per}\,(\mathcal{C}_1) \leq \mathrm{per}\,(\mathcal{C}_2)$ [1, 12.10.2]).

Let δ be a constant satisfying $\delta < \min\{|XB|, |XC|\}$. Define the points X^1 and X^2 by translating X by δ towards C and B, respectively. Let Z^1 and Z^2 be such that they are contained on the side AB with $|YZ^1| = |YX^1|$ and $|YZ^2| = |YX^2|$, see Figure 10. Let M be the Minkowski mean of X^1YZ^1 and X^2YZ^2. The vertex Y is contained in both triangles,

[1]Hinge theorem: Let XYZ be a triangle and let $X'Y'Z'$ be another triangle such that $XY = X'Y', YZ = Y'Z'$, and $\sphericalangle XYZ < \sphericalangle X'Y'Z'$. Then $\mathrm{per}\,(XYZ) < \mathrm{per}\,(X'Y'Z')$.

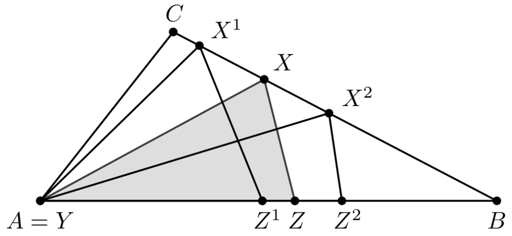


Figure 10: Illustration for Subcase B.1.

thus it is also contained in M. It is also easy to see that $X \in M$ since $X = \frac{1}{2}(X^1 + X^2)$. We show that Z is contained in the segment between Y and $\frac{1}{2}(Z^1 + Z^2)$, which implies $Z \in M$. To this end, observe that the segment YX is a median of the triangle X^1YX^2 and thus $|YX| < \frac{1}{2}(|YX^1| + |YX^2|)$, which directly implies that $|YZ| < \frac{1}{2}(|YZ^1| + |YZ^2|)$.

Subcase B.2: *The common vertex of ABC and XYZ is $A = Z$ and both X and Y are in the interior of the side of ABC opposite to Z.*

Define the points X^1 and X^2 by translating X by δ towards C and B, respectively. We choose δ to be small enough such that there are points Y^1, Y^2 in the segment BC with $|Y^1Z| = |Y^1X^1|$ and $|Y^2Z| = |Y^2X^2|$, see Figure 11. Let M be the Minkowski mean of X^1YZ^1 and X^2YZ^2. As before, it is clear that the vertices X and Z are contained in M.

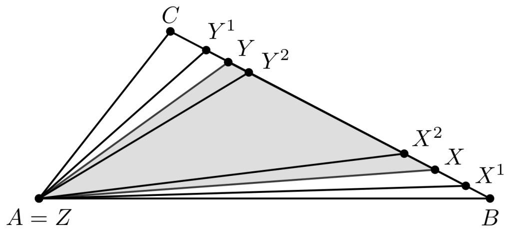


Figure 11: Illustration for Subcase B.2.

To argue that $Y \in M$, we shall show that Y is contained in the segment between X and $\frac{1}{2}(Y^1 + Y^2)$. To simplify the calculations, we move and scale the triangle so that $A = (0, 1)$, $B = (b, 0)$, $C = (c, 0)$ and $X = (x', 0)$ with $b < x' < c$. Note that since $\sphericalangle ZXY$ is acute, $x' < 0$. For each $b < x < 0$, let $X_x = (x, 0)$ and Y_x be the point in BC such that $|ZY_x| = |Y_xX_x|$ and define $f(x) = |X_xY_x|$, see Figure 12. Observe that Y is contained in the segment between X and $\frac{1}{2}(Y^1 + Y^2)$ iff $\frac{1}{2}(f(x' - \delta) + f(x' + \delta)) > f(x')$. Thus, it is sufficient to show that $f(x)$ is a convex function on $(b, 0)$. To find an analytic formula for $f(x)$, we introduce some auxiliary points. Let $O = (0, 0)$ and P_x be the orthogonal projection of Y_x to the segment X_xZ. Note that P_x is the midpoint of XZ. Then the triangles $X_xP_xY_x$ and X_xOZ are similar, which yields

$$f(x) = |Y_xX_x| = |X_xZ| \cdot \frac{|X_xP_x|}{|X_xO|} = \sqrt{1 + x^2} \cdot \frac{\sqrt{1 + x^2}/2}{-x} = \frac{1 + x^2}{-2x}$$

The second derivative of f is $f''(x) = -1/x^3$, thus $f(x)$ is convex on the interval $(b, 0)$.

Subcase B.3: *The common vertex of ABC and XYZ is $A = Z$ and X, Y lie in the interior of different sides of ABC.*

Firstly, since X and Z lie on different sides of ABC, we get that X is on the side opposite to Z, see Figure 13. If $\sphericalangle AXB$ is obtuse, then we can rotate the triangle XYZ about Z and obtain a copy of XYZ which has two vertices in the interior of ABC, thus by Lemma 4.1,

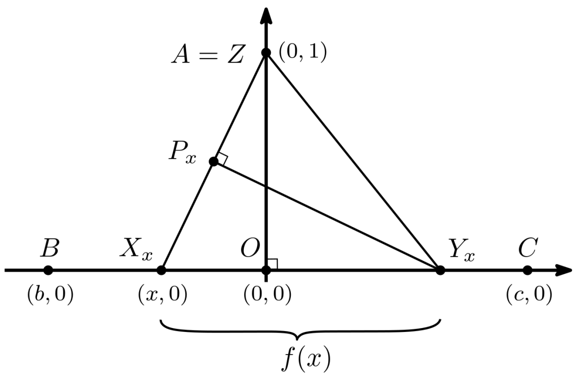


Figure 12: Embedding the instance in $\mathbb{R}^2$.

