Eötvös Loránd University
Institute of Mathematics
Department of Analysis

Ph.D. thesis

# Linear functional equations

Gergely Kiss

Doctoral School: Mathematics
Director: Miklós Laczkovich
member of the Hungarian Academy of Sciences

Doctoral Program: Pure Mathematics
Director: András Szűcs
Professor, Doctor of Sciences

Supervisor: Miklós Laczkovich
member of the Hungarian Academy of Sciences

April 2014

# Contents

**1 Introduction** **5**

**2 Preliminaries** **7**

2.1 Additive functions, polynomials, generalized polynomials, local polynomials 7

2.2 Discrete spectral analysis and spectral synthesis . . . . . . . . . . . . . . . 13

2.3 Linear functional equations . . . . . . . . . . . . . . . . . . . . . . . . . . . 17

**3 Existence of nonzero solutions of linear functional equations** **23**

3.1 Some varieties related to linear functional equations . . . . . . . . . . . . . 23

3.2 Non-zero additive solutions . . . . . . . . . . . . . . . . . . . . . . . . . . . 25

3.3 Existence of solutions of higher degree . . . . . . . . . . . . . . . . . . . . . 29

**4 Linear functional equations with algebraic parameters** **35**

4.1 The space of additive solutions in the algebraic case . . . . . . . . . . . . . 35

4.2 The space of solutions of higher degree . . . . . . . . . . . . . . . . . . . . 37

4.3 Trivial functional equations . . . . . . . . . . . . . . . . . . . . . . . . . . . 39

4.4 Further developments . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 42

**5 Spectral synthesis in varieties of additive functions** **45**

5.1 Differential operators . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 45

5.2 Proof of spectral synthesis in varieties of additive functions . . . . . . . . . 50

5.3 The space of additive solutions . . . . . . . . . . . . . . . . . . . . . . . . . 51

**6 Spectral synthesis of higher degree** **58**

6.1 Proof of spectral synthesis in varieties of $k$-additive functions . . . . . . . . 58

6.2 The space of solutions of linear functional equations . . . . . . . . . . . . . 61

6.3 An example and concluding remarks . . . . . . . . . . . . . . . . . . . . . . 63

**7 The discrete Pompeiu problem** **68**

**8 Summary** **73**

**9 Összefoglaló** **75**

**References** **77**

## Acknowledgement

I would like to express my sincere gratitude to my supervisor, Miklós Laczkovich for the continuous support as well as for the interesting research topics. I think without his support this work would be never finished and may be never started. His advice and encouragement (both in mathematical and personal matters) meant a lot to me. I could not have imagined having a better supervisor.

I also would like to thank to my parents for their constant care, love and support, to my brother and some of my friends for their encouragement.

Last but not least, I would like to thank to all members of Department of Stochastic in BME where I got lot of support for my work.

# 1 Introduction

Let $\mathbb{C}$ denote the field of complex numbers. We are concerned with the linear functional equation

$$\sum_{i=1}^{n} a_i f(b_i x + c_i y) = 0 \qquad (x, y \in \mathbb{C}), \tag{1}$$

where $a_i, b_i, c_i$ are given complex numbers, and $f : \mathbb{C} \to \mathbb{C}$ is the unknown function. By a well-known result of L. Székelyhidi [36] (see Theorem 2.26), under some mild conditions on the equation, every solution of (1) is a generalized polynomial. (For the definition of generalized polynomials see the next section.) But the finer structure of the solutions has been investigated only recently. During the past couple of years several papers have been devoted to the problem of finding the generalized polynomial (in particular, additive) solutions of some special cases of equation (1). Most of these recent investigations were started independently by A. Varga and the author of the present thesis around 2009. See [13], [38], [39], [42]; see also the references of [38] and [42]. Results concerning the additive solutions of the general equation (1) can be found in [15], [40] and [41]. In this thesis we continue these investigations and attempt to give a complete description of those solutions of the equations of type (1) which are generalized polynomials.

There are important special cases when every solutions is automatically a generalized polynomial. Such equations are, e.g.,

$$\sum_{i=1}^{n} a_i f(b_i x + y) = 0$$

and

$$\sum_{i=1}^{n} a_i f(b_i x + (1 - b_i) y) = 0.$$

Therefore, in these special cases our results gives the complete description of the space of solutions, at least in principle.

Let $S$ denote the set of solutions defined on the field generated by the parameters $b_i, c_i$. It is clear that $S$ is a linear space over $\mathbb{C}$ and is closed under pointwise convergence. If $S$ is also translation invariant, then $S$ is a variety. We want to describe $S$ by presenting solutions of simple structure that span $S$.

The situation is that of *spectral analysis and synthesis*. In fact, the most important contribution of our results to the theory of linear functional equations is the application of spectral analysis and synthesis to some varieties related to the spaces of solutions of the equations. The idea of applying spectral analysis to these varieties first appeared in [15]. The method of spectral synthesis was used in [13] and [14]. The main difficulty in applying spectral synthesis in these varieties stems from the fact that the underlying groups are of infinite torsion free rank, where spectral synthesis fails in some varieties. We overcome this difficulty by observing that a more general form of spectral synthesis, called local spectral synthesis holds on every countable group, and that in the varieties in question the polynomial-exponentials and local polynomial-exponentials coincide.

In our description of solutions of simple structure derivations and field automorphisms play a key role. The observation that derivations can be used in the description of solutions was first made in [13].

We shall apply the general results to several equations with special properties; e.g. having algebraic parameters etc.

We also give an application of discrete spectral synthesis in Section 7, where we introduce the so-called discrete Pompeiu problem, and present a solution to a special case posed by L. Pósa.

# 2 Preliminaries

## 2.1 Additive functions, polynomials, generalized polynomials, local polynomials

Let $G$ and $G'$ be Abelian groups. The difference operator $\Delta_h$ is defined by

$$\Delta_h f(x) = f(x+h) - f(x) \qquad (x, h \in G)$$

for every $f : G \to G'$. The operator $\Delta_h^n$ denotes the composition $\underbrace{\Delta_h \dots \Delta_h}_{n}$. It is easy to check that $\Delta_{h_1}\Delta_{h_2} = \Delta_{h_2}\Delta_{h_1}$ for every $h_1, h_2 \in G$.

By an additive function $f : G \to G'$ we mean a function satisfying the functional equation $f(x) + f(y) = f(x+y)$ for every $x, y \in G$. In other words, $f$ is additive if it is a homomorphism from $G$ to $G'$.

A map $F : G^i \to G'$ is called $i$-additive if it is a homomorphism in each of its variables, the other variables being fixed. The function $F : G^i \to G'$ is said to be symmetric, if it takes the same value at every permutation of its variables.

The function $f$ is a (generalized) *monomial* of degree $i$ if there exists a symmetric, $i$-additive map $F : G^i \to G'$ such that $f(x) = F(x, \dots, x)$ for every $x \in G$. For every $F : G^i \to G'$, the function $f(x) = F(x, \dots, x)$ $(x \in G)$ is said to be the *diagonal* of $F$, denoted by $\operatorname{diag}(F)$. If $G'$ is uniquely divisible by $i!$; that is, if the equation $i! \cdot x = a$ has a unique solution for every $a \in G'$ then the symmetric $i$-additive function $F$ is uniquely determined by $f$. This is a consequence of the following well-known fact.

**Lemma 2.1.** *Let $f(x) = F(\underbrace{x, \dots, x}_{i})$ be a monomial, where $F$ is symmetric and $i$-additive. Then*

$$i! \cdot F(x_1, x_2, \dots, x_i) = \Delta_{x_1}\Delta_{x_2} \dots \Delta_{x_i} f(t) \tag{2}$$

*for every $t, x_1, x_2, \dots, x_i \in G$.*

**Proof.** See [16, Lemma 15.9.2, p. 394].

We shall use the lemma in the special case when $G' = \mathbb{C}$. We get that a function $f : G \to \mathbb{C}$ is a generalized monomial of degree $i$ if and only if the function $F$ defined by (2) is $i$-additive, and $f(x) = F(x, \dots, x)$ for every $x \in G$.

A function $f$ is called a *generalized polynomial* if $f = \sum_{i=0}^{n} f_i$, where $f_i$ is a monomial of degree $i$ $(i = 1, \dots, n)$ and $f_0$ is a constant.

The following theorem is a classical result of Mazur and Orlitz [23].

**Theorem 2.2.** *Let $f : \mathbb{C} \to \mathbb{C}$ be a function and $n \in \mathbb{N}$. Then the following are equivalent:*

(i)

$$\Delta_{h_1} \dots \Delta_{h_{n+1}} f(x) = 0$$

*for every* $h_1, \dots, h_{n+1}, x \in \mathbb{C}$.

(ii)

$$\Delta_h^{n+1} f(x) = 0$$

*for every* $h \in \mathbb{C}$.

(iii)

$$f = \sum_{i=0}^{n} f_i$$

*where* $f_i$ *is a monomial of degree* $i$ $(i = 1, \dots, n)$ *and* $f_0$ *is a constant, thus* $f$ *is a generalized polynomial.*

In [17] Laczkovich studies these properties in a more general context. For the sake of completeness we recall some of the definitions and results of [17].

