# Solutions to the discrete Pompeiu problem and to the finite Steinhaus tiling problem

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**Abstract**

Let $K$ be a nonempty finite subset of the Euclidean space $\mathbb{R}^k$ ($k \geq 2$). We prove that if a function $f\colon \mathbb{R}^k \to \mathbb{C}$ is such that the sum of $f$ on every congruent copy of $K$ is zero, then $f$ vanishes everywhere. In fact, a stronger, weighted version is proved. As a corollary we find that every finite subset of $\mathbb{R}^k$ having at least two elements is a Jackson set; that is, no subset of $\mathbb{R}^k$ intersects every congruent copy of $K$ in exactly one point.

## 1 Introduction and main results

A compact subset $K$ of the plane having positive Lebesgue measure is said to have the *Pompeiu property* if the following condition is satisfied: if $f\colon \mathbb{R}^2 \to \mathbb{C}$ is a continuous function such that $\int_{\sigma(K)} f\, d\lambda_2 = 0$ for every isometry $\sigma$ of the plane, then $f \equiv 0$. It is known that the disc does not have the Pompeiu property, while all polygons have. The Pompeiu problem asks if a connected compact set with a smooth boundary that does not have the Pompeiu property is necessarily a disc. As for the history of the problem, see [16], [18], [21].

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Replacing the Lebesgue measure $\lambda_2$ by the counting measure and the isometry group by an arbitrary family $\mathcal{F}$ of functions mapping a set $X$ into itself, we obtain the following notion.

Let $\mathcal{F}$ be a family of functions mapping a set $X$ into itself, and let $K$ be a nonempty finite subset of $X$. We say that $K$ *has the discrete Pompeiu property with respect to the family $\mathcal{F}$ if the following condition is satisfied: whenever $f\colon X \to \mathbb{C}$ is such that*

$$\sum_{x \in K} f(\phi(x)) = 0 \tag{1}$$

*for every $\phi \in \mathcal{F}$, then $f \equiv 0$.*

We also define a stronger property as follows.

We say that an $n$-tuple $(a_1, \ldots, a_n)$ of (not necessarily distinct) elements of $X$ *has the weighted discrete Pompeiu property with respect to the family $\mathcal{F}$ if the following condition is satisfied: whenever $c_1, \ldots, c_n$ are complex numbers with $\sum_{j=1}^{n} c_j \neq 0$ and $f\colon X \to \mathbb{C}$ is such that $\sum_{j=1}^{n} c_j \cdot f(\phi(a_j)) = 0$ for every $\phi \in \mathcal{F}$, then $f \equiv 0$.*

Note that the condition $\sum_{j=1}^{n} c_j \neq 0$ is necessary: if $\sum_{j=1}^{n} c_j = 0$, then every constant function satisfies the condition.

The problem of characterizing sets with the discrete Pompeiu property has been investigated in several contexts. The case of translations in groups are treated in [10], [15], [17], [22]. As it turns out, no finite subset having at least two elements of a torsion free Abelian group (in particular, of $\mathbb{R}^k$) has the discrete Pompeiu property with respect to translations [9, Proposition 1.1]).

The case of similarities in $\mathbb{R}^2$ was considered by C. De Groote and M. Duerinckx. They proved in [4] that every finite and nonempty subset of $\mathbb{R}^2$ has the discrete Pompeiu property with respect to direct similarities. The weighted version in the plane is proved in [9, Theorem 3.3]. The notion of similarity can be defined in every Abelian group as follows. We say that the map $\phi\colon G \to G$ is a *simple similarity* of the Abelian group $G$ if there is an element $a \in G$ and there is a positive integer $k$ such that $\phi(x) = a + k \cdot x$ for every $x \in G$. The following statement generalizes the results of [4] and [9] cited above.

**Proposition 1.** *In every Abelian group $G$, every $n$-tuple of points of $G$ has the weighted discrete Pompeiu property with respect to the family of simple similarities.*

We prove Proposition 1 at the end of this section.

In the sequel we consider the case when $X = \mathbb{R}^k$ and $\mathcal{F} = G_k$ is the family of rigid motions of $\mathbb{R}^k$. In this context the first relevant result appeared in [8], stating that the vertices of the unit square has the discrete Pompeiu property with respect to $G_2$. Later the discrete Pompeiu property of all parallelograms and some other special four-element subsets of the plane was established in [9]. Our main result is the following.

**Theorem 2.** *For every $k \geq 2$ and $a_1, \dots, a_n \in \mathbb{R}^k$, the $n$-tuple $(a_1, \dots, a_n)$ has the weighted discrete Pompeiu property with respect to the group $G_k$ of rigid motions of $\mathbb{R}^k$. In particular, every nonempty finite subset of $\mathbb{R}^k$ has the discrete Pompeiu property with respect to $G_k$.*

The proof of Theorem 2 will be given in Sections 3 and 4.

**Remark 3.** For $k = 1$ the statement of Theorem 2 is false: if $K = \{1, \dots, n\}$ and $f(x) = e^{2\pi x/n}$, then (1) holds for every $\phi \in G_1$. More generally, if $n \geq 2$, $K = \{a_1, \dots, a_n\}$, $\lambda$ is a root of the entire function $\sum_{j=1}^n e^{a_j z}$ and $f(x) = e^{\lambda x}$, then (1) holds for every translation $\phi$. If $K$ is symmetric; that is, if $K = -K$, then (1) holds for every isometry $\phi$ of $\mathbb{R}$.

Let $a_1, \dots, a_n$ and $z_0$ be given points in $\mathbb{R}^k$, and let $c_1, \dots, c_n$ be complex numbers with $\sum_{j=1}^n c_j \neq 0$. Let $\Sigma$ denote the system of linear equations

$$\sum_{j=1}^{n} c_j \cdot x_{\phi(a_j)} = 0 \quad (\phi \in G_k), \qquad x_{z_0} = 1,$$

where $x_z$ is an unknown for every $z \in \mathbb{R}^k$. By Theorem 2, $\Sigma$ has no solution. Then, by a well-known fact of linear algebra, there is a finite subsystem of $\Sigma$ that has no solution. Therefore, we obtain the following.