XYZ cannot be of maximum perimeter. Therefore, $\sphericalangle AXB$ and consequently $\sphericalangle ACB$ are acute.

Define the points X^1 and X^2 by translating X by δ towards C and B, respectively. We choose an increment $\delta \in (0, 1/c)$ which is small enough such that there are points Y^1, Y^2 in the segment AC with $|Y^1Z| = |Y^1X^1|$ and $|Y^2Z| = |Y^2X^2|$. Let M be the Minkowski mean of X^1YZ^1 and X^2YZ^2. The vertices X and Z are clearly contained in M.

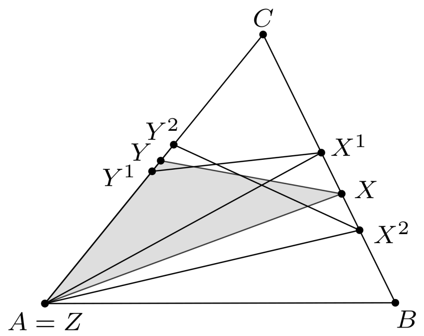


Figure 13: Illustration for Case B.3.

To prove that $Y \in M$, we shall show that Y is contained in the segment between Z and $\frac{1}{2}(Y^1 + Y^2)$. Again, we translate and scale of the triangle so that $A = (0, 1)$, $B = (b, 0)$, $C = (c, 0)$ and $X = (x', 0)$ with $b < x' < c$. Since $\sphericalangle AXB$ is acute, we have $x' \geq 0$. For each $x \in [x' - \delta, x' + \delta]$, let $X_x = (x, 0)$ and Y_x be the point in ZC such that $|ZY_x| = |Y_xX_x|$ and define $f(x) = |ZY_x|$, see Figure 14. Note that since $x' \geq 0$ and δ is smaller than $1/c$, each $x \in [x' - \delta, x' + \delta]$ satisfies $-1/c < x$. We want to show that the function $f(x)$ is convex, which then directly implies that Y is contained in the segment between Z and $\frac{1}{2}(Y^1 + Y^2)$. Let $P_x = (p_1(x), p_2(x))$ be a point on AC such that the segment P_xX_x is orthogonal to AX_x. Note that P_x satisfies $|ZP_x| = 2f(x)$.

Let γ denote the angle $\sphericalangle ACX_x$, then $p_2(x) = 1 - 2\sin(\gamma) \cdot f(x)$ which is concave if and only if $f(x)$ is convex. Since X_xP_x is orthogonal to AX_x and P_x is contained in AC, we get the following equations on $p_1(x)$ and $p_2(x)$

$$p_1(x) \cdot x - p_2(x) = x^2, \quad p_1(x) + cp_2(x) = c,$$

which gives $p_2(x) = \frac{cx - x^2}{cx + 1}$. Taking the second derivative, we get

$$p_2''(x) = -\frac{2(1 + c^2)}{(1 + cx)^3} < 0 \text{ for all } x \in \left(-\frac{1}{c}, \infty\right).$$

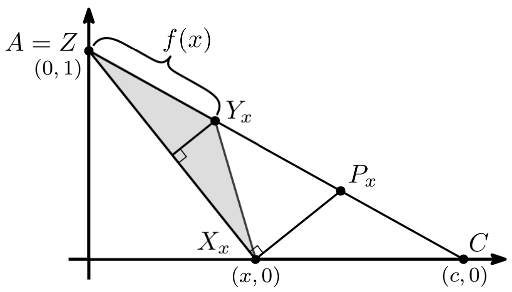


Figure 14: Embedding the instance in $\mathbb{R}^2$.

We proved that none of the triangles of types A.1-A.2 and B.1-B.3 is a maximum perimeter embedded isosceles triangle of ABC, which completes the proof of Theorem 1.2(iii). $\square$

5 Minimum perimeter enclosing triangles – Proof of Theorem 1.3

In this section, we prove that any smallest perimeter isosceles container of a triangle is either a special container or one of two non-special containers defined in the next subsection. We also show that this is the shortest possible characterization of isosceles containers, that is, any of the five examples can be realized as a minimum perimeter isosceles container for some triangle ABC. Now, we define two non-special isosceles containers that can be optimal.

5.1 Two examples for non-special minimum perimeter containers of a triangle

Let P be a point in $\mathbb{R}^2$ and l a line such that $P \notin l$ and let m denote the distance of P from l. Define an isosceles triangle PRS such that S and R lie on l and its apex angle γ is in R, see Figure 15. Let p denote the perimeter of PRS. Note that p can be considered as a function of γ.