Let $f$ be a map from $G$ to $G'$. In [17] the following properties of $f$ are defined.

$\mathbf{P(n)}$: $f = \sum_{i=0}^{n} f_i$, where $f_i$ is a monomial of degree $i$ $(i = 1, \dots, n)$ and $f_0$ is constant.

$\mathbf{P_{loc}(n)}$ : For every $g_1, \dots, g_s \in G$ there are element $c_i \in G'$ $(i = (i_1, \dots, i_s), i_1, \dots, i_s \geq 0, |i| = i_1 + \dots + i_s \leq n)$ such that

$$f(k_1 g_1 + \dots + k_s g_s) = \sum_{|i| \leq n} k_1^{i_1} \cdots k_s^{i_s} c_i$$

for every $k_1, \dots, k_s \in \mathbb{Z}$.

$\mathbf{P_{lin}(n)}$ : For every $a, b \in G$ there are elements $c_i \in G' (i = 0, \dots, n)$ such that

$$f(a + kb) = \sum_{i=0}^{n} k^i \cdot c_i$$

for every $k \in \mathbb{Z}$.

$\mathbf{MD(n)}$ : $\Delta_{h_1} \dots \Delta_{h_{n+1}} f(x) = 0$ for every $h_1, \dots, h_{n+1}, x \in G$.

$\mathbf{ID(n)}$ : $\Delta_h^{n+1} f(x) = 0$ for every $h, x \in G$.

$\mathbf{R(n)}$ : There are functions $f_1, \dots, f_{n+1} : G \to G'$ and integers $a_i$ and $b_i$ such that $b_i \neq 0$ $(i = 1, \dots, n+1)$, and

$$f(x) = \sum_{i=1}^{n+1} f_i(a_i x + b_i y)$$

for every $x, y \in G$.

Proposition 1 of [17] states the following.

**Proposition 2.3.** *For every function $f : G \to G'$ we have*

$$P(n) \Rightarrow P_{loc}(n) \Rightarrow MD(n) \Rightarrow ID(n) \Rightarrow R(n),$$

*and*

$$P_{loc}(n) \Rightarrow P_{lin}(n) \Rightarrow ID(n).$$

In [17] it is also shown that the reverse implications do not hold in general. However, in a large class of special cases, when $G$ is divisible, all of them are equivalent.

From this point on we shall only consider functions mapping to the field of complex numbers $\mathbb{C}$.

**Theorem 2.4.** *If the group $G'$ is the additive group of $\mathbb{C}$, then we have*

$$P(n) \iff P_{loc}(n) \iff P_{lin}(n) \iff MD(n) \iff ID(n).$$

Indeed, Z. Djoković [8] proved $MD(n) \Longrightarrow P(n)$ assuming that $G'$ is uniquely divisible by $(n+1)!$. (Another proof, supposing that $G$, $G'$ are divisible and $G'$ is torsion-free was given by L. Székelyhidi in [31].) In particular, $MD(n) \Longrightarrow P(n)$ holds if $G' = \mathbb{C}$. Thus, by Proposition 2.3,

$$P(n) \iff P_{loc}(n) \iff MD(n).$$

The equivalence of $MD(n)$ and $ID(n)$ in the case of $G' = \mathbb{C}$ follows from [17, Theorem 14], and this implies the equivalence of the remaining conditions.

Therefore, *a function $f : G \to \mathbb{C}$ is a generalized polynomial if and only if it satisfies any of the conditions $P(n)$, $P_{loc}(n)$, $P_{lin}(n)$, $MD(n)$, $ID(n)$.* The smallest $n$ for which $f$ satisfies any of these conditions is called the degree of $f$. The degree of the identically zero function is, by definition, $-1$.

It is easy to see that the generalized polynomials of degree 0 are the nonzero constant functions. The generalized polynomials of degree 1 are of the form $a(x)+b$ where $a : G \to \mathbb{C}$ is a nonzero additive function and $b$ is constant.

**Proposition 2.5.** *The set of all generalized polynomials is an algebra over $\mathbb{C}$.*

**Proof.** We prove that if $f$ and $g$ are generalized polynomials of degree $\leq n$, then $c\cdot f$, $f+g$ and $f \cdot g$ are generalized polynomials. This easily follows by induction on $n$, using the formulas

$$\Delta_h(cf) = c\Delta_h f,$$

$$\Delta_h(f+g) = \Delta_h f + \Delta_h g,$$

$$\Delta_h(f \cdot g) = \Delta_h f \cdot \Delta_h g + f \cdot \Delta_h g + \Delta_h f \cdot g.$$

□

A function $f : G \to \mathbb{C}$ is a *polynomial* if it belongs to the algebra generated by the additive functions and the constant functions.

**Proposition 2.6.** *Let $G$ be an Abelian group. Then every polynomial function $f : G \to \mathbb{C}$ is a generalized polynomial.*

**Proof.** Let $q$ be a polynomial. Then it is of the form $Q(a_1(x), \dots, a_n(x))$, where $Q \in \mathbb{C}[x_1, \dots, x_n]$ and $a_1, \dots, a_n$ are additive functions. Thus $q$ is in the algebra generated by the generalized polynomials $a_1, \dots, a_n$ and it follows that $q$ is a generalized polynomial.

□

The reverse implication is not true in general. Székelyhidi [35] constructed the following counterexample.
Let $F^\omega$ denote the free Abelian group generated by countably infinitely many generators. We shall represent $F^\omega$ as

$$F^\omega = \{(x_1, x_2, \dots) : x_k \in \mathbb{Z}\ (k = 1, 2, \dots, ), \exists i_0, x_i = 0\ (i > i_0)\},$$

where the sum of the elements $(x_1, x_2, , \dots, )$ and $(y_1, y_2, , \dots, )$ is $(x_1 + y_1, x_2 + y_2, , \dots, )$.

We denote by $C(G)$ the linear space (over $\mathbb{C}$) of all functions mapping $G$ into $\mathbb{C}$.

**Proposition 2.7.** *The translates of a polynomial function generates a finite dimensional linear subspace of $C(G)$.*

**Proof.** It is easy to see that the translates of a nonzero additive function generate a linear space of dimension two. On the other hand, if the translates of $f$ and $h$ generate an $n$ and an $m$ dimensional linear subspace of $C(G)$, respectively, then the translates of $f + h$ generate an at most $n + m$ dimensional space, and the translates of $f \cdot h$ generate an at most $n \cdot m$ dimensional space. The statement of the Proposition clearly follows from these observations. □

**Theorem 2.8** (Székelyhidi [35])**.** *There is a generalized polynomial $g : F^\omega \to \mathbb{C}$ which is not a polynomial.*

**Proof.** Let

$$g(x) = \sum_{i=1}^{\infty} x_i^2 \qquad (\forall x = (x_i)_{i=1}^{\infty} \in F^\omega).$$

The function $g(x)$ is a generalized polynomial since $B(x, y) = \sum_{i=1}^{\infty} x_i y_i$ is a biadditive function and $\operatorname{diag}(B)(x) = g(x)$ holds. Suppose $g$ is a polynomial. Then, by Proposition 2.7, the translates of $g$ generate a finite dimensional linear subspace of $C(G)$.

The functions of the form $x \mapsto B(x,y)$ $(x \in G)$ are additive for every $y \in G$. If $y_{(k)} = (0,\dots,0,1,0\dots)$, where the 1 appears at the $k^{\text{th}}$ coordinate, then $B(x, y_{(k)}) = x_k$, the $k^{\text{th}}$ coordinate of $x$. It is clear that the maps $x \mapsto x_k$ $(k = 1, 2, \dots)$ are linearly independent over $\mathbb{C}$. Since $2 \cdot B(x,y) = g(x+y) - g(x)$, it follows that the functions $x \mapsto x_k$ belong to the linear space generated by the translates of $g$. This implies that the translates of $g$ generate an infinite dimensional space. Therefore, as we saw above, $g$ cannot be a polynomial.

□

By the torsion free rank of $G$ we mean the cardinality of a maximal independent system of elements of infinite order. In other words the torsion free rank of $G$ is the maximal cardinality $\kappa$ such that $G$ contains the free Abelian group of rank $\kappa$ as a subgroup.

The statement of Theorem 2.8 immediately implies the following theorem [35, Theorem 2.].

**Theorem 2.9.** *For every Abelian group $G$ with infinite torsion free rank there is a generalized polynomial $f : G \to \mathbb{C}$ which is not a polynomial.*

**Proof.** Indeed, an Abelian group $G$ with infinite torsion free rank contains a subgroup $H$ which is isomorphic to $F^{\omega}$. The proof of Theorem 2.8 gives a symmetric biadditive function $B : (H \times H) \to \mathbb{C}$ such that $g = \operatorname{diag}(B)$ is not a polynomial on $H$. Since $\mathbb{C}$ is divisible, it follows from a well-known fact of group theory that $B$ can be extended to $G \times G$ as a symmetric biadditive function.

Then $B(x,x)$ is a generalized polynomial on $G$ which is not a polynomial, since its restriction to $H$ is not a polynomial.

□

On the other hand, it is easy to prove the following theorem.