**Corollary 4.** *Suppose $k \geq 2$, $a_1, \dots, a_n, z_0 \in \mathbb{R}^k$ and $c_1, \dots, c_n \in \mathbb{C}$ are given such that $\sum_{j=1}^n c_j \neq 0$. Then there is a finite set $E \subset \mathbb{R}^k$ containing $z_0$ with the following property: whenever $f \colon E \to \mathbb{C}$ is such that $\sum_{j=1}^n c_j \cdot f(\phi(a_j)) = 0$ for every $\phi \in G_k$ satisfying $\phi(a_j) \in E$ for every $j = 1, \dots, n$, then $f(z_0) = 0$.*

Similarly to the classical Pompeiu problem, the main tool in proving Theorem 2 is harmonic analysis. Since our objects are finite, we need harmonic analysis on discrete groups. Let $G$ be an Abelian group equipped with the discrete topology. We denote by $C(G)$ the set of all maps from $G$ into $\mathbb{C}$ equipped with the topology of pointwise convergence. More precisely, a set $U \subset C(G)$ is open if, for every $f \in U$ there is a finite set $F \subset G$ and an $\varepsilon > 0$ such that, if $g \in C(G)$ and if $|g(x) - f(x)| < \varepsilon$ for every $x \in F$, then $g \in U$. (In fact, this is the same as the product topology of $\mathbb{C}^G$.) A nonzero function $m \in C(G)$ is called an *exponential*, if $m$ is multiplicative; that is, if $m(x+y) = m(x) \cdot m(y)$ for every $x, y \in G$. By a *variety* we mean a translation invariant closed linear subspace of $C(G)$. We say that *harmonic analysis* holds on $G$ if every nonzero variety contains an exponential.

By [14, Theorem 1], *harmonic analysis holds on a discrete Abelian group $G$ if and only if the torsion free rank of $G$ is less than continuum.* Therefore, harmonic analysis does not hold on the additive group of $\mathbb{R}^k$. On the other hand, it holds on every countable Abelian group by the theorem above, and so we have to work on suitable countable subgroups of $\mathbb{R}^k$. The next proof of Proposition 1 is hardly more than an application of this fact.

**Proof of Proposition 1.** Let $G$ be an Abelian group, and let $a_1, \ldots, a_n \in G$ and $c_1, \ldots, c_n \in \mathbb{C}$ are given such that $\sum_{j=1}^n c_j \neq 0$. Let $f \colon G \to \mathbb{C}$ be such that $\sum_{j=1}^n c_j \cdot f(b + k \cdot a_j) = 0$ for every $b \in G$ and $k = 1, 2, \ldots$. We have to prove that $f \equiv 0$. Suppose this is not true, and let $x \in G$ be such that $f(x) \neq 0$. Let $H$ denote the subgroup of $G$ generated by $x$ and $a_1, \ldots, a_n$, and let $V$ denote the set of all functions $g \colon H \to \mathbb{C}$ such that $\sum_{j=1}^n c_j \cdot g(b + k \cdot a_j) = 0$ for every $b \in H$ and $k = 1, 2, \ldots$. It is clear that $V$ is a linear space over $\mathbb{C}$, and that $V$ is invariant under translations by elements of $H$. It is also easy to see that $V$ is closed in the set $\mathbb{C}^H$ equipped with the product topology. This means that $V$ is a variety on the discrete, countable additive group $H$.

Since $f|_H \in V$, we have $V \neq \{0\}$. Then, by [14, Theorem 1], $V$ contains an exponential; that is, a function $m \colon H \to \mathbb{C}$ such that $m \neq 0$ and $m(x+y) = m(x) \cdot m(y)$ for every $x, y \in H$. Since $m \in V$, we have

$$\sum_{j=1}^n c_j \cdot m(a_j)^k = \sum_{j=1}^n c_j \cdot m(k \cdot a_j) = 0 \qquad (k = 1, 2, \ldots). \tag{2}$$

Permuting the elements $a_1, \ldots, a_n$ if necessary, we may assume that there is an $1 \le s \le n$ such that $m(a_1), \ldots, m(a_s)$ are distinct, and for every $s < j \le n$ $m(a_j)$ equals one of $m(a_1), \ldots, m(a_s)$. Then, by (2) we have

$$\sum_{j=1}^{s} d_j \cdot m(a_j)^k = 0 \tag{3}$$

for every $k = 1, 2, \ldots$, where $d_j = \sum\{c_\nu \colon m(a_\nu) = m(a_j)\}$ for every $j = 1, \ldots, s$. Then we have $\sum_{j=1}^{s} d_j = \sum_{j=1}^{n} c_j \ne 0$. Now, (3) with $k = 1, \ldots, s$ constitute a system of linear equations with unknowns $d_1, \ldots, d_s$. The determinant of this system is nonzero by the nonvanishing of Vandermonde determinants. Therefore, we have $d_1 = \ldots = d_s = 0$, which is impossible. This contradiction proves the statement. $\square$

## 2 Applications to coloring problems and to the finite Steinhaus tiling problem

Theorem 2 has the following obvious consequence.

**Corollary 5.** *If $k \ge 2$, $K \subset \mathbb{R}^k$ has $n$ elements, $d \mid n$ and $\mathbb{R}^k$ is colored with $d$ colors, then there is a congruent copy of $K$ containing more than $n/d$ points of the same color.*

Indeed, otherwise there is a partition $\mathbb{R}^k = A_1 \cup \ldots \cup A_d$ such that every congruent copy of $K$ intersects each of the sets $A_1, \ldots, A_d$ in exactly $n/d$ points. Let $b_1, \ldots, b_d$ be nonzero complex numbers with $\sum_{j=1}^{d} b_j = 0$. If we define $f(x) = b_j$ for every $x \in A_j$ $(j = 1, \ldots, d)$, then $\sum_{x \in K} f(\phi(x)) = \sum_{j=1}^{d} (n/d) \cdot b_j = 0$ for every $\phi \in G_k$, contradicting Theorem 2.

In the case of $n = 4$, $d = 2$ we obtain the following.