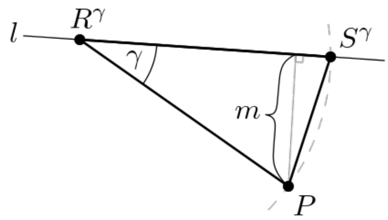


Figure 15: Illustration for Proposition 5.1

Proposition 5.1. *The function p has a unique minimum at*

$$\gamma^* = 4\tan^{-1}\left(\frac{1}{2}\left(1+\sqrt{5}-\sqrt{2(1+\sqrt{5})}\right)\right) \approx 76.3466°. \tag{1}$$

Proof outline. It is easy to see that $|PR| = |RS| = \frac{m}{\sin\gamma}$ and $|PS| = \frac{m}{\sin(90°-\gamma/2)} = \frac{m}{\cos(\gamma/2)}$,. Hence $\text{per}\,(PRS) = m\left(\frac{2}{\sin\gamma} + \frac{1}{\cos(\gamma/2)}\right)$. Elementary analysis shows that the function $f(x) = \frac{2}{\sin x} + \frac{1}{\cos(x/2)}$ is strictly decreasing in $(0°, \gamma^*]$ and strictly increasing in $[\gamma^*, 180°)$. Thus it has a unique minimum in $0 \le x \le 180°$ that is taken at the value specified in Equation (1). $\square$

Example 5.2. Let PRS be an isosceles triangle with apex angle γ^* which is defined as in Proposition 5.1. Now we take an acute triangle ABC in PRS such that ABC and PRS have exactly one common vertex at $A = P$ and $B, C \in SR$ (see Figure 16). Moreover, the largest angle γ of ABC is at C with $\gamma < \gamma^*$ being close to γ^* (e.g., $76°$) and ABC is almost isosceles ($|AC| \approx |BC|$).

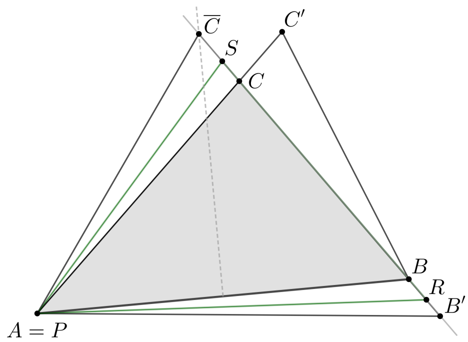


Figure 16: Illustration for Example 5.2.

Claim 5.3. *The perimeter of PRS is strictly smaller than the perimeter of any special container of ABC.*

Proof outline. By Corollary 2.6, it is enough to show that the special containers $AB'C$, ABC', and $AB\overline{C}$ have larger perimeter than PRS. First observe that, since $a \approx b < c$, ABC is an 'almost' isosceles triangle, thus the perimeter $\text{per}\,(AB'C) \approx \text{per}\,(ABC)$ and $\text{per}\,(ABC') > d \cdot \text{per}\,(ABC)$, for a fixed $d > 1$. This implies that $\text{per}\,(AB'C) < \text{per}\,(ABC')$. Now we show that PRS has perimeter smaller than $\text{per}\,(AB'C)$ and $\text{per}\,\big(AB\overline{C}\big)$. Note that, each of PRS, $AB'C$ and $AB\overline{C}$ are isosceles triangles with base vertex $A = P$ and legs on the line RS. By Proposition 5.1, the smallest perimeter isosceles triangle under these conditions is PRS. Thus, it is enough to guarantee that the triangles $AB'C$ and $AB\overline{C}$ do not coincide with PRS which follows from the fact that ABC and PRS has exactly one common vertex. □

Now we turn to our second example. We start by taking the points $A = P = (0,0)$, $C = (1, v)$ and $S_x = (x, 0)$ and define R_x to be the point on the $\overrightarrow{S_x C}$ ray so that $|PR_x| = |R_x S_x|$. The next claim follows by elementary calculations, its proof is omitted.

Proposition 5.4. *For any $x \in (1,2)$, the perimeter of PR_xS_x can be expressed as*

$$\text{per}\,(PR_xS_x) = f_v(x) = x\left(1 + \sqrt{1 + \frac{v^2}{(1-x)^2}}\right). \tag{2}$$

and for any $v \in [0.56, \sqrt{3})$, the function f_v has a unique minimum in $(1,2)$ denoted by x_v^*.[2]

Example 5.5. Consider a triangle ABC that can be embedded in $\mathbb{R}^2$ as $A = (0,0)$, $C = (1, v)$ and $B = (x_b, 0)$ with $1 < x_b < x_v^*$ (the value x_v^* is defined in Proposition 5.4; see also Figure 17). Let PRS be the an isosceles triangle with $P = A$, $S = (x_v^*, 0)$, and R defined as the point on the $\overrightarrow{S_x C}$ ray with $|PR| = |RS|$. By definition, SPR is an isosceles container of ABC.

[2]The formula for x_v^* is given in the Appendix.