**Theorem 2.10** (M. Laczkovich [18])**.** *If $G$ is a finitely generated Abelian group and $f : G \to \mathbb{C}$ is a generalized polynomial, then $f$ is a polynomial.*

**Proof.** Every finitely generated Abelian group is isomorphic to $\mathbb{Z}^k \times H$ for a suitable $k$, where $H$ is a finite Abelian group. Thus, it is enough to deal with the following two special cases.

On the group $\mathbb{Z}^k$ every generalized polynomial is a polynomial. Indeed, the properties $P(n)$ and $P_{loc}(n)$ are equivalent. Thus the generalized polynomial $f$ is a restriction of a polynomial $p \in \mathbb{C}[x_1, \dots, x_k]$ to $\mathbb{Z}^k$. Since the projection onto any coordinate $\pi_i : x \mapsto x_i$ is additive for every $i = 1, \dots, k$, therefore $f(x) = P(\pi_1(x), \dots, \pi_k(x))$.

If $H$ is a finite Abelian group, then every generalized polynomial is constant on $H$. Indeed, on $H$ every additive function is zero. Thus every $k$-additive function is zero as well, and thus every generalized polynomial is constant on $H$.

If $(x, y) \mapsto f(x, y)$ ($(x, y) \in \mathbb{Z}^k \times H$) is a generalized polynomial, then $y \mapsto f(x, y)$ ($y \in H$) is a generalized polynomial on $H$ for every fixed $x \in \mathbb{Z}^k$. Thus $f(x, y) = f(x, 0)$ for every $x \in \mathbb{Z}^k$. Since $f(x, 0)$ is a generalized polynomial on $\mathbb{Z}^k$, it is a polynomial. $\square$

Now we take one step further by generalizing the notation of polynomials. We say that the function $f : G \to \mathbb{C}$ is a *local polynomial* if, for every finitely generated subgroup $H$ of $G$, the restriction $f|_H$ is a polynomial on $H$. Property $P_{loc}(n)$ guarantees that every generalized polynomial is a local polynomial.

The following example shows that the reverse implication is not necessary true.

**Proposition 2.11** (M. Laczkovich [18]). *There is an $f : F^\omega \to \mathbb{C}$ which is a local polynomial and is not a generalized polynomial.*

**Proof.** Let

$$P(x) = \sum_{i=1}^{\infty} x_i^i$$

for every $x = (x_1, x_2, \dots) \in F^\omega$. It is easy to check that $P(x)$ is a local polynomial.

The elements $(x_1, x_2, \dots)$ with $x_k = 0$ for every $k > n$ constitute a subgroup $H_n$ of $F^\omega$. It is easy to see that the restriction of $P(x)$ to $H_n$ is a generalized polynomial of degree $n$. Therefore, $\Delta_{h_1} \dots \Delta_{h_n} P(x) \neq 0$ for some $h_1, \dots, h_n, x \in H_n$. This implies that $P(x)$ is not a generalized polynomial of degree $\leq n$. Since this is true for every $n$, $P(x)$ is not a generalized polynomial. $\square$

As above, it is easy to generalize this result in the following way.

**Theorem 2.12.** *For every Abelian group $G$ with infinite torsion free rank there is a local polynomial $g : G \to \mathbb{C}$ which is not a generalized polynomial.*

It is clear that if $G$ is a finitely generated Abelian group, then every local polynomial on $G$ is a polynomial. Comparing with Theorem 2.10, we obtain the following.

**Proposition 2.13.** *If $G$ is a finitely generated Abelian group then, for every $f : G \to \mathbb{C}$ we have*

*$f$ is a polynomial $\Leftrightarrow$ $f$ is a generalized polynomial $\Leftrightarrow$ $f$ is a local polynomial.*

## 2.2 Discrete spectral analysis and spectral synthesis

Let $G$ be a locally compact group and let $C(G)$ denote the linear space of all complex valued functions defined on $G$ equipped with the topology of uniform convergence on compact sets. In several different areas of mathematics (differential and difference equations, theory of group representation, harmonic analysis, etc.) some special classes of subspaces of $C(G)$ play fundamental roles. One of them is the class of translation invariant, closed subspaces of $C(G)$. Spectral analysis and synthesis deal with the description of translation invariant functions spaces over locally compact groups. In order to summarize the main motivation of this section we quote from the article [35]:
"The fundamental problem is to discover the structure of such space of functions, or more exactly, to find an appropriate class of basic functions, the building blocks, which serve as "typical elements" of the space, a kind of basis."

In the sequel we only consider Abelian groups with the discrete topology. Then $C(G)$ is the same as $\mathbb{C}^G$, the linear space of all complex valued functions defined on $G$. We equip $C(G) = \mathbb{C}^G$ with the product topology.

A nonzero function $m \in C(G)$ is called an *exponential* if $m$ is multiplicative; that is, if $m(x + y) = m(x) \cdot m(y)$ for every $x, y \in G$. An *exponential monomial* is the product of a polynomial and an exponential, a *polynomial-exponential function* is a finite sum of exponential monomials.

The translation operator $T_h$ is defined by $T_h f(x) = f(x + h)$ $(x \in G)$ for every $h \in G$ and $f \in C(G)$. We call $T_h f(x)$ $(h, x \in G)$ a translate of $f$. The following theorem [10, 19, 34] gives a characterization of polynomial-exponential functions.

**Theorem 2.14.** *Let $G$ be a topological group (as a special case, locally compact). The translates of $f : G \to \mathbb{C}$ generate a finite dimensional space if and only if $f$ is a polynomial-exponential function.*

Note that the translates of an exponential function generate a one dimensional subspace of $C(G)$. Thus the argument used in the proof of Proposition 2.7 gives the 'if' part of the theorem.

We shall frequently use the following well-known result.

**Lemma 2.15.** *Let $(G, *)$ be an Abelian group, $V$ be a translation invariant linear subspace of $C(G)$, and let $\sum_{i=1}^{M} p_i \cdot m_i \in V$, where $p_1, \ldots, p_M$ are nonzero generalized polynomials and $m_1, \ldots, m_i$ are distinct nonzero exponentials on $G$. Then $(\Delta_{h_1} \ldots \Delta_{h_k} p_i) \cdot m_i \in V$ for every $i$ and for every $h_1, \ldots, h_k \in G$.*

*If $p_i$ is not constant, then there is a nonzero additive function $A : G \to \mathbb{C}$ such that $A \cdot m_i \in V$ and $m_i \in V$.*

**Proof.** Since $m_i$ is a nonzero exponential, $m_i(x) \neq 0$ for any $x \in G$. If $p(x)m(x) \in V$, then $p(x*h)m(x*h)$ and $c \cdot p(x)m(x)$ is in $V$ for any $c \neq 0 \in \mathbb{C}$, because $V$ is a translation invariant linear space. Thus, the equation

$$\begin{aligned} &p(x*h)m(x*h) - m(h)p(x)m(x) = \\ =&(p(x*h) - p(x))m(x)m(h) = (\Delta_h p(x)) \cdot m(x)m(h) \end{aligned}$$

shows that $\Delta_h p(x)m(x) \in V$. The iteration of this process proves the first statement of the lemma. It is well known that for every generalized polynomial $p$ there exist an integer $k$ and an additive function $A$ such that $\Delta_{h_1} \ldots \Delta_{h_k} p - A$ is constant, and $\Delta_{h_1} \ldots \Delta_{h_k} \Delta_{h_{k+1}} p$ is a nonzero constant for some $h_1, \ldots, h_k, h_{k+1} \in G$. Thus, if $p \cdot m \in V$, then $A \cdot m$ and $m$ are in $V$, which completes the proof. □

By a *variety* we mean a translation invariant closed subspace of $C(G)$.

If a variety contains an exponential function, then we say that *spectral analysis holds in this variety.* If spectral analysis holds in every variety on $G$, then we say that *spectral analysis holds on* $G$.

If a variety $V$ is spanned by exponential monomials belonging to $V$, then we say that *spectral synthesis holds in variety* $V$. If spectral synthesis holds in every variety on $G$, then we say that *spectral synthesis holds on* $G$. It is clear that if spectral synthesis holds in a variety $V$ then spectral analysis holds on $V$, as well.

For example, spectral synthesis holds on $\mathbb{Z}$ by a classical fact on sequences satisfying a linear recursion. Indeed, suppose that $V$ is a proper variety of $C(\mathbb{Z})$ meaning that $V \neq C(\mathbb{Z})$. As $V$ is closed, it follows that there is a $k$ such that $L = \{f|_{\{0,\ldots,k-1\}} : f \in V\}$ is a proper linear subspace of $\mathbb{C}^k = \mathbb{C}^{\{0,\ldots,k-1\}}$. Let $(c_0, \ldots, c_{k-1})$ be a nonzero element of $\mathbb{C}^k$ perpendicular to $L$. Then one can show, using translation invariance of $V$ that

$$\sum_{i=0}^{k-1} c_i f(n-i) = 0 \tag{3}$$

for every $n$. Therefore, the sequence $f(n)$ satisfies a linear recursion. Let $p(x) = \sum_{i=0}^{k-1} c_i x^i$ denote the characteristic polynomial of the linear recursion satisfying (3), and let $\lambda_j$ $(1 \leq j \leq l)$ denote the roots of $p(x)$ with multiplicity $m_j \geq 1$. Then $\sum_{j=1}^{l} m_j = k$. We may assume that $\lambda_j \neq 0$ which is equivalent to $c_0 \neq 0$. It is well-known that every solution $f$ of (3) is of the form

$$f(n) = \sum_{j=1}^{l} p_j(n)\lambda_j^n,$$

where $p_j \in \mathbb{C}[x]$ is a polynomial and $\deg p_j \leq m_j$ for every $j$. It is easy to check that, for every $j$, the map $n \mapsto \lambda_j^n$ is an exponential function, and that $p_j(n)$ is a polynomial function on $\mathbb{Z}$. Thus every solution is a linear combination of polynomial-exponential functions, thus spectral synthesis holds on $\mathbb{Z}$.