**Corollary 6.** *If $k \ge 2$, $|K| = 4$ and if $\mathbb{R}^k$ is colored with two colors, then there is a congruent copy of $K$ containing at least three points of the same color.*

If $k = 2$ and $K$ is a rectangle, then we obtain the following: *For every right triangle $T$ and for every coloring the plane with two colors, there is always a monochromatic triangle congruent to $T$.* This is L.E. Shader's theorem [20].

The special case $d = n$ in Corollary 5 is closely connected to the general Steinhaus problem: decide, for a given set $K \subset \mathbb{R}^k$ if there is a set $S$ that intersects every congruent copy of $K$ in exactly one point. The original question of Hugo Steinhaus, posed in the 1950s, was the following. Is there a set $S$ in the plane such that every set congruent to $\mathbb{Z}^2$ has exactly one point in common with $S$? This question was answered in the affirmative by S. Jackson and R.D. Mauldin in 2002 [6] (see also [7]). Analogous results were obtained by P. Komjáth [12], [13] and J.H. Schmerl [19] for $\mathbb{Z}, \mathbb{Q}$ and $\mathbb{Q}^n$.

These results motivated S. Jackson to ask if there is a finite set $K \subset \mathbb{R}^2$ having at least two points such that for a suitable set $S \subset \mathbb{R}^2$, every isometric copy of $S$ meets $K$ in exactly one point. A finite set $K \subset \mathbb{R}^k$ is called a *Jackson set* if there is no such set $S$ (see [3]). It is clear that singletons are not Jackson sets (as $S = \mathbb{R}^k$ works), and it is easy to see that all 2-element sets are Jackson sets. It is known that every set of cardinality $3, 4, 5$ or $7$ is a Jackson set (see [5]). It is also known that for every finite set $K \subset \mathbb{R}^k$ having at least two elements there is no measurable sets that intersect each congruent copy of $K$ in exactly one point [11].

Now, we show that *if a finite set of cardinality at least two has the Pompeiu property, then it is a Jackson set.* We apply the argument of [3, Proposition 1.3]. Suppose that $K \subset \mathbb{R}^k$, $|K| \geq 2$, and that $S \subset \mathbb{R}^k$ is such that $|S \cap \sigma(K)| = 1$ for every $\sigma \in G_k$. Then the sets $S + a$ $(a \in K)$ are pairwise disjoint. Indeed, if $c \in (S + a) \cap (S + b)$, where $a, b \in K$ and $a \neq b$, then $c - a, c - b \in S$. In this case, however, $|S \cap \sigma(K)| \geq 2$ for every rigid motion $\sigma$ such that $\sigma(a) = c - a$ and $\sigma(b) = c - b$, which is impossible. Since every congruent copy of $K$ must be covered by $\bigcup_{a \in K}(S + a)$, it follows that the sets $S + a$ $(a \in K)$ constitute a partition of $\mathbb{R}^k$ such that every congruent copy of $K$ intersects each of the sets $S + a$ in exactly one point. As we saw in the proof of Corollary 5, this contradicts the Pompeiu property of the set $K$.

By Theorem 2 we obtain the following:

**Corollary 7.** *Every finite subset of $\mathbb{R}^k$ $(k \geq 2)$ having at least two elements* *is a Jackson set.*

**Remark 8.** For $k = 1$ the statement of the corollary is false: if $K = \{1, \ldots, n\}$, then $S = \bigcup_{t \in \mathbb{Z}}([0,1) + n \cdot t)$ intersects every congruent copy of $K$ in exactly one point, so $K$ is not a Jackson set. For more on Jackson sets in $\mathbb{R}$, see [3].

**Remark 9.** Note that the definition of Jackson set uses isometries and not just rigid motions, while Theorem 2 is about the Pompeiu property with respect to the family of rigid motions. Therefore, Corollaries 5, 6, 7 remain true if we replace congruent copies of $K$ by images of $K$ under rigid motions, and, in the definition of Jackson sets we replace isometries by rigid motions.

**Remark 10.** Let $K \subset \mathbb{R}^k$ be given, and let $m$ be a positive integer. We say that the set $S \subset \mathbb{R}^k$ is an *$m$-Steinhaus set for $K$* if every congruent copy of $S$ intersects $K$ in exactly $m$ points. The finite set $K$ is called an *$m$-Jackson set*, if there is no $m$-Steinhaus set for $K$. Obviously, the sets of cardinality $< m$ are $m$-Jackson sets, and if $|K| = m$, then $K$ is not an $m$-Jackson set, as $S = \mathbb{R}^k$ is an $m$-Steinhaus set for $K$. The following generalization of Corollary 7 can be obtained by a similar argument.

**Corollary 11.** *Every finite subset of $\mathbb{R}^k$ $(k \geq 2)$ having more than $m$ elements is an $m$-Jackson set.*

We sketch the proof. Suppose $S$ is an $m$-Steinhaus set for $K$. Then the sets $S + a$ $(a \in K)$ constitute an $m$-cover of $\mathbb{R}^k$. Indeed, if $x \in \mathbb{R}^k$, then $x \in S + a$ $(a \in K) \iff x - a \in S$. Since $|(x - K) \cap S| = m$, it follows that every point of $\mathbb{R}^k$ is contained in exactly $m$ of the sets $S + a$ $(a \in K)$.

Let $|K| = n > m$, and let $c_1, \ldots, c_n$ be complex numbers such that $\sum_{j=1}^{n} c_j = 0$ and $\sum_{j \in I} c_j \neq 0$ for every $m$-element subset $I$ of the set of indices $\{1, \ldots, n\}$. For $x \in \mathbb{R}^k$ let $f(x)$ be the sum of those numbers $c_j$ for which $x \in S + a_j$. (Formally, let $f(x) = \sum\{c_j \colon x \in S + a_j\}$.) Then $f$ is nowhere zero, but $\sum_{x \in K} f(\sigma(x)) = 0$ for every $\sigma \in G_k$. By Theorem 2, this is impossible.

(Note that if $k$ is odd, then the argument above needs isometries and not just rigid motions.)