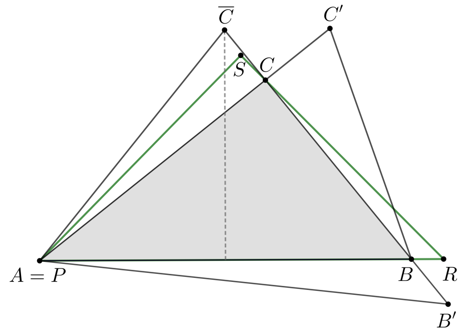


Figure 17: Illustration for Example 5.5

Claim 5.6. *The perimeter of PRS is smaller than the perimeter of any special container of ABC.*

Proof outline. By Corollary 2.6, we only need to show that PRS has a smaller perimeter than the special containers $AB'C, ABC'$, and $AB\overline{C}$. Observe that by the choices of x_v^* and x_b, we have $\text{per}\left(AB\overline{C}\right) = f_v(x_b) < f_v(x_v^*) = \text{per}\,(PRS)$.

We verify the remaining cases only for the fixed value $v = 0.7$. The function $f_{0.7}(x)$ takes its minimum at $x_{0.7}^* \approx 1.57517$, and thus $\text{per}\,(PRS) = f_{0.7}(x_{0.7}^*) \approx 4.056333$. On the other hand, if we set e.g. $x_b = 1.57$, we have $\text{per}\,(AB'C) \approx 4.229145$ and $\text{per}\,(ABC') \approx 4.084007$. □

5.2 Proof of Theorem 1.3

We start by proving that every smallest perimeter isosceles container of a triangle $\Delta = ABC$ is either a special container or one of the two triangles constructed in the Examples 5.2 and 5.5. Later, we will show that each of these five containers is realized as the unique minimum perimeter isosceles container for some triangle ABC. By Lemmas 2.1 and 2.2, we have the following statements on minimum perimeter isosceles containers.

Lemma 5.7. *Let PRS be any minimum area isosceles triangle enclosing ABC. Then*

(i) a side of PRS contains a side of ABC;
(ii) each side of PRS contains a vertex of ABC;
(iii) ABC and PRS share a common vertex;
(iv) no vertex of ABC lies in the interior of PRS.

In what follows, PRS is labeled so that $|PR| = |RS|$. If PRS shares the vertex R with ABC, but it does not share the angle at R, then we can get a smaller perimeter container by decreasing $\sphericalangle SRP$ (while keeping $|PR| = |RS|$ unchanged). Thus without loss of generality, we can assume that PRS shares the vertex P with ABC. The above restrictions allow only the following types of minimum perimeter isosceles containers that do not share an angle with ABC (see also Figure 18)[3]:

Case 1: *If two vertices of ABC lie in the interior of RS, or one of the vertices of ABC lies in the interior of the side RS and one lies in the interior of PS.*

The smallest perimeter isosceles containers of these types are precisely the non-special optimal containers shown in Examples 5.2 and 5.5.

[3]Note that in this subsection, we do not assume a special labeling of ABC, in particular, we do not necessarily have $|BC| < |AC| < |AB|$.

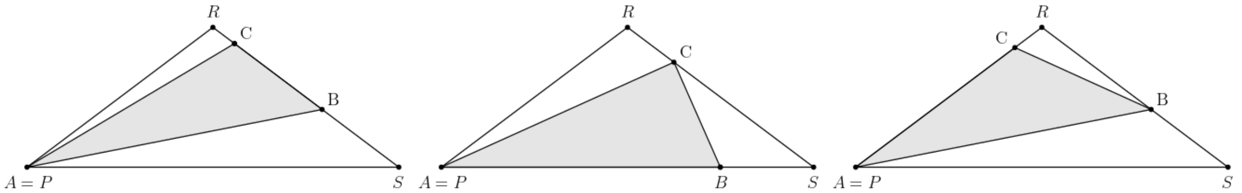
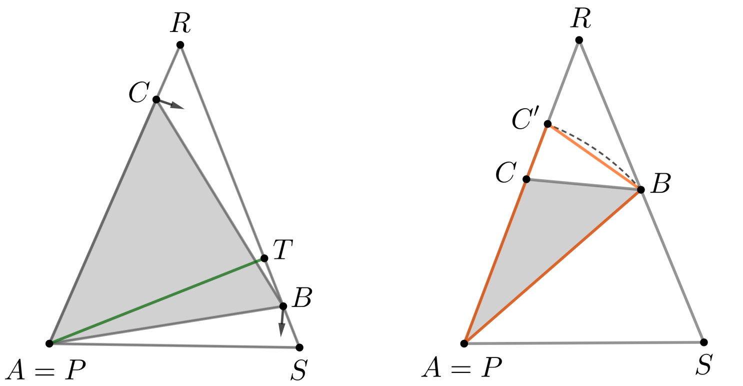


Figure 18: Illustration for Case 1 (left and middle) and Case 2 (right).

Case 2: *One vertex of ABC is in the interior of PR and one is in the interior of RS.*

Let T denote the base of the altitude perpendicular to RS and let B denote the vertex in RS. If $|SB| \leq |ST|$, then $\sphericalangle SBP \geq 90°$, hence we can rotate ABC with center $A = P$ such that the triangle remains in PRS and hence PRS was not minimal, see Figure 19. Note that this happens if PRS is not acute. From now on, we assume that $\sphericalangle SBP < 90°$. Hence $|AB| < |AR|$ if $B \neq R$.