The following theorem is a generalization of the case above and first was proved by M. Lefranc [22].

**Theorem 2.16.** *For every finite $n$, spectral synthesis holds in every variety on the group $\mathbb{Z}^n$ equipped with the discrete topology.*

The proof of this theorem is based on Krull's theorem and other ring theoretical results. By the following proposition, we obtain the same result for any finitely generated Abelian group, since any finitely generated Abelian group is a homomorphic image of $\mathbb{Z}^n$ for some $n$.

**Proposition 2.17.** *If spectral synthesis holds on an Abelian group $G$, then it holds for any homomorphic image of $G$.*

The proof can be found in [36, p. 21].

We denote by $r_0(G)$ the torsion free rank of $G$. The following theorem is the main result of [20].

**Theorem 2.18.** *Spectral analysis holds on a discrete Abelian group $G$ if and only if $r_0(G) < 2^\omega$.*

R. J. Elliot claimed in [9] that spectral synthesis holds on every Abelian group. Elloit's proof, however, was defective. In fact, the statement is false, as the following theorem shows.

**Theorem 2.19** (L. Székelyhidi [35])**.** *Spectral synthesis does not hold on any discrete Abelian group $G$ with $r_0(G) \geq \omega$.*

The proof is based on the generalized polynomial $P(x) = \sum_{i=0}^{\infty} x_i^2$ $(x \in F^\omega)$ that was constructed in Theorem 2.9. Let $V$ denote the variety generated by $P$. We proved in Theorem 2.9 that $P$ is not a polynomial. On the other hand, it is easy to show that every element of the variety $V$ is of the form $cP(x) + a(x) + b$, where $c, b$ are constant and $a$ is additive. This implies that the only exponential in $V$ is the identically 1 function, and every polynomial in $V$ is of the form $a(x) + b$. Therefore, the function $P$ is not in the closure of polynomial-exponential functions of $V$. Indeed, every polynomial in $V$ vanishes under the operation $\Delta_h^2$ but $P$ does not.

On the other hand Laczkovich and Székelyhidi proved the following useful result in [21].

**Theorem 2.20.** *Spectral synthesis holds on a discrete Abelian group $G$ if and only if $r_0(G)$ is finite.*

We shall say that the function $f$ is a *generalized polynomial-exponential*, if $f = \sum_{i=1}^{n} p_i \cdot m_i$, where $p_1, \dots, p_n$ are generalized polynomials and $m_1, \dots, m_n$ are exponentials. If a variety $V$ is spanned by generalized polynomial-exponentials belonging to $V$, then we say that *generalized spectral synthesis holds in the variety $V$*.

Since every function in the variety $V$ of the previous example is a generalized polynomial, it follows that generalized spectral synthesis holds in $V$. However, generalized spectral synthesis does not hold on any Abelian group $G$ for which $r_0(G) \geq \omega$. This is a consequence of the following theorem.

**Theorem 2.21** (M. Laczkovich [18]). *Let $Q(x) = \sum_{i=1}^{\infty} x_i^i$ for every $x = (x_1, x_2, \dots,) \in F^{\omega}$, and let $W$ denote the variety generated by $Q$. Then the set of generalized polynomial-exponentials contained in $W$ is not dense in $W$.*

We shall say that the function $f$ is a *local polynomial-exponential* if $f = \sum_{i=1}^{n} p_i \cdot m_i$, where $p_1, \dots, p_n$ are local polynomials and $m_1, \dots, m_n$ are exponentials. Let $V$ be a variety on $G$. We say that *local spectral synthesis* holds in $V$ if the set of local polynomial-exponentials contained in $V$ is dense in $V$. We say that *local spectral synthesis* holds on a group $G$ if local spectral synthesis holds in every variety on $G$. The following result was proved in [18].

**Theorem 2.22.** *There exists a cardinal $\omega_1 \leq \kappa \leq 2^{\omega}$ such that, for every Abelian group $G$, local spectral synthesis holds on $G$ if and only if $r_0(G) < \kappa$. In particular, local spectral synthesis holds on every countable Abelian group $G$.*

As the proof of Lefranc's theorem based on the reduction to a consequence of a Krull's intersection theorem, the proof of this statement is based on a generalization of Krull's intersection theorem.

We just remark that the exact value of $\kappa$ is unknown. The conjecture formulated in [18] is that $\kappa = \omega_1$, which is equivalent to the statement that local spectral synthesis does not hold on the free Abelian group generated by $\omega_1$ elements.

For the sake of completeness we quote the following result from [18] which is the analogue of the case of 'standard' spectral synthesis.

**Proposition 2.23.** *If local spectral synthesis holds on the Abelian group $G$, then the same is true for every homomorphic image of $G$.*

## 2.3 Linear functional equations

Let $a_i, b_i, c_i$ be fixed complex numbers and $K$ be a subfield of the field of complex numbers which contains the parameters $b_i, c_i$. In this thesis our main concern is the investigation of the solutions of the functional equations of the form

$$\sum_{i=1}^{n} a_i f(b_i x + c_i y) = 0 \qquad (x, y \in K) \tag{4}$$

where $f\colon K \to \mathbb{C}$ is the unknown function.

Our aim is to describe the solutions, at least on the field $K$. In the following sections we give a description of all solutions of equation (4) satisfying certain natural conditions.

Note that *every solution defined on $K$ can be extended to a solution on* $\mathbb{C}$. Indeed, since $\mathbb{C}$ is a linear space over the field $K$, the identity on $K$ can be extended to $\mathbb{C}$ as a linear function over $K$. Let $\phi : \mathbb{C} \to K$ be such an extension. It is clear that if $f$ satisfies (4) for every $x, y \in K$, then $f \circ \phi$ satisfies (4) for every $x, y \in \mathbb{C}$. We note also that *if there is a non-identically zero solution on* $\mathbb{C}$, *then there is one on* $K$. This is obvious from the fact that if $f$ is a solution, then so is $f(c \cdot x)$ for every $c \in \mathbb{C}$.

First we shall restrict our attention to those equations (4) whose parameters satisfy the following condition.

*The numbers $a_1, \ldots, a_n$ are nonzero, and* **there exists** *an $1 \le i \le n$ such that $b_i c_j \ne b_j c_i$ holds for any $1 \le j \le n$, $j \ne i$.* (5)

The following result was rediscovered several times. There are much more general variants (see the Remarks following the theorem). Since the result is of basic importance in our investigations, we provide a simple proof.

**Theorem 2.24.** *Suppose that*

$$\textbf{for every } 1 \le i \le n,\ b_i c_j \ne b_j c_i \textit{ holds for any } 1 \le j \le n,\ j \ne i. \tag{6}$$

*Let $K$ be a subfield of complex numbers which contains $b_i$ and $c_i$. If the functions $f_i : K \to \mathbb{C}$ $(i = 1, \ldots, n)$ satisfy*

$$\sum_{i=1}^{n} f_i(b_i x + c_i y) = 0 \tag{7}$$

*for every $x, y \in K$, then every $f_i$ is a generalized polynomial on $K$ of degree at most $n-2$.*

**Proof.** The proof is based on the following idea. Let $d_{i,j} = b_i c_j - b_j c_i$ for every $i \ne j \in \{1, \ldots, n\}$. By condition (6), $d_{i,j} \ne 0$. Then

$$b_k(x + c_n h) + c_k(y - b_n h) = b_k x + c_k y + d_{k,n} h$$

for every $x, y, h \in K$ and $k = 1, \dots, n$. Therefore,

$$\begin{aligned} 0 &= \sum_{i=1}^{n} f_i(b_i(x + c_n h) + c_i(y - b_n h)) - \sum_{i=1}^{n} f_i(b_i x + c_i y) \\ &= \sum_{i=1}^{n} (f_i(b_i x + c_i y + d_{i,n} h) - f_i(b_i x + c_i y)) \\ &= \sum_{i=1}^{n} \Delta_{d_{i,n} h} f_i(b_i x + c_i y) = \sum_{i=1}^{n-1} \Delta_{d_{i,n} h} f_i(b_i x + c_i y). \end{aligned}$$

The last equation holds, since $d_{n,n} = 0$. Thus there are $n - 1$ terms remained. By continuing the process we get the following:

$$0 = \sum_{i=1}^{n-j-1} \Delta_{d_{i,n-j} h_j} \Delta_{d_{i,n-j+1} h_{j-1}} \dots \Delta_{d_{i,n} h_1} f_i(b_i x + c_i y),$$

for every $j = 1, \dots, n-1$. At the end of this process ($j = n - 1$) we obtain

$$0 = \Delta_{d_{1,2} h_{n-1}} \Delta_{d_{1,3} h_{n-2}} \dots \Delta_{d_{1,n} h_1} f_1(b_1 x + c_1 y).$$

Since $d_{1,2}, \dots, d_{1,n}$ are nonzero and the numbers $h_i$ ($i = 1, \dots, n-1$) can be chosen arbitrarily, this shows that $f_1$ is a generalized polynomial of degree at most $n - 2$. By symmetry, we obtain that $f_i$ is a generalized polynomial of degree at most $n - 2$ for every $i = 1, \dots, n$. □

**Remark 2.25.** The previous theorem has several generalizations. One of them is the result of L. Székelyhidi [33, Theorem 3.9] (see also [1], [17] and [36]).