# 3 Proof of Theorem 2 for $k = 2$

Let $(a_1, \ldots, a_n)$ be a fixed $n$-tuple of elements of $\mathbb{R}^2$. We identify $\mathbb{R}^2$ with $\mathbb{C}$, and denote by $S^1$ the unit circle $\{u \in \mathbb{C} \colon |u| = 1\}$. Let $c_1, \ldots, c_n$ be complex numbers with $\sum_{j=1}^n c_j \neq 0$, and suppose the function $f \colon \mathbb{C} \to \mathbb{C}$ satisfies $\sum_{j=1}^n c_j \cdot f(x + a_j y) = 0$ for every $x \in \mathbb{C}$ and $y \in S^1$. We have to show that $f \equiv 0$. Let $z_0 \in \mathbb{C}$ be arbitrary, and let $E$ denote the subfield of $\mathbb{C}$ generated by $z_0$, $a_1, \ldots, a_n$ and the set $S_a^1 = \{z \in S^1 \colon z \text{ is algebraic}\}$. Then $E$ is a countable subfield of $\mathbb{C}$ containing $S_a^1 \cup \{a_1, \ldots, a_n\}$. Therefore, in order to prove Theorem 2, it is enough to prove the following.

**Theorem 12.** *Let $E$ be a countable subfield of $\mathbb{C}$ containing $S_a^1 \cup \{a_1, \ldots, a_n\}$. If $c_1, \ldots, c_n \in \mathbb{C}$, $\sum_{j=1}^n c_j \neq 0$ and $f \colon E \to \mathbb{C}$ is such that $\sum_{j=1}^n c_j \cdot f(x + a_j y) = 0$ for every $x \in E$ and $y \in S_a^1$, then $f \equiv 0$.*

The structure of the proof of Theorem 12 is the following. Suppose $f$ satisfies the condition, but $f \not\equiv 0$. Applying harmonic analysis on the countable additive group $E$, we find a multiplicative function $m \colon E \to \mathbb{C}$ such that $\sum_{j=1}^n c_j \cdot m(u \cdot a_j) = 0$ for every $u \in S_a^1$. Then we apply this equation with many $u$ having rational coordinates, and obtain, by applying a theorem of J.-H. Evertse, H.P. Schlickewei and W.M. Schmidt on the number of solutions of linear equations, that there are indices $j_1 \neq j_2$ and there is an integer $d$ such that $m(u \cdot (a_{j_2} - a_{j_1})/d)$ and $m(u \cdot i \cdot (a_{j_2} - a_{j_1})/d)$ are roots of unity of bounded degree for every algebraic $u$ with $|u| = 1$ (Lemma 13). In the final step we show that this contradicts the fact that $m(x_1) \cdots m(x_s) = 1$ whenever $x_1 + \ldots + x_s = 0$. Now we turn to the details.

Fix $c_1, \ldots, c_n \in \mathbb{C}$ with $\sum_{j=1}^n c_j \neq 0$. Clearly, we may assume that $c_j \neq 0$ for every $j = 1, \ldots, n$. Let $\Omega$ denote the set of all functions $f \colon E \to \mathbb{C}$ such that $\sum_{j=1}^n c_j \cdot f(x + a_j y) = 0$ for every $x \in E$ and $y \in S_a^1$. It is clear that $\Omega$ is a linear space over $\mathbb{C}$, and that $\Omega$ is invariant under translations by elements of $E$. It is also easy to see that $\Omega$ is closed in the set $\mathbb{C}^E$ equipped with the product topology. This means that $\Omega$ is a variety on the discrete additive group $E$.

Suppose that the statement of the theorem is false; that is, $\Omega \neq \{0\}$. Clearly, this implies $n \geq 2$. Then, by [14, Theorem 1], $\Omega$ contains an exponential; that is, a function $m \colon E \to \mathbb{C}$ such that $m \neq 0$ and $m(x + y) = m(x) \cdot m(y)$

for every $x, y \in E$. Since $m \in \Omega$, we have

$$\sum_{j=1}^{n} c_j \cdot m(u \cdot a_j) = 0 \qquad (u \in S_a^1).$$

In the sequel we fix an exponential function $m$ with the properties above, and look for a contradiction.

We shall need the following result. There exists a positive integer $A(n)$ that only depends on $n$ and has the following property: whenever $\Gamma$ is a multiplicative subgroup of $\mathbb{C}^*$ of rank at most $n$ and $1 \le r \le n$, then the number of solutions of the equation

$$x_1 + \ldots + x_r = 1$$

such that $x_1, \ldots, x_r \in \Gamma$ and no subsum of $x_1 + \ldots + x_r$ equals zero is at most $A(n)$. (See [1, Theorem 1.1] and [2, Theorem 6.1.3].)

**Lemma 13.** *There are positive integers $d$ and $D$ only depending on $n$ such that for every $u \in S_a^1$ there are indices $1 \le j_1, j_2 \le n$ with the following property: $m(u \cdot (a_{j_2} - a_{j_1})/d)$ and $m(u \cdot i \cdot (a_{j_2} - a_{j_1})/d)$ are roots of unity of degree dividing $D$, and at least one of $m(u \cdot (a_{j_2} - a_{j_1})/d)$ and $m(u \cdot i \cdot (a_{j_2} - a_{j_1})/d)$ is different from 1.*

**Proof.** It is enough to prove the statement for $u = 1$. Indeed, if this special case is true and $u \in S_a^1$ is arbitrary, then we obtain the statement for $u$ by applying the special case for the $n$-tuple $(ua_1, \ldots, ua_n)$.

We put $\gamma_k = ((1 - k^2) + i \cdot 2k)/(1 + k^2)$ for every $k = 1, 2, \ldots$. Then $\gamma_k \in S_a^1$ for every $k$.

For every $k$ there exists a partition $\{1, \ldots, n\} = I_1 \cup \ldots \cup I_m$ with the following property: for every $1 \le \mu \le m$,

$$\sum_{j \in I_\mu} c_j \cdot m(\gamma_k \cdot a_j) = 0,$$

and $\sum_{j \in I} c_j \cdot m(\gamma_k \cdot a_j) \ne 0$ whenever $\emptyset \ne I \subsetneq I_\mu$. For a given $k$ there can be more than one such partition; we select one for each $k$, and denote it by $\mathcal{I}_{\gamma_k}$.