If $|AC| < |AB|$, then we take $C' \in AR$ such that $|AB| = |AC'| < |AR|$ so $AC' \subset AR$ (Figure 19). Thus, ABC' is an isosceles container of ABC and $ABC' \subsetneq PRS$. Hence, PRS was not minimal. Therefore, we may assume that $|AC| > |AB|$, as $|AC| = |AB|$ would imply that ABC was isosceles.



Subcase when $|SB| \leq |ST|$ Subcase when $|AC| < |AB|$

Figure 19: Simple configurations of Case 2.

If $\sphericalangle RAB < \sphericalangle BRA$ holds, let B' be the point on the line AB such that $|AC| = |AB'|$ then we have $|AB'| = |AC| < |AR| = |RS|$, and hence $\text{per}\,(AB'C) < \text{per}\,(PRS)$, thus PRS was not minimal. Thus, assume that $\sphericalangle BRA \leq \sphericalangle RAB$ and as $\sphericalangle RAB + \sphericalangle BRA = \sphericalangle SBP < 90°$, we get that $\sphericalangle BRA = \sphericalangle PRS < 45°$.

For the remaining part, we embed the configuration in $\mathbb{R}^2$ such that $P = A = (0,0)$, $R = (x,0)$ and $B = (1,h)$, where $x > 1$ and $h > 0$, see Figure 20. Under the assumptions that $|AC| > |AB|$ and $\sphericalangle PRS < \min(\sphericalangle RAB, 45°)$, we show that $\text{per}\,(PRS)$ as a function of x is increasing. Thus, as $B \neq R, C \neq R$ there exists a smaller perimeter isosceles container of ABC than PRS (e.g. $PR'S'$ in Figure 20). The condition $\sphericalangle PRS < \sphericalangle RAB$ implies that $|PT'| < |RT'|$, where T' is the base of the altitude of PR, hence $x > 2$.

Clearly, $|BR| = \sqrt{h^2 + (x-1)^2}$ and $\sin(\sphericalangle PRS) = \frac{h}{\sqrt{h^2+(x-1)^2}}$. Hence $\text{per}\,(PRS) =$

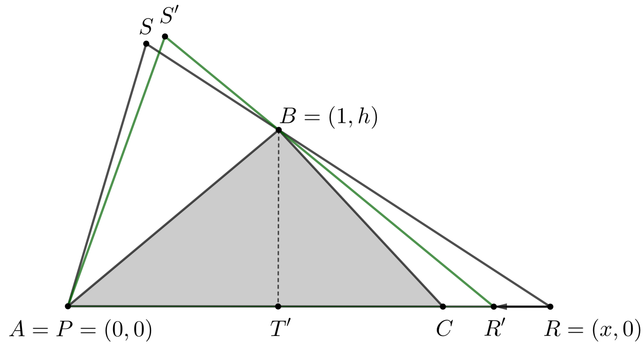


Figure 20: Case 3 in a coordinate system.

$2x(1 + \sin(\frac{\sphericalangle PRS}{2}))$. As $\sin\delta = 2\sin(\frac{\delta}{2})\sqrt{1-\sin^2(\frac{\delta}{2})}$, we get

$$\sin\left(\frac{\sphericalangle PRS}{2}\right) = \frac{1}{\sqrt{2}}\sqrt{1 \pm \frac{x-1}{h}\sqrt{\frac{1}{1+\left(\frac{x-1}{h}\right)^2}}},$$

where $\pm$ is taken to be $-$ sign, since $\sphericalangle PRS < 45°$. Therefore,

$$\text{per}\,(PRS) = 2x + \sqrt{2}x\sqrt{1 - \frac{x-1}{h}\sqrt{\frac{1}{1+(\frac{x-1}{h})^2}}}.$$

Let $y = \frac{x-1}{h}$ and let $f_h(y) = (1+hy)\left(1+\sqrt{\frac{1-y}{2}\sqrt{\frac{1}{1+y^2}}}\right)$. It follows from our assumptions that $y > 1/h$. We show that $f_h(y)$ is strictly increasing in y, which implies that PRS is not a minimum perimeter isosceles container of ABC. For $g(y) := 1 + \sqrt{\frac{1-y}{2}\sqrt{\frac{1}{1+y^2}}}$, we show that $f_h'(y) = ((1+hy)g(y))' > 0$, equivalently $-g'(y) < \frac{hg(y)}{1+hy}$. Simple calculation shows that

$$-g'(y) = \frac{1}{2\sqrt{2}(1+y^2)}\sqrt{1+\sqrt{\frac{y^2}{1+y^2}}} < \frac{1}{2(1+y^2)},$$

where the last inequality holds as $\frac{y^2}{1+y^2} < 1$ for all $y \in \mathbb{R}$. Note that $g(y) > 1$, hence $hg(y) > h$. Thus, it is enough to show that

$$\frac{1}{2(1+y^2)} < \frac{h}{1+hy} \qquad \text{if } y = \frac{x-1}{h} > \frac{1}{h}.$$

This is true if and only if $0 < 2hy^2 - hy + 2h - 1$. This holds if the roots of this polynomial satisfy $y_1 < y_2 = \frac{h+\sqrt{-15h^2+8h}}{2h} \leq \frac{1}{h}$. The last inequality is equivalent to $0 \leq 4h^2 - 3h + 1 = (2h-1)^2 + h$, which is true for $h > 0$. Therefore, the argument above verifies that in this case PRS is not minimal. This concludes the proof in Case 2.