**Theorem 2.26.** *Let $G, S$ be Abelian groups, and suppose that $G$ is divisible and $S$ is torsion-free. Let $n$ be a non-negative integer, and let $\phi_i, \psi_i \colon G \to G$ be homomorphisms of $G$ onto itself such that* $\mathrm{Rg}(\psi_j \circ \psi_i^{-1} - \phi_j \circ \phi_i^{-1}) = G$ *for $i \neq j$ ($i, j = 1, \dots, n$), where* $\mathrm{Rg}(\phi)$ *denotes the range of $\phi$. If the functions $f_i : G \to S$ ($i = 0, \dots, n$) satisfy*

$$f_0(x) + \sum_{i=1}^{n} f_i(\phi_i(x) + \psi_i(y)) = 0,$$

*then each $f_i$ is a generalized polynomial of degree at most $n - 1$.*

As a special case of this theorem we get the following. *Let $V$ and $W$ be vector spaces over one of the fields $\mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$, and let $a_i, b_i, c_i$ ($i = 1, \dots, n$) be scalars which satisfy* (6). *If the functions $f_i : V \to W$ ($i = 1, \dots, n$) satisfy equation* (7) *for every $x, y \in V$, then every $f_i$ is a generalized polynomial of degree at most $n - 2$.*

In our investigations we shall need the following version of Theorem 2.24. The argument used in the proof of Theorem 2.24 in the case when condition (6) is replaced by $d_{i,j} \neq 0$ ($j \neq i$) gives that $f_i$ is a generalized polynomial of degree $\leq n-2$. If $f_1 = \ldots = f_n$ then we obtain the following.

**Theorem 2.27.** *Let $n$ be a positive integer, and $b_i, c_i$ $(i = 1, \ldots, n)$ complex numbers which satisfy condition* (5)*. Let $K$ be a subfield of the complex numbers containing $b_i, c_i$, and let $f : K \to \mathbb{C}$ be a function satisfying*

$$\sum_{i=1}^{n} a_i f(b_i x + c_i y) = 0 \tag{8}$$

*for every $x, y \in K$. Then $f$ is a generalized polynomial on $K$ of degree at most $n-2$.*

Condition (5) is automatically satisfied in some important special cases, such as

$$\sum_{i=1}^{n} a_i f(b_i x + y) = 0 \tag{9}$$

and

$$\sum_{i=1}^{n} a_i f(b_i x + (1-b_i)y) = 0, \tag{10}$$

where $b_1, \ldots, b_n$ are distinct.

The following example shows that condition (5) is necessary in Theorem 2.27.

**Proposition 2.28.** *The functional equation*

$$f(x) - f(2x) = 0 \qquad (x \in \mathbb{C}) \tag{11}$$

*has nonconstant solutions, although every generalized polynomial solution of this equation is constant.*

**Proof.** Let $H_x$ denote the set $\{2^n x : n \in \mathbb{Z}\}$ for every $x \in \mathbb{C}$. It is clear that for $x \neq x'$ either $H_x = H_{x'}$ or $H_x \cap H_{x'} = \emptyset$. By Zorn's lemma, there exists a maximal subset $X$ of $\mathbb{C}$ such that $\bigcup_{x \in X} H_x = \mathbb{C}$ and $H_x \cap H_{x'} = \emptyset$ for every $x, x' \in X$. If we define $f$ on every $H_x$ $(x \in X)$ as an arbitrary constant, then we obtain a solution of (11).

On the other hand we show that every generalized polynomial solution of $f(x) - f(2x) = 0$ is constant. Let us assume that $g$ is a nonconstant generalized polynomial solution. Then $g$ is of the form $\sum_{j=0}^{m} f_j$, where $f_j$ is a monomial of degree $j$ for $j = 0, \ldots, m$. This means that $f_j$ is a diagonal of a $j$-additive function. Suppose that $m \geq 1$ and $f_m$ is not identically zero. Since any $j$-additive function has the rational homogeneity property in every coordinate, we have

$$g(rx) = (f_0 + r \cdot f_1(x) + \ldots + r^m \cdot f_m(x)) = (f_0 + 2r \cdot f_1(x) + \ldots + 2^m r^m \cdot f_m(x)) = g(2rx).$$

for every $x \in \mathbb{C},\ r \in \mathbb{Q}$. Let $x \in \mathbb{C}$ be fixed such that $f_m(x) \neq 0$. Then we get

$$r \cdot f_1(x) + \ldots + (2^m - 1)r^m \cdot f_m(x) = 0.$$

This is a nonzero polynomial of the variable $r$, which has infinitely many roots. This contradiction shows that $g$ is constant. $\square$

We note that the statement of Proposition 2.28 can be generalized as follows. *Suppose that the pairs of parameters $(b_i, c_i)$ lie on a line going through the origin. Then the functional equation* (8) *can be reduced to one of the form $\sum_{i=1}^{n} a_i f(d_i x) = 0$, which only has constant generalized polynomial solutions.*

The space of the solutions of (8) defined on $\mathbb{C}$ is always a linear space over $\mathbb{C}$. It is easy to see that *the space of solutions of any of the equations* (9), (10) *is translation invariant.* However, in general the space of the solutions of (8) is not necessary translation invariant as shown by the equation of Proposition 2.28.

In general, *if the points $(b_i, c_i) \in \mathbb{C}^2$ lie on a line not going through the origin* $(0,0)$, *then the space of solutions is translation invariant.*

Indeed, it follows from the condition that there are constants $\beta, \gamma \in \mathbb{C}$ such that $\beta b_i + \gamma c_i = 1$ $(i = 1, \ldots, n)$. If $f\colon \mathbb{C} \to \mathbb{C}$ is a solution of (8), then we have

$$\begin{aligned}\sum_{i=1}^{n} a_i T_h f(b_i x + c_i y) = \sum_{i=1}^{n} a_i f(b_i x + c_i y + h) = \\ = \sum_{i=1}^{n} a_i f(b_i x + c_i y + \beta b_i h + \gamma c_i h) = \\ = \sum_{i=1}^{n} a_i f(b_i(x + \beta h) + c_i(y + \gamma h)) = 0\end{aligned}$$

for every $h, x, y \in \mathbb{C}$. Thus $T_h f$ is also a solution, which proves translation invariance.

We note that any equation for which the points $(b_i, c_i) \in \mathbb{C}^2$ lie on a line not going through the origin $(0,0)$ can be transformed into an equation of type (9) using a suitable linear substitution. For instance, the substitution $x \mapsto x + y$ transforms (10) into (9). The translation invariance of the space of solutions of the equations satisfying this condition follows from this observation.

However, the collinearity of $(b_i, c_i)$ is not necessary for the subspace of solutions to be translation invariant. This is shown by the following proposition.

**Proposition 2.29.** *There is a functional equation of the type* (4) *which only has constant solutions, and the pairs $(b_i, c_i)$ are not collinear.*

The proof of this statement uses some results of the next section, thus we postpone the proof until Proposition 3.23.

**From now on we concentrate on the generalized polynomial solutions of the equations** (8)**.** (By Theorem 2.27, if condition (5) is satisfied then all solution are automatically generalized polynomials.)

**Lemma 2.30** ([15])**.** *Let* $f = f_0 + \sum_{k=1}^{m} f_k$*, where* $f_0$ *is constant and* $f_k$ *is a monomial of degree* $k$ $(k = 1, \ldots, m)$*. Then* $f$ *is a solution of* (8) *if and only if each of* $f_0, \ldots, f_m$ *is a solution of* (8)*.*

**Proof.** If each of $f_0, \ldots, f_m$ is a solution then so is $f$ as the space of solutions is linear.

Suppose $f$ is a solution. Since any $k$-additive function has the rational homogeneity property in every coordinate, we have

$$\sum_{i=1}^{n} a_i \cdot (f_0 + r \cdot f_1(b_i x + c_i y) + \ldots + r^m \cdot f_m(b_i x + c_i y)) = 0$$

for every $x, y \in \mathbb{C},\ r \in \mathbb{Q}$. Let $x, y$ be fixed. Putting $F_j = \sum_{i=1}^{n} a_i f_j(b_i x + c_i y)$ we obtain $\sum_{j=0}^{m} F_j \cdot r^j = 0$ for every $r \in \mathbb{Q}$. Thus $F_j = 0$ for every $j$, proving that $f_j$ is a solution of (8). □

The constant solutions of (8) are trivial to find: if $\sum_{i=1}^{n} a_i = 0$ then every constant function is a solution; otherwise there is no nonzero constant solution.