Let $P(n)$ denote the number of partitions of $\{1, \ldots, n\}$, and put $B(n) = 2 \cdot P(n) \cdot A(3n) + 1$. Then there is a set $H \subset \{1, \ldots, B(n)\}$ such that $|H| > 2 \cdot A(3n)$, and the partitions $\mathcal{I}_{\gamma_k}$ $(k \in H)$ are the same. Let $\mathcal{I}_{\gamma_k} = \{I_1, \ldots, I_m\}$ for every $k \in H$.

Let $d = (1 + B(n)^2)!$. Then $d$ is a common multiple of the numbers $1 + k^2$ $(k \in H)$, and thus $\gamma_k = (e_k + i \cdot f_k)/d$ for every $k \in H$, where $|e_k|, |f_k| \le d$. Let $\mu \in \{1, \ldots, m\}$ be given. Then, for every $k \in H$ we have

$$\begin{aligned} 0 &= \sum_{j \in I_\mu} c_j \cdot m(\gamma_k \cdot a_j) = \sum_{j \in I_\mu} c_j \cdot m\left( e_k \cdot \frac{a_j}{d} + f_k \cdot \frac{i \cdot a_j}{d} \right) \\ &= \sum_{j \in I_\mu} c_j \cdot m(a_j/d)^{e_k} \cdot m(i \cdot a_j/d)^{f_k} = \sum_{j \in I_\mu} c_j \cdot u_j^{e_k} \cdot v_j^{f_k}, \end{aligned} \tag{4}$$

where $u_j = m(a_j/d)$ and $v_j = m(i \cdot a_j/d)$. Select an index $j_\mu \in I_\mu$. Then, by (4), we have

$$\sum_{j \in I_\mu,\ j \neq j_\mu} \beta_j \cdot (u_j/u_{j_\mu})^{e_k} \cdot (v_j/v_{j_\mu})^{f_k} = 1 \tag{5}$$

for every $k \in H$, where $\beta_j = -c_j/c_{j_\mu}$. Put $\overline{u}_j = u_j/u_{j_\mu}$ and $\overline{v}_j = v_j/v_{j_\mu}$ $(j \in I_\mu)$, and let $\Gamma$ be the multiplicative group generated by the elements $\beta_j, \overline{u}_j$ and $\overline{v}_j$. Then the rank of $\Gamma$ is at most $3n$, and $\beta_j \cdot \overline{u}_j^{e_k} \cdot \overline{v}_j^{f_k} \in \Gamma$ for every $j \in I_\mu$ and $k \in H$. By the choice of $A(3n)$, the equation

$$\sum_{j \in I_\mu,\ j \neq j_\mu} x_j = 1 \tag{6}$$

has at most $A(3n)$ solutions having the property that $x_j \in \Gamma$ for every $j$, and no subsum of the left hand side of (6) is zero. However, (5) gives such a solution for every $k \in H$. Since $|H| > 2 \cdot A(3n)$, there must exist three distinct indices $s, t, z \in H$ giving the same solution. Then

$$\overline{u}_j^{e_s} \cdot \overline{v}_j^{f_s} = \overline{u}_j^{e_t} \cdot \overline{v}_j^{f_t} = \overline{u}_j^{e_z} \cdot \overline{v}_j^{f_z}$$

for every $j \in I_\mu$, $j \neq j_\mu$. The equations above are also true if $j = j_\mu$, since $\overline{u}_{j_\mu} = \overline{v}_{j_\mu} = 1$. Then we have

$$\overline{u}_j^{e_t - e_s} \cdot \overline{v}_j^{f_t - f_s} = 1 \text{ and } \overline{u}_j^{e_z - e_s} \cdot \overline{v}_j^{f_z - f_s} = 1 \qquad (j \in I_\mu). \tag{7}$$

From (7) we obtain $\overline{u}_j^C = 1$ and $\overline{v}_j^C = 1$ for every $j \in I_\mu$, where $C = (e_z - e_s)(f_t - f_s) - (e_t - e_s)(f_z - f_s)$. We show that $C \neq 0$.

The points $(e_s, f_s), (e_t, f_t), (e_z, f_z)$ are distinct, and lie on a circle of radius $d$. Consequently, they are not collinear; that is,

$$\frac{f_t - f_s}{e_t - e_s} \neq \frac{f_z - f_s}{e_z - e_s}.$$

Multiplying by the denominators we obtain $C \neq 0$. Note that $|C| \leq 8 \cdot d^2$.

We find that $\overline{u}_j$ and $\overline{v}_j$ are roots of unity of order $\leq C \leq 8 \cdot d^2$ for every $j \in I_\mu$. Putting $D = (8 \cdot d^2)!$, the orders of $\overline{u}_j$ and $\overline{v}_j$ will be divisors of $D$.

Now we prove that there is an index $j \in \{1, \ldots, n\}$ such that at least one of $\overline{u}_j$ and $\overline{v}_j$ is different from 1. Suppose not. Then, for every $\mu = 1, \ldots, m$ we have $\overline{u}_j = \overline{v}_j = 1$ for every $j \in I_\mu$. By (5), we have $\sum_{j \in I_\mu,\ j \neq j_\mu} \beta_j = 1$ and $\sum_{j \in I_\mu} c_j = 0$ $(\mu = 1, \ldots, m)$. However, this would imply $\sum_{j=1}^{n} c_j = \sum_{\mu=1}^{m} \sum_{j \in I_\mu} c_j = 0$, which is impossible.