Note on realizability. Now we briefly discuss that each of the special containers $AB'C$, ABC', $AB\overline{C}$, and triangles constructed in Examples 5.2 and 5.5 can occur as a minimum perimeter container for some ABC. It is easy to find triangles for which one of the special containers is the best among the five options.

To see that the container of Example 5.5 is optimal for some triangles, note that the construction presented in Example 5.2 works only if the special containers of ABC satisfy $\gamma^* \in (\sphericalangle AB'C, \sphericalangle AB\overline{C})$. Now consider the example from the proof of Claim 5.6. It can be easily calculated that under these choices $\sphericalangle(B\overline{C}A) \approx 78,310868°$. This (together with Claim 5.6) implies that for the example presented in the proof of Claim 5.6, the container described in Example 5.5 is better the one given in Example 5.2 and than any special container.

Finally, we show that the container presented in Example 5.2 is optimal for some triangles. Following the construction in the proof of Claim 5.6, consider the triangle ABC with $A = (0, 0)$, and $C = (1, 0.8)$ and $B = (0, x^*_{0.8})$ such that $f_{0.8}(x)$ takes its minimum at $x^*_{0.8} \approx 1.62474$. We get that the container constructed in Example 5.5 coincides with the special container $AB\overline{C}$ and $\operatorname{per}\left(AB\overline{C}\right) = f_{0.8}(x^*_{0.8}) \approx 4.264511$. Simple calculation shows that $\operatorname{per}(ABC') \approx 4.3250804$, thus $\operatorname{per}\left(AB\overline{C}\right) < \operatorname{per}(ABC')$. Since $\sphericalangle(B\overline{C}A) \approx 75.974334° < \gamma* < \sphericalangle(BCA) = \sphericalangle(B'CA) \approx 89.327359°$, the construction of Example 5.2 provides smaller perimeter than any of the special containers, indeed if we let SPR to be the container constructed in Example 5.2 for out choice of ABC, then we get $\operatorname{per}(PRS) \approx 4.264431$.

This concludes the proof of Theorem 1.3. □

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Appendix - Basic inequalities for embedded triangles

Proof of Lemma 2.1

Observe that if Δ and Δ' are similar triangles then $\text{per}(\Delta) < \text{per}(\Delta')$ and $\text{area}(\Delta) < \text{area}(\Delta')$ hold if and only if $\text{diam}(\Delta) < \text{diam}(\Delta')$. Therefore, it is sufficient to prove following the statements:

Claim 5.8. *Let Δ_1 be a triangle inside the triangle Δ_2. If Δ_1' is a maximum diameter triangle similar to Δ_1 contained in Δ_2, then*

(A) Δ_1' has two vertices on a side of Δ_2;
(B) each side of Δ_2 contains a vertex of Δ_1';
(C) Δ_1' and Δ_2 have a common vertex.

Proof. Let $\Delta_1 = ABC$, $\Delta_1' = A'B'C'$, and $\Delta_2 = DEF$.

(A) The statement was proved by Post [21].

(B) Assume indirectly that the side DE does not contain any of the vertices A', B', C' and let A' the closest one to DE. Let l' denote the line parallel to DE which contains A'. Let $D' = l' \cap DF$ and $E' = l' \cap EF$. Then the triangle $D'E'F \subset DEF$ is similar to DEF and it contains $A'B'C'$. If $\mathcal{S}$ is the homothety with center F and ratio $|DF|/|D'F|$, then $\mathcal{S}(A'B'C') \subset DEF$ is a similar copy of $A'B'C'$ which has larger diameter than $A'B'C'$, a contradiction. Hence, DE contains a vertex of $A'B'C'$. The argument applies for any other side of DEF.

(C) By part (A), $A'B'C'$ has two vertices that are on the same side of DEF. If any of these vertices coincide with a vertex of DEF, then the second part of the statement holds. Otherwise, the third vertex of $A'B'C'$ must be contained in two sides of DEF, thus it is a vertex of DEF. □

Proof of Lemma 2.2

Let $\Delta = ABC$, $\Delta_1 = DEF$, and $\Delta_2 = PRS$.

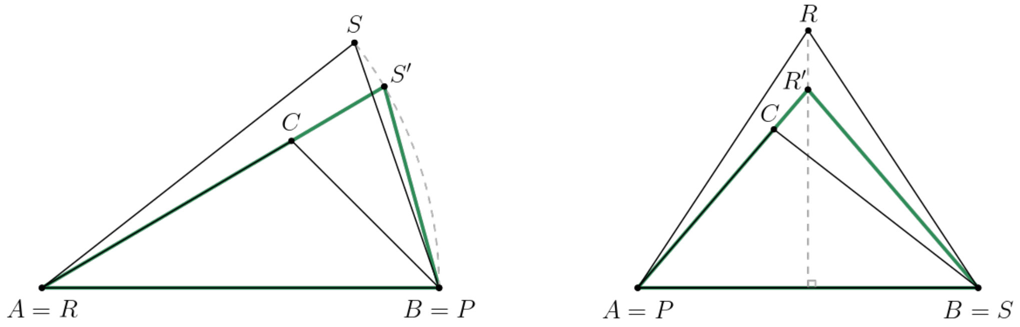
(i) We prove that the vertices of minimum area (resp. perimeter) isosceles containers are contained in a compact subset of the plane. (A similar result for the set of embedded triangles is trivial.)