It is also clear that if $f$ is additive, then $f$ is a solution of (8) if and only if

$$\sum_{i=1}^{n} a_i \cdot f(b_i x) = \sum_{i=1}^{n} a_i \cdot f(c_i x) = 0$$

for every $x$. Note that in the case of (9) and (10), $\sum_{i=1}^{n} a_i = 0$ is a necessary condition of the existence of nonzero additive solutions.

The $k$-additive solutions of more general equations have been investigated by Székelyhidi [36] and Varga and Vincze [42].

In [38] the equation

$$\sum_{i=1}^{n} a_i \cdot f(b_i x) = 0 \tag{12}$$

is called Daróczy's functional equation. This terminology comes from the following theorem of Z. Daróczy [7].

**Theorem 2.31.** *The equation* $a \cdot f(x) - f(bx) = 0$ *has a nonzero additive solution if and only if either* $a$ *and* $b$ *are transcendental or they are algebraically conjugates.*

(Daróczy's theorem implies that the equation $f(x)-f(2x)=0$ has no nonzero additive solution, as we have seen already in Proposition 2.28.)

We note that the functional equations (10) were investigated by Adrienn Varga [40] in the case when $a_i, b_i \in \mathbb{R}$ and (10) is satisfied for $x, y \in I$ where $I \subseteq \mathbb{R}$ is an interval. She used the extension theorem of Páles [24] and showed that if $f$ is a solution on $I$ then there exists a unique extension of $f$ to $\mathbb{R}$ which satisfies (10) for every $x, y \in \mathbb{R}$.

We close this section with a nice result of A. Varga and Cs. Vincze [41] about Daróczy's functional equation containing several terms.

**Theorem 2.32.** *Suppose that $n \geq 2$.*

(i) *Let us assume that the parameters $b_1, \ldots, b_n$ are algebraically independent over $\mathbb{Q}$. Then the equation*

$$f(x) + \sum_{i=1}^{n-1} a_i f(b_i x) = 0 \tag{13}$$

*has nonzero additive solutions if and only if at least one of the parameters $a_1, \ldots, a_n$ is a transcendental number.*

(ii) *Let us assume that the parameters $a_1, \ldots, a_n$ are algebraically independent over $\mathbb{Q}$. Then the equation* (13) *has nonzero additive solutions if and only if at least one of the parameters $b_1, \ldots, b_n$ is transcendental number.*

As we shall see in the next section, Theorems 2.31 and 2.32 are easy consequences of Theorems 3.7 and 3.14 (see Remark 3.9). For a further generalization, see Theorem 5.14.

# 3 Existence of nonzero solutions of linear functional equations

## 3.1 Some varieties related to linear functional equations

In this subsection we introduce some notation and basic results that will be needed in the sequel. We consider the linear functional equation

$$\sum_{i=1}^{n} a_i f(b_i x + c_i y) = 0. \tag{14}$$

We fix a finitely generated subfield $K$ of $\mathbb{C}$ containing the parameters $b_i, c_i$ $(i = 1, \dots, n)$. We denote by $S$ the set of solutions of (14) defined on $K$. It is clear that $S$ is a closed linear subspace of the product space $\mathbb{C}^K$.

Let $S_k$ denote the set of those solutions of (14) defined on $K$ which are generalized monomials of degree $k$. Thus $S_1$ is the set of additive solutions on $K$.

We denote by $V_k$ the set of $k$-additive functions defined on the additive group $K^k$. Thus $V_1$ is the set of additive functions on $K$.

**Lemma 3.1.** *$V_k$ is a closed linear subspace of the product space $\mathbb{C}^{K^k}$, and $S_k$ is a closed linear subspace of the product space $\mathbb{C}^K$.*

**Proof.** It is clear that $V_k$ is a linear space. We prove that $V_1$ is closed; a similar argument works for $V_k$.

Let $g\colon K \to \mathbb{C}$ be a function in the closure of $V_1$. Since $g \in clV_1$, for every $\varepsilon > 0$ there exists $f \in V_1$ such that

$$|f(a) - g(a)| < \varepsilon,\ \ |f(b) - g(b)| < \varepsilon \text{ and } |f(a+b) - g(a+b)| < \varepsilon.$$

Now, $f(a) + f(b) = f(a+b)$ because $f$ is additive, so $|g(a+b) - g(a) - g(b)| < 3\varepsilon$. This is true for every $\varepsilon$, therefore $g(a+b) = g(a) + g(b)$. This holds for every $a, b \in K$ with $a, b, a+b \neq 0$ and then, by $g(0) = 0$, it follows that $g$ is additive on $K$. This means that $V_1$ is closed.

Let $\operatorname{diag} V_k = \{\operatorname{diag} F : F \in V_k\}$. Then $\operatorname{diag} V_k$ is a closed linear subspace of $\mathbb{C}^K$: this follows from the formula (2) and the fact that $V_k$ is closed. Since $S_k = S \cap \operatorname{diag} V_k$, we obtain that $S_k$ is closed. □

Let $M_k$ denote the set of the functions $F\colon K^k \to \mathbb{C}$ such that $F$ is $k$-additive, and the function $x \mapsto F(s_1 x, s_2 x, \dots, s_k x)$ is a solution of (14) on $K$ for every $s_1, s_2, \dots s_k \in K \setminus \{0\}$.

We put $K^* = \{x \in K : x \neq 0\}$; then $K^*$ is an Abelian group under multiplication. The Abelian group $\underbrace{K^* \times \ldots \times K^*}_{k}$ will be denoted by $(K^*)^k$, where the operation is multiplication in every coordinate of the vectors of $(K^*)^k$. We put $V_k^* = \{F|_{(K^*)^k} : F \in V_k\}$ and

$$M_k^* = \{F|_{(K^*)^k} : F \in M_k\}.$$

**Lemma 3.2.** *$V_k^*$ and $M_k^*$ are varieties on $(K^*)^k$.*

**Proof.** It is clear that $V_k$ and $V_k^*$ are linear spaces over $\mathbb{C}$. Translation invariance of $V_k^*$ on $(K^*)^k$ follows from the fact that if $F : K^k \to \mathbb{C}$ is $k$-additive, then so is

$$(x_1, \ldots, x_k) \mapsto F(c_1 x_1, \ldots, c_k x_k) \qquad ((x_1, \ldots, x_k) \in K^k)$$

for every $(c_1, \ldots, c_k) \in (K^*)^k$. The statement that $V_k$ and $V_k^*$ are closed can be proved as in Lemma 3.1 and is left to the reader.

It is easy to see that $M_k$ is a closed linear subspace of $V_k$, and $M_k^*$ is a closed linear subspace of $V_k^*$. Translation invariance of $M_k^*$ on $(K^*)^k$ means that if $F \in M_k^*$, then the map $(x_1, \ldots, x_k) \mapsto F(c_1 x_1, \ldots, c_k x_k)$ $(x_1, \ldots, x_k \in K^*)$ also belongs to $M_k^*$ for every $c_1, \ldots, c_K \in K^*$, which is easily seen from the definition of $M_k^*$. $\square$

**Lemma 3.3.**

$$S_k = \{\operatorname{diag} F : F \in M_k\}.$$

**Proof.** It is clear that $\operatorname{diag} F \in S_k$ for every $F \in M_k$. We prove the converse. Let $f : K \to \mathbb{C}$ be an element of $S_k$. Then $f$ is a solution (14) on $K$, and $f = \operatorname{diag} F$ for a symmetric $k$-additive function $F : K^k \to \mathbb{C}$. We prove $F \in M_k$. We have to show that for every $(s_1, \ldots, s_k) \in (K^*)^k$ the diagonal of the function

$$G(x_1, \ldots, x_k) = F(s_1 x_1, \ldots, s_k x_k)$$

belongs to $S_k$. Let $\operatorname{diag} G = g$. Then, by Lemma 2.1 we have

$$g(x) = F(s_1 x, \ldots, s_k x) = \frac{1}{k!} \cdot \Delta_{s_1 x} \Delta_{s_2 x} \ldots \Delta_{s_k x} f(0) = \sum_{j=1}^{M} \pm f(e_j x)$$

with suitable $e_1, \ldots, e_M \in K$. Since $x \mapsto f(ex)$ belongs to $S_k$ for every $e \in K$, it follows that $g \in S_k$, and thus $F \in M_k$. $\square$

The following proposition will be used frequently (see [16, Theorem 14.5.1, p. 358]).