Therefore, we can find a $\mu$ and a $j \in I_\mu$ such that $\overline{u}_j = u_j / u_{j_\mu} \neq 1$ or $\overline{v}_j = v_j / v_{j_\mu} \neq 1$. Choosing $j_1 = j_\mu$ and $j_2 = j$ we find that $m((a_{j_2} - a_{j_1})/d) = \overline{u}_j \neq 1$ or $m(i \cdot (a_{j_2} - a_{j_1})/d) = \overline{v}_j \neq 1$, completing the proof. $\square$

**Lemma 14.** *Let $S^1_a = A_1 \cup \ldots \cup A_N$ be a cover of $S^1_a$, and let $c > 1$ be an integer. Then there is a $j \in \{1, \ldots, N\}$ and there are elements $u_1, u_2, u_3 \in A_j$ and integers $n_1, n_2, n_3$ such that $n_1 u_1 + n_2 u_2 + n_3 u_3 = 0$ and $n_1 + n_2 + n_3$ is prime to $c$.*

**Proof.** The polynomial $p(x) = cx^2 + x + c$ is irreducible over $\mathbb{Q}$, and its roots belong to $S^1_a$. Let $\alpha$ be one of the roots of $p$. Since $\alpha^n \in S^1_a$ for every $n$, there is a $j \in \{1, \ldots, N\}$ such that $\alpha^n \in A_j$ holds for at least three distinct nonnegative exponents $n$. Suppose $\alpha^r, \alpha^s, \alpha^t \in A_j$, where $0 \leq r < s < t$ are integers.

Since $\alpha^r, \alpha^s, \alpha^t \in \mathbb{Q}(\alpha)$ and $\mathbb{Q}(\alpha)$ is a linear space of dimension two over $\mathbb{Q}$, there are rational numbers $n_1, n_2, n_3$, not all zero, such that $n_1 \alpha^r + n_2 \alpha^s + n_3 \alpha^t = 0$. Then $\alpha$ is a root of the polynomial $n_1 x^r + n_2 x^s + n_3 x^t$, hence we have

$$n_1 x^r + n_2 x^s + n_3 x^t = (cx^2 + x + c) \cdot q(x), \tag{8}$$

where $q$ is a polynomial with rational coefficients. Let $q(x) = \sum_{i=u}^{v} b_i x^i$, where $u \leq v$ and $b_u \neq 0$, $b_v \neq 0$. Multiplying by the common denominator of the coefficients $b_i$, we may assume that $b_u, \dots, b_v$ are integers, and the polynomial $q$ is primitive, meaning that the greatest common divisor of $b_u, \dots, b_v$ is 1. Then $n_1, n_2, n_3$ are integers. Since $cx^2+x+c$ is also primitive, it follows from Gauss' lemma that $n_1x^r + n_2x^s + n_3x^t$ is primitive as well. It follows from (8) that either $n_3 = 0$ or $n_3 = c \cdot b_v$. In both cases we have $c \mid n_3$. We obtain $c \mid n_1$ similarly. Then $n_2$ must be prime to $c$, and thus the same is true for $n_1 + n_2 + n_3$. $\square$

**Proof of Theorem 12.** Let $d$ and $D$ be as in Lemma 13. We put

$$A_{j_1,j_2,k} = \{u \in S_a^1 \colon m(u \cdot (a_{j_2} - a_{j_1})/d) = e^{2\pi i \cdot k/D}\}$$

and

$$B_{j_1,j_2,k} = \{u \in S_a^1 \colon m(u \cdot i \cdot (a_{j_2} - a_{j_1})/d) = e^{2\pi i \cdot k/D}\}.$$

By Lemma 13, we have

$$S_a^1 = \bigcup_{j_1=1}^{n} \bigcup_{j_2=1}^{n} \bigcup_{k=1}^{D-1} (A_{j_1,j_2,k} \cup B_{j_1,j_2,k}).$$

Then, by Lemma 14, there are elements $u_1, u_2, u_3$ and integers $n_1, n_2, n_3$ such that $n_1u_1 + n_2u_2 + n_3u_3 = 0$, $n_1 + n_2 + n_3$ is prime to $D$, and $u_1, u_2, u_3$ belong to one of the sets $A_{j_1,j_2,k}$ and $B_{j_1,j_2,k}$.

Suppose they belong to $A_{j_1,j_2,k}$. We have $\sum_{t=1}^{3} n_t \cdot u_t \cdot (a_{j_2} - a_{j_1})/d = 0$, and thus

$$1 = m\left(\sum_{t=1}^{3} n_t \cdot u_t \cdot (a_{j_2} - a_{j_1})/d\right) = \prod_{t=1}^{3} m(u_t \cdot (a_{j_2} - a_{j_1})/d)^{n_t} =$$
$$= \left(e^{2\pi i \cdot k/D}\right)^{n_1+n_2+n_3} = e^{2\pi i \cdot k \cdot (n_1+n_2+n_3)/D}.$$

This implies $D \mid k \cdot (n_1 + n_2 + n_3)$. However, $n_1 + n_2 + n_3$ is prime to $D$ and $1 \le k \le D - 1$, which is a contradiction. If $u_1, u_2, u_3 \in B_{j_1,j_2,k}$, then we reach a contradiction by a similar computation. $\square$

# 4 Proof of Theorem 2 for $k > 2$

We prove by induction on $k$ the following statement: *for every* $a_1, \dots, a_n, z_0 \in \mathbb{R}^k$ *there exists a countable additive subgroup* $E$ *of* $\mathbb{R}^k$ *containing* $a_1, \dots, a_n, z_0$

*and having the following property: whenever* $c_1, \ldots, c_n \in \mathbb{C}$, $\sum_{j=1}^n c_j \neq 0$, $f\colon E \to \mathbb{C}$ *and* $\sum_{j=1}^n c_j \cdot f(\phi(a_j)) = 0$ *for every* $\phi \in G_k$ *satisfying* $\phi(a_j) \in E$ *for every* $j = 1, \ldots, n$, *then* $f(z_0) = 0$.

By Theorem 12, the statement above is true for $k = 2$. Let $k \geq 2$, and suppose that the statement is true in $\mathbb{R}^k$. We prove the statement in $\mathbb{R}^{k+1}$. Let $a_1, \ldots, a_n, z_0 \in \mathbb{R}^{k+1}$ be given. Since the statement is obvious if $a_1 = \ldots = a_n$, we may assume that $n \geq 2$, $a_1 = 0$ and $a_n \neq 0$.