The case of the area was handled in [12]. Concerning the perimeter, it is easy to see that there is a special container whose perimeter is at most twice as large as the one of ABC. Thus any minimum perimeter container of ABC is contained in the closed ball $\mathcal{B}$ centered at A with radius $2\text{per}(ABC)$.

Thus the collection of those isosceles triangles $\overline{\Delta}$ satisfies $ABC \subset \overline{\Delta} \subset \mathcal{B}$ can be considered as a compact subset of $\mathbb{R}^6$ with respect to the Euclidean topology. The result simply follows from the fact that both the area and the perimeter are continuous functions on the parameter set.

(iii) The statement for the minimum area has been proved in [12][Lemma 3.2].

Let PRS be a minimum perimeter isosceles container of ABC. Then, by Claim 5.8, one side of PRS contains two vertices of ABC, and by Lemma 2.1 every side of PRS contains a vertex of ABC and the triangles share a common vertex. Thus either each vertex of ABC is on a side of PRS, as we stated, or two vertices of ABC coincide with two vertices of PRS (i.e., the triangles share a common side) as in Figure 21. In the latter case we distinguish two subcases when the common side is a leg (Case a.) or a base (Case b.) of PRS. As Figure 21 illustrates, in both subcases we can find a smaller isosceles triangle (green in Figure 21) by shrinking the original triangle PRS so that the modified isosceles triangle contains ABC and has smaller area and perimeter.



Case a) Case b)

Figure 21: Illustration for the proof of Lemma 2.2 (iii).

(ii) The proof of this case is analogous to the proof of case (iii) for the perimeter and the proof of [12][Lemma 3.2] for the area.

□

Proof of Lemma 2.3

The areas of special embedded triangles of the first, the second and the third type are the following:

$$\text{area}\,(A'BC) = \frac{a^2 \sin\gamma}{2}, \quad \text{area}\,(AB'C) = \frac{b^2 \sin\alpha}{2}, \quad \text{area}\,(A''BC) = \frac{a^2 \sin\beta}{2}.$$

$$\text{area}\,(A_1BC) = \frac{a^2 \tan\beta}{4}, \quad \text{area}\,(AB_1C) = \frac{b^2 \tan\alpha}{4}, \quad \text{area}\,(ABC_1) = \frac{c^2 \tan\alpha}{4}.$$

$$\text{area}\left(\overline{A}BC\right) = \frac{a^2 \sin(2\beta)}{2}, \quad \text{area}\left(\overline{\overline{A}}BC\right) = \frac{a^2 \sin(2\gamma)}{2}, \quad \text{area}\left(A\overline{B}C\right) = \frac{b^2 \sin(2\gamma)}{2}.$$

1. It follows easily from the previous equalities that

$$\text{area}\,(A''BC) = \frac{a^2 \sin\beta}{2} < \frac{a^2 \sin\beta}{2}\frac{c}{b} = \frac{a^2 \sin\beta}{2}\frac{\sin\gamma}{\sin\beta} = \text{area}\,(A'BC)\,.$$

2. We have seen that $\text{area}\,(A_1BC) = \frac{a^2 \tan\beta}{4}$ and $\text{area}\,(AB'C) = \frac{b^2 \sin\alpha}{2}$. Now $\frac{a^2 \tan\beta}{4} < \frac{b^2 \sin\alpha}{2}$ is equivalent to $\frac{a^2}{2b^2} = \frac{\sin^2\alpha}{2\sin^2\beta} < \frac{\sin\alpha\cos\beta}{\sin\beta}$, by using the law of sines. Reformulating this we have $\sin\alpha < \sin(2\beta)$, which always holds as $\alpha < \min\{2\beta, 180 - 2\beta\}$.

   The inequality $\frac{b^2 \tan\alpha}{4} < \frac{c^2 \tan\alpha}{4}$ implies that $\text{area}\,(AB_1C) < \text{area}\,(ABC_1)$.

3. We first prove

$$\text{area}\left(\overline{A}BC\right) < \text{area}\,(ABC_1)\,.$$

Using the equations given above this is equivalent to

$$\frac{a^2 \sin(2\beta)}{2} < \frac{c^2 \tan\alpha}{4} \iff a^2 \sin\beta \cos\beta < \frac{c^2 \sin\alpha}{4\cos\alpha}.$$

By replacing $\frac{\sin\alpha}{\sin\beta}$ with $\frac{a}{b}$ we obtain $\frac{4ab}{c^2}\cos\alpha\cos\beta < 1$. Law of cosines gives

$$\frac{4ab}{c^2}\frac{2bc}{b^2+c^2-a^2}\frac{2ac}{a^2+c^2-b^2} < 1.$$

After a simple rearrangement we obtain

$$(c^2 + (a^2 - b^2))(c^2 - (a^2 - b^2)) < c^4,$$

which holds by $a \neq b$.