**Proposition 3.4.** *Let $K \subset \mathbb{C}$ be a finitely generated field and $\phi \colon K \to \mathbb{C}$ be an injective homomorphism. Then there exists an automorphism $\psi$ of $\mathbb{C}$ such that $\psi|_K = \phi$.*

**Lemma 3.5.** *If $m \in V_1^*$ is an exponential on $K^*$, then $m$ can be extended to $\mathbb{C}$ as an automorphism of $\mathbb{C}$.*

**Proof.** The condition $m \in V_1^*$ means that extending $m$ to $K$ by $m(0) = 0$, we obtain an additive function. Now $m$ is an exponential on $K^*$, and thus $m$ satisfies $m(xy) = m(x)m(y)$ for every $x, y \in K^*$. Consequently, the extended $m$ is a homomorphism of $K$. By $m \neq 0$ on $K^*$, it follows that $m$ is injective. Since the transcendence degree of $K$ over $\mathbb{Q}$ is finite, it follows, by Proposition 3.4, that $m$ can be extended to $\mathbb{C}$ as an automorphism of $\mathbb{C}$. □

**Lemma 3.6.** *Suppose that $m \in V_k$ and $m|_{(K^*)^k}$ is an exponential; i.e. $m$ is nonzero on $(K^*)^k$, and $m(xy) = m(x)m(y)$ for every $x, y \in (K^*)^k$. Then there are field automorphisms $m_1, \ldots, m_k$ of $\mathbb{C}$ such that*

$$m(x) = m(x_1, \ldots, x_k) = m_1(x_1) \cdots m_k(x_k) \qquad (x_1, \ldots, x_k \in K).$$

**Proof.** By the multiplicativity of $m$,

$$m(x_1, \ldots, x_k) = m(x_1, 1, \ldots, 1) \cdot m(1, x_2, 1, \ldots, 1) \cdot \ldots \cdot m(1, \ldots, 1, x_k).$$

Since $x_i \mapsto m(1, \ldots, 1, x_i, 1, \ldots, 1)$ is additive on $K$ and exponential on $K^*$, it is an injective homomorphism, which we denote by $m_i$. By Proposition 3.4, $m_1, \ldots, m_k$ can be extended to $\mathbb{C}$ as automorphisms of $\mathbb{C}$. □

## 3.2 Non-zero additive solutions

Our main result in this subsection is Theorem 3.7.

Let $K$ be a finitely generated subfield of $\mathbb{C}$ containing $b_i, c_i$ $(i = 1, \ldots, n)$. Recall that $S_1$ denotes the set of additive solutions of (14) defined on $K$. Clearly, $f : K \to \mathbb{C}$ belongs to $S_1$ if and only if

$$\sum_{i=1}^n a_i f(b_i x) = 0, \qquad \sum_{i=1}^n a_i f(c_i x) = 0 \tag{15}$$

holds for every $x \in K$.

It is easy to see that if $\phi$ is an automorphism of $\mathbb{C}$ then $\phi$ is a solution of (14) if and only if

$$\sum_{i=1}^n a_i \phi(b_i) = 0 \text{ and } \sum_{i=1}^n a_i \phi(c_i) = 0. \tag{16}$$

Indeed, we have, for every $x, h \in \mathbb{C}$,

$$\sum_{i=1}^n a_i \phi(b_i x + c_i h) = \sum_{i=1}^n a_i \left(\phi(b_i x) + \phi(c_i h)\right) =$$

$$= \left( \sum_{i=1}^{n} a_i \phi(b_i) \right) \phi(x) + \left( \sum_{i=1}^{n} a_i \phi(c_i) \right) \phi(h).$$

**Theorem 3.7.** *There is a nonzero additive solution of* (14) *if and only if there exists a solution of* (14) *which is an automorphism* $\phi\colon \mathbb{C} \to \mathbb{C}$ *or, equivalently, an automorphism satisfying* (16).

**Proof.** The 'if' statement is obvious.

Let $f_1$ be a nonzero additive solution of (14) and let $d \in \mathbb{C}$ be such that $f_1(d) \neq 0$. Put $K = \mathbb{Q}(b_1, \ldots, b_n, c_1, \ldots, c_n, d)$, the extension of $\mathbb{Q}$ by the complex numbers $b_i, c_i, d$ $(i = 1, \ldots, n)$.

Note that $S_1 = M_1$, and thus $S_1^* = M_1^*$. Therefore, by Lemma 3.2, $S_1^*$ is a variety on the Abelian group $K^*$.

We have $S_1^* \neq \{0\}$ because $f_1|_{K^*} \in S_1^*$ and $f_1$ is not identically zero on $K^*$. By Theorem 2.18, if $G$ is a discrete Abelian group of torsion free rank less than continuum, then harmonic analysis holds on $G$. This means that every nonzero variety on $\mathbb{C}^G$ contains an exponential. Since $K^*$ is countable, we find that $S_1^*$ contains an exponential.

Let $\phi$ be an exponential element of $S_1^*$. Since $\phi \in S_1^*$, we have

$$0 = \sum_{i=1}^{n} a_i \phi(b_i x) = \sum_{i=1}^{n} a_i \phi(b_i) \phi(x) \text{ and } 0 = \sum_{i=1}^{n} a_i \phi(c_i x) = \sum_{i=1}^{n} a_i \phi(c_i) \phi(x)$$

for every $x \in K^*$. This implies $0 = \sum_{i=1}^{n} a_i \phi(b_i) = \sum_{i=1}^{n} a_i \phi(c_i)$. Using Lemma 3.5, $\phi$ can be extended to an automorphism of $\mathbb{C}$. This completes the proof. □

**Remark 3.8.** Essentially the same proof shows the following more general result.

*The functional equation*

$$\sum_{i=1}^{n} a_i f(b_{i,1} x_1 + b_{i,2} x_2 + \cdots + b_{i,k} x_k) = 0 \qquad (x_1, \ldots, x_k \in \mathbb{C}) \tag{17}$$

*has a nonzero additive solution if and only if there exists an automorphism* $\phi$ *of* $\mathbb{C}$ *which is a solution.*

In fact, *all of our results to be presented about the generalized polynomial solutions of* (14) *can be generalized, with the same proofs, to the equations of the form* (17) *for any* $k$.

**Remark 3.9.** Theorem 2.31 is an immediate consequence of Remark 3.8. Indeed, if there is a nonzero additive solution of equation $a \cdot f(x) - f(bx) = 0$, then there exists an automorphism $\phi$ of $\mathbb{C}$ which is a solution of this equation. Thus, the equation $a \cdot \phi(x) - \phi(bx) = 0$ implies that $a = \phi(b)$, which means that either $a$ and $b$ are transcendental or they are algebraically conjugates.

**Theorem 3.10.** *Suppose that $a_1, \dots, a_n$ are nonzero and $b_1, \dots, b_n$ are distinct complex numbers. The following are equivalent:*

(i) *There is a nonconstant solution of*

$$\sum_{i=1}^{n} a_i f(b_i x + y) = 0 \quad x, y \in \mathbb{C}. \tag{18}$$

(ii) *There is a solution of* (18) *which is an automorphism of* $\mathbb{C}$.

(iii) $\sum_{i=1}^{n} a_i = 0$, *and there exists an automorphism* $\phi\colon \mathbb{C} \to \mathbb{C}$ *such that*

$$\sum_{i=1}^{n} a_i \phi(b_i) = 0.$$

**Proof.** (i)$\Longrightarrow$ (ii): The substitution $x = 0$ shows that $\sum_{i=1}^{n} a_i f(y) = 0$, thus either $\sum_{i=1}^{n} a_i = 0$ or the 0 function is the only solution. By (i), we have $\sum_{i=1}^{n} a_i = 0$. This condition implies that the constant functions are solutions.

Let $f(x)$ be a nonconstant solution. By Theorem 2.24, every solution is a generalized polynomial. If the degree of $f$ is $k$, then for suitable $h_1, \dots, h_{k-1}$, the degree of the function $g = \Delta_{h_1} \dots \Delta_{h_{k-1}} f$ is 1. Then $g(x) = a(x) + c$, where $a(x)$ is a nonzero additive function and $c$ is a constant. Since $g - c$ is a solution, we may assume that $f$ is a not identically zero additive function. Then, by Theorem 3.7, there is an automorphism of $\mathbb{C}$ which is a solution of (52).

(ii) $\Longrightarrow$ (iii): If $\phi$ is an automorphism which is a solution of (52), then $\sum_{i=1}^{n} a_i \phi(1) = 0$ and $\sum_{i=1}^{n} a_i \phi(b_i) = 0$.

(iii) $\Longrightarrow$ (i): It is clear by the previous theorem. □

**Remark 3.11.** Theorem 3.10 does not hold in the general case, because the existence of a nonzero solution of (14) does not imply the existence of a nonzero additive solution. For example, the equation

$$f(x + y) + f(x - y) - 2f(x) - 2f(y) = 0$$

is satisfied by $x^2$. It is easy to check that every additive solution must be zero.

**Remark 3.12.** We shall need the following, slightly stronger version of the implication (i)$\Longrightarrow$(ii) of Theorem 3.10. *Suppose that there is a nonconstant function $f\colon K \to \mathbb{C}$ such that* (18) *holds for every $x \in K^*$, $y \in K$. Then there is an automorphism solution to* (18). This can be proved by using the argument of Theorems 3.7 and 3.10, since the set of functions satisfying the condition above also forms a variety on $K^*$.

**Proposition 3.13.** *Suppose* $\sum_{i=1}^{n} a_i = 0$ *and* $\sum_{i=1}^{n} a_i b_i \neq 0$. *If either*

(i) $b_i \in \mathbb{Q}$ $(i = 1, \ldots, n)$, *or*

(ii) $a_i \in \mathbb{Q}$ $(i = 1, \ldots, n)$,

*then every solution of* $\sum_{i=1}^{n} a_i f(b_i x + y) = 0$ *is constant.*

**Proof.** We may assume that $a_1, \ldots, a_n$ are nonzero and $b_1, \ldots, b_n$ are distinct. Indeed, by deleting the terms corresponding to $a_i = 0$ and adding the terms corresponding identical $b_i$'s neither the conditions, nor the conclusion change. Due to Theorem 3.10, if there is a nonconstant solution, then $\sum_{i=1}^{n} a_i \phi(b_i) = 0$, where $\phi \colon \mathbb{Q}(b_1, \ldots, b_n) \to \mathbb{C}$ is an injective homomorphism.