Let $S^k$ denote the unit sphere in $\mathbb{R}^{k+1}$; that is, let $S^k = \{x \in \mathbb{R}^{k+1} \colon |x| = 1\}$. If $v \in S^k$, then we denote by $v^\perp$ the linear subspace of $\mathbb{R}^{k+1}$ of dimension $k$ and perpendicular to $v$. If $V$ is a linear subspace of $\mathbb{R}^{k+1}$, then we denote by $G(V)$ the family of rigid motions mapping $V$ into itself. Thus $G_{k+1} = G(\mathbb{R}^{k+1})$.

Let a unit vector $v_0 \in S^k$ be selected such that $\langle v_0, a_n \rangle \neq 0$, and put $t_j = \langle v_0, a_j \rangle$ $(j = 1, \ldots, n)$. Note that $t_1 = \langle v_0, 0 \rangle = 0$ and $t_n \neq 0$. Let $V_0$ be a linear subspace of $\mathbb{R}^{k+1}$ of dimension $k$ containing $z_0$ and $v_0$. By the induction hypothesis applied to the points $t_1 v_0, \ldots, t_n v_0, z_0 \in V_0$, we find a countable additive group $E_0 \subset V_0$ containing $t_1 v_0, \ldots, t_n v_0, z_0$ and having the following property: whenever $c_1, \ldots, c_n \in \mathbb{C}$, $\sum_{j=1}^n c_j \neq 0$, and $f\colon E_0 \to \mathbb{C}$ is such that $\sum_{j=1}^n c_j \cdot f(\phi(t_j v_0)) = 0$ for every $\phi \in G(V_0)$ satisfying $\phi(t_j v_0) \in E_0$ for every $j = 1, \ldots, n$, then $f(z_0) = 0$.

Let $W = \{v \in S^k \colon t_j v \in E_0 \ (j = 1, \ldots, n)\}$. Since $t_n \neq 0$, $W$ is a countable set of unit vectors. For every $v \in W$ let a rigid motion $\phi_v \in G_{k+1}$ be selected such that

$$\phi_v(0) = 0 \quad \text{and } v = \phi_v(v_0). \tag{9}$$

Then $\phi_v$ is a linear transformation of $\mathbb{R}^{k+1}$. Let $b_{v,j}$ denote the orthogonal projection of $\phi_v(a_j)$ onto $v^\perp$ $(j = 1, \ldots, n)$.

Let $v \in W$. Applying the induction hypothesis again, we find a countable additive group $E_v \subset v^\perp$ containing $b_{v,1}, \ldots, b_{v,n}$ and having the following property: whenever $d_1, \ldots, d_n \in \mathbb{C}$, $\sum_{j=1}^n d_j \neq 0$, and $f\colon E_v \to \mathbb{C}$ is such that $\sum_{j=1}^n d_j \cdot f(\psi(b_{v,j})) = 0$ for every $\psi \in G(v^\perp)$ satisfying $\psi(b_{v,j}) \in E_v$ for every $j = 1, \ldots, n$, then $f(b_{v,1}) = 0$.

Let $E$ be the additive group generated by $E_0 \cup \bigcup_{v \in W} \bigcup_{j=1}^n (E_v + t_j v)$. Then $E$ is countable, and contains $z_0$. We show that $E$ satisfies the requirements.

In fact, we shall prove the following, stronger statement.

*If $c_1, \ldots, c_n \in \mathbb{C}$, $\sum_{j=1}^n c_j \neq 0$, and $f\colon E \to \mathbb{C}$ is such that $\sum_{j=1}^n c_j \cdot f(\phi(a_j)) = 0$ for every $\phi \in G_{k+1}$ satisfying $\phi(a_j) \in E$ for every $j = 1, \ldots, n$, then $f \equiv 0$.*

Let $c_1, \ldots, c_n$ be fixed complex numbers satisfying $\sum_{j=1}^n c_j \neq 0$, and let $\Lambda$ denote the set of all functions $f\colon E \to \mathbb{C}$ such that $\sum_{j=1}^n c_j \cdot f(\phi(a_j)) = 0$ for every $\phi \in G_{k+1}$ satisfying $\phi(a_j) \in E$ for every $j = 1, \ldots, n$. It is clear that $\Lambda$ is a linear space over $\mathbb{C}$, and that $\Lambda$ is invariant under translations by elements of $E$. It is also easy to see that $\Lambda$ is closed in the set $\mathbb{C}^E$ equipped with the product topology. This means that $\Lambda$ is a variety on the discrete additive group $E$.

Suppose $\Lambda \neq \{0\}$. Then, by [14, Theorem 1], $\Lambda$ contains an exponential; that is, a function $m\colon E \to \mathbb{C}$ such that $m \neq 0$ and $m(x+y) = m(x) \cdot m(y)$ for every $x, y \in E$. Since $m \in \Lambda$, we have $\sum_{j=1}^n c_j \cdot m(\phi(a_j)) = 0$ for every $\phi \in G_{k+1}$ such that $\phi(a_j) \in E$ $(j = 1, \ldots, n)$.

Now $m$ is defined on $E_0$, and $m(z_0) \neq 0$. Then it follows from the choice of $E_0$ that there exists a $\phi \in G(V_0)$ satisfying $\phi(t_j v_0) \in E_0$ for every $j = 1, \ldots, n$, and such that $\sum_{j=1}^n c_j \cdot m(\phi(t_j v_0)) \neq 0$. Since $t_1 = 0$, we have $\phi(0) \in E_0$. Put $\sigma = \phi - \phi(0)$. Then $\sigma \in G(V_0)$ and $\sigma(t_j v_0) \in E_0$ for every $j = 1, \ldots, n$, as $E_0$ is an additive group. Note that $\sigma$ is a linear transformation of $V_0$. We put $d_j = c_j \cdot m(\sigma(t_j v_0))$ $(j = 1, \ldots, n)$. Then $\sum_{j=1}^n d_j \neq 0$, since $m(\sigma(t_j v_0)) = m(\phi(t_j v_0))/m(\phi(0))$ for every $j$.