The inequality area $\left(A\overline{B}C\right) >$ area $\left(\overline{\overline{A}}BC\right)$ follows from

$$\text{area}\left(\overline{\overline{A}}BC\right) = \frac{a^2 \sin(2\gamma)}{2} < \frac{b^2 \sin(2\gamma)}{2} = \text{area}\left(A\overline{B}C\right).$$

The length of the legs of the isosceles triangles $A\overline{B}C$ and $AB'C$ are equal to b, but the apex angle is greater in the latter triangle. As both apex angles are upper bounded by $\alpha < 90°$, we get area $\left(A\overline{B}C\right) <$ area $(AB'C)$.

4. If ABC is obtuse, then $0° < \alpha < 45°$. The function $\sin(2\alpha)$ is strictly monotonically increasing on the interval $[0°45°]$. Since $\alpha < \beta$ we have $\alpha < 180° - \gamma < 90°$. Thus

$$\sin(2\alpha) < \sin(180 - \gamma) = \sin\gamma,$$

$$2\sin\alpha\cos\alpha < \sin\gamma$$

$$\sin\alpha < \frac{\sin\gamma}{2\cos\alpha}.$$

We multiply both sides of the inequality with the positive number $\frac{\sin\alpha\sin\gamma}{2}$:

$$\frac{\sin^2\alpha\sin\gamma}{2} < \frac{\sin\alpha\sin^2\gamma}{4\cos\alpha}.$$

By the law of sines this is equivalent to

$$\frac{a^2\sin\gamma}{2} < \frac{c^2\sin\alpha}{4\cos\alpha},$$

which implies

$$\text{area}(A'BC) = \frac{a^2\sin\gamma}{2} \leq \frac{c^2\tan\alpha}{4} = \text{area}(ABC_1).$$

□

Proof of Lemma 2.5

1. $\mathrm{per}\,(ABC') < \mathrm{per}\,(ABC'')$ follows from the Hinge theorem since $|AB| = |AC'| = |AC''| = c$ and $\alpha = \sphericalangle C'AB < \sphericalangle C''BA = \beta < 90°$.

2. Notice that $|AB| = |AC_1| = |AC_2| = c$. On the other hand $\sphericalangle BAC' = \alpha < \sphericalangle BAC_2 = 180° - 2\beta < 180° - 2\alpha$ since $\alpha + \beta + \gamma = 180°$ and $\alpha < \beta < \gamma$. Thus the Hinge theorem gives $\mathrm{per}\,(ABC') < \mathrm{per}\,(ABC_2) < \mathrm{per}\,(ABC_1)$.

3. Similarly to the previous cases, we have $|AC| = |CB'| = |CB_1| = b$ and $\sphericalangle ACB' < \sphericalangle ACB_1$ so the perimeter of $AB'C$ is smaller than that of AB_1C.

4. First note that the triangles $\overline{A}BC$ and $A\overline{B}C$ do exist if and only if ABC is acute, hence we assume $\gamma < 90°$. First we show that $\mathrm{per}\,(ABC') > \mathrm{per}\left(\overline{A}BC\right)$. Indeed, $\mathrm{per}\,(ABC') = 2c(1 + \sin(\alpha/2))$ and $\mathrm{per}\left(\overline{A}BC\right) = a(1 + 1/\cos\gamma)$, thus it is enough to show that
$$2c(1 + \sin(\alpha/2)) < a(1 + 1/\cos\gamma).$$
Using the law of sines we obtain
$$\frac{2(1 + \sin(\alpha/2))}{1 + 1/\cos\gamma} < \frac{a}{c} = \frac{\sin\alpha}{\sin\gamma}.$$
Equivalently,
$$\frac{2(1 + \sin(\alpha/2))}{\sin\alpha} < \frac{1 + 1/\cos\gamma}{\sin\gamma} = \frac{2(1 + \cos\gamma)}{\sin(2\gamma)} = \frac{2(1 + \sin(90° - \gamma))}{\sin(180° - 2\gamma)}.$$
Note that $(1 + \sin x)/\sin(2x)$ is strictly decreasing on the interval $(0°, 60°)$. It is clear that $90° - \gamma < \alpha$, otherwise $90° < \beta < \gamma$, contradicting the assumption that $\gamma < 90°$. Hence we get that $\mathrm{per}\,(ABC') < \mathrm{per}\left(\overline{A}BC\right)$.

   The inequality $\mathrm{per}\left(\overline{A}BC\right) < \mathrm{per}\left(A\overline{B}C\right)$ simply follows from that fact that $\overline{A}BC$ and $A\overline{B}C$ are similar triangles such that the base of $\overline{A}BC$, is of length a so it shorter than the base of $A\overline{B}C$, which is of length b.

□

Formula for x_v^*

Let $\delta_v := \sqrt{48v^6 + 81v^4} - 9v^2$, then x_v^* can be expressed as

$$x_v^* = \frac{1}{2}\left(1 + \sqrt{1 + \sqrt[3]{\frac{2\delta_v}{9}} - \sqrt[3]{\frac{2^5v^6}{3\delta_v}}} + \sqrt{2 + \sqrt[3]{\frac{2^5v^6}{3\delta_v}} - \sqrt[3]{\frac{2\delta_v}{9}} - \frac{2}{\sqrt{1 + \sqrt[3]{\frac{2\delta_v}{9}} - \sqrt[3]{\frac{2^5v^6}{3\delta_v}}}}}\right).$$