Let $v = \sigma(v_0)$. Then $v \in W$, as $t_j \cdot v = t_j \cdot \sigma(v_0) = \sigma(t_j v_0) \in E_0$ for every $j = 1, \ldots, n$. We show that if $\psi \in G(v^\perp)$ is such that $\psi(b_{v,j}) \in E_v$ for every $j = 1, \ldots, n$, then $\sum_{j=1}^n d_j \cdot m(\psi(b_{v,j})) = 0$. As $m$ is defined on $E_v$, and is nowhere zero, this will contradict the choice of $E_v$, proving the theorem.

Every element of $\mathbb{R}^{k+1}$ has a unique representation of the form $b + tv$, where $b \in v^\perp$ and $t \in \mathbb{R}$. Putting $\overline{\psi}(b + tv) = \psi(b) + tv$, we define the rigid motion $\overline{\psi} \in G_{k+1}$. We prove that $\phi_v(a_j) = b_{v,j} + t_j v$ for every $j = 1, \ldots, n$. (As for $\phi_v$, see (9).) Indeed, if $\phi_v(a_j) = b_{v,j} + tv$, then

$$t = \langle v, \phi_v(a_j) \rangle = \langle \phi_v(v_0), \phi_v(a_j) \rangle = \langle v_0, a_j \rangle = t_j.$$

Therefore, we have $(\overline{\psi} \circ \phi_v)(a_j) = \psi(b_{v,j}) + t_j \cdot v$ and

$$m((\overline{\psi} \circ \phi_v)(a_j)) = m(\psi(b_{v,j})) \cdot m(t_j \cdot v).$$

Now $d_j = c_j \cdot m(\sigma(t_j v_0)) = c_j \cdot m(t_j v)$, and thus

$$\sum_{j=1}^{n} d_j \cdot m(\psi(b_{v,j})) = \sum_{j=1}^{n} c_j \cdot m(t_j \cdot v) \cdot m(\psi(b_{v,j})) = \sum_{j=1}^{n} c_j \cdot m((\overline{\psi} \circ \phi_v)(a_j) = 0,$$

as $\overline{\psi} \circ \phi_v \in G_{k+1}$, and $(\overline{\psi} \circ \phi_v)(a_j) \in E$ for every $j = 1, \ldots, n$. This completes the proof. □


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## References


[1] J.-H. Evertse, H.P. Schlickewei and W.M. Schmidt, Linear equations in variables which lie in a multiplicative group, *Ann. of Math.* (2) **155** (2002), no. 3, 807-836.

[2] J.-H. Evertse and K. Győry: *Unit equations in Diophantine number theory.* Cambridge Stud. Adv. Math., 146. Cambridge University Press, Cambridge, 2015.

[3] S. Gao, A.W. Miller and W.A.R. Weiss, Steinhaus sets and Jackson sets *Advances in logic*, 127-145. Contemp. Math., 425.

[4] C. de Groote, M. Duerinckx, Functions with constant mean on similar countable subsets of $\mathbb{R}^2$, *Amer. Math. Monthly* **119** (2012), 603-605.

[5] D. Henkis, S. Jackson, J. Lobe, The Finite Steinhaus Problem, *The Quarterly Journal of Mathematics* **67**(4) (2016), 551-564.

[6] S. Jackson and R.D. Mauldin, On a lattice problem of H. Steinhaus *J. Amer. Math. Soc.* **15** (2002), no.4, 817-856.

[7] S. Jackson and R.D. Mauldin, Survey of the Steinhaus tiling problem, *Bull. Symbolic Logic* **9** (2003) no.3, 335-361.

[8] R. Katz, M. Krebs and A. Shaheen, Zero sums on unit square vertex sets and plane colorings, *Amer. Math. Monthly* **121** (2014), no. 7, 610-618.

[9] G. Kiss, M. Laczkovich and Cs. Vincze, The discrete Pompeiu problem on the plane, *Monatsh. Math.* **186** (2018), no. 2, 299-314.

[10] G. Kiss, R. D. Malikiosis, G. Somlai and M. Vizer, On the discrete Fuglede and Pompeiu problems, *Analysis & PDE*, **13**(3) (2020), 765-788.

[11] M. Kolountzakis and M. Papadimitrakis, Measurable Steinhaus sets do not exist for finite sets or the integers in the plane, *Bulletin of the London Mathematical Society* **49**(5) (2017), 798-805.

[12] P. Komjáth, A lattice-point problem of Steinhaus, *Quart. J. Math. Oxford* Ser. (2) 43 (1992), no. 170, 235-241.

[13] P. Komjáth, A coloring result for the plane, *J. Appl. Anal.* **5** (1999), no. 1, 113-117.

[14] M. Laczkovich and G. Székelyhidi, Harmonic analysis on discrete Abelian groups, *Proc. Amer. Math. Soc.* **133** (2005), no. 6, 1581-1586.

[15] P. A. Linnell and M. J. Puls, The two-sided Pompeiu problem for discrete groups, *Proc. Amer. Math. Soc.*, Series B, **9** (2-22) (2022), 221-229.

[16] F.C. Machado and S. Robins, The null set of a polytope, and the Pompeiu property for polytopes *J. Anal. Math.* **150** (2023), no.2, 673-683.

[17] M. J. Puls, The Pompeiu problem and discrete groups, *Monatsh. Math.* **172**(3-4) (2013), 415-429.

[18] A. G. Ramm, The Pompeiu problem, *Applicable Analysis* **64** no. 1-2 (1997), 19-26.

[19] J.H. Schmerl, Coloring $\mathbb{R}^n$, *Trans. Amer. Math. Soc.* **354** (2002), no. 3, 967-974.

[20] L.E. Shader, All right triangles are Ramsey in $E^2$! *J. Combinatorial Theory* Ser. **A20** (1976), no.3, 385-389.

[21] L. Zalcman, A bibliographical survey of the Pompeiu Problem, in the book *Approximation by solutions of partial differential equations.* Edited by B. Fuglede. Kluwer Acad., Dordrecht, 1992, 177-186.

[22] D. Zeilberger, Pompeiu's problem on discrete space, *Proc. Natl. Acad. Sci. USA*, **75**(8) (1978), 3555-3556.

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