Assuming (i) and taking into consideration that the identity is the only isomorphism over $\mathbb{Q}$, we find $\sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n} a_i \phi(b_i) = 0$, contradicting the assumption.

Assuming (ii), again we use the fact that $\phi$ fixes the elements of $\mathbb{Q}$. We obtain

$$\phi\left(\sum_{i=1}^{n} a_i b_i\right) = \sum_{i=1}^{n} a_i \phi(b_i) = 0$$

and $\sum_{i=1}^{n} a_i b_i = 0$, a contradiction. □

Now we show that Theorem 2.32 is a consequence of Theorem 3.10.

**Theorem 3.14.** (i) *Suppose that the parameters* $b_1, \ldots, b_s$ *are algebraic numbers and* $b_{s+1}, \ldots, b_n$ *are algebraically independent over* $\mathbb{Q}$, *where* $0 \leq s < n$. *If the parameters* $a_1, \ldots, a_n$ *are algebraic numbers, then*

$$\sum_{i=1}^{n} a_i f(b_i x) = 0 \tag{19}$$

*has no nonzero additive solution.*

(ii) *Suppose that the parameters* $a_1, \ldots, a_s$ *are algebraic numbers and* $a_{s+1}, \ldots, a_n$ *are algebraically independent over* $\mathbb{Q}$, *where* $0 \leq s < n$. *If the parameters* $b_1, \ldots, b_n$ *are algebraic numbers, then* (19) *has no nonzero additive solution.*

**Proof.** (i) If $f$ is a nonzero additive solution of (19), then by Theorem 3.7 there exists an automorphism $\phi$ of $\mathbb{C}$ which is also a solution of (19). Thus, $\sum_{i=1}^{n} a_i \phi(b_i) = 0$. Let us assume that every $a_i$ is algebraic. Since $b_1, \ldots, b_s$ are algebraic numbers, the algebraic independence of $b_{s+1}, \ldots, b_n$ $(s < n)$, implies that $b_n$ is transcendental over the algebraic closure of $\mathbb{Q}(b_1, \ldots, b_{n-1})$. The exact statement and the proof can be found in [16, Lemma 3.9.1, p. 100]. Since $\phi$ is an automorphism, so is $\phi^{-1}$, and we can reformulate the previous

equation as $\sum_{i=1}^{n} \phi^{-1}(a_i)b_i = 0$. This is a linear combination of the $b_i$'s with algebraic coefficients, since $\phi^{-1}(a)$ is algebraic, if $a$ is algebraic. This is a contradiction.
(ii) The proof is similar. The only difference is that we use $\phi$ instead of $\phi^{-1}$, otherwise the argument is the same. □

If we put $a_1 = b_1 = 1$, we obtain the 'only if' part of Theorem 2.32. We will generalize Theorem 3.14 in Theorem 5.14.

## 3.3 Existence of solutions of higher degree

We start with some important ideas which we use to investigate the cases when the solutions are generalized polynomials of degree $k > 1$. In this section our main result is Theorem 3.18.

**Lemma 3.15.** *If $\phi_1, \dots, \phi_k$ are distinct injective homomorphisms of the field $K$ into $\mathbb{C}$, then there exists an element $x \in K$ such that $\phi_i(x) \neq \phi_j(x)$ for every $1 \leq i < j \leq k$.*

**Proof.** We use induction on $k$. For $k = 1$ and $k = 2$ the statement is clear. Suppose $k > 2$ and the statement is true for $k - 1$. Let $\phi_1, \dots, \phi_k$ be distinct injective homomorphisms. By the induction hypothesis, there exists an $x \in K$ such that $\phi_1(x), \phi_2(x), \dots, \phi_{k-1}(x)$ are distinct. If they are different from $\phi_k(x)$, then we are done. Suppose for example that $\phi_1(x) = \phi_k(x)$. Since $\phi_1 \neq \phi_k$, there is an $x'$ such that $\phi_1(x') \neq \phi_k(x')$. For every $i < k$ the number of integers $m_i$ satisfying $\phi_i(x + m_i x') = \phi_1(x + m_i x')$ is finite. Thus there remains a suitable element of the form $x + mx'$. □

**Lemma 3.16.** *Let $\phi_1, \dots, \phi_m$ be distinct injective homomorphisms of the field $K$ into $\mathbb{C}$, and let $k$ be a positive integer. Then there exists an element $h \in K$ such that*

$$\prod_{j \in J} \phi_j(h) \neq \prod_{j' \in J'} \phi_{j'}(h)$$

*whenever $J$ and $J'$ are distinct multisets of the elements $1, \dots, m$ containing each of $1, \dots, m$ at most $k$ times.*

**Proof.** Let $x$ be as in Lemma 3.15. For every multiset $J$ let

$$P_J(r) = \prod_{j \in J} \phi_j(r - x) = \prod_{j \in J} (r - \phi_j(x))$$

for every $r \in \mathbb{Q}$. Then $P_J$ is a polynomial of the variable $r \in \mathbb{Q}$. If the multisets $J, J'$ are distinct, then the polynomials $P_J, P_{J'}$ are also distinct, because, by the choice of $x$, the numbers $\phi_j(x)$ $(j = 1, \dots, m)$ are distinct, and thus the set of roots of $P_J$ with multiplicities is different from that of $P_{J'}$.

If the multisets $J, J'$ are distinct then $P_J(r) \neq P_{J'}(r)$ for all but a finite number of $r \in \mathbb{Q}$. Therefore, we can choose an $r \in \mathbb{Q}$ such that all the values $P_J(r)$ are distinct as $J$ runs through the possible multisets. Then $h = r - x$ satisfies the requirements. □

**Lemma 3.17.** *For every field automorphisms $\phi_1, \ldots, \phi_k$ of $\mathbb{C}$, the product $\phi_1 \cdot \ldots \cdot \phi_k$ is a solution of*

$$\sum_{i=1}^{n} a_i f(b_i x + c_i y) = 0 \qquad (x, y \in \mathbb{C}) \tag{20}$$

*if and only if*

$$\sum_{i=1}^{n} a_i \prod_{j \in J} \phi_j(b_i) \prod_{j' \notin J} \phi_{j'}(c_i) = 0 \tag{21}$$

*for every $J \subseteq \{1, \ldots, k\}$.*

**Proof.** Let

$$H_J = \sum_{i=1}^{n} a_i \prod_{j \in J} \phi_j(b_i) \prod_{j' \notin J} \phi_{j'}(c_i) \qquad (J \subset I).$$

($\Longrightarrow$) We denote $I = \{1, \ldots k\}$. If $\phi_1 \cdot \ldots \cdot \phi_k$ is a solution then, applying (20) with $x = 1$ and $h = y \in \mathbb{C}$ we obtain

$$\begin{aligned} 0 &= \sum_{i=1}^{n} a_i \prod_{j=1}^{k} (\phi_j(b_i) + \phi_j(c_i)\phi_j(y)) = \\ &= \sum_{i=1}^{n} a_i \cdot \sum_{J \subset I} \prod_{j \in J} \phi_j(b_i) \prod_{j' \notin J} \phi_{j'}(c_i)\phi_{j'}(y) = \\ &= \sum_{J \subset I} \left( \sum_{i=1}^{n} a_i \prod_{j \in J} \phi_j(b_i) \prod_{j' \notin J} \phi_{j'}(c_i) \right) \prod_{j' \notin J} \phi_{j'}(y) = \\ &= \sum_{J \subset I} H_J \prod_{j' \notin J} \phi_{j'}(y). \end{aligned} \tag{22}$$

We may assume that the automorphisms $\phi_1, \ldots, \phi_m$ are distinct, and each of $\phi_{m+1}, \ldots, \phi_k$ equal one of $\phi_1, \ldots, \phi_m$. If $\overline{J}$ is a multiset containing the element $i$ $q_i$ times $(i = 1, \ldots, m)$ then we put $H_{\overline{J}} = H_J$, where $J \subset I$ is such that, for every $i = 1, \ldots, m$, the cardinality of the set $\{j : \phi_j = \phi_i\}$ equals $q_i$. Clearly, (22) can be written in the form

$$\sum_{\overline{J}} n_{\overline{J}} H_{\overline{J}} \prod_{j' \notin \overline{J}} \phi_{j'}(y) = 0, \tag{23}$$

where $\overline{J}$ runs through the distinct multisets of size at most $k$ consisting of elements $1 \ldots m$, and the coefficient $n_{\overline{J}}$ are suitable positive integers. By Lemma 3.16, we can choose $h$ such that the values of the products of these injective homomorphisms at $h$ are distinct.

Then, applying (23) with $y = h, h^2, \dots, h^N$, we get the following system with the notation $\prod_{\overline{J}} \phi_j(h) = M_J$:

$$\sum_{\overline{J}} n_{\overline{J}} H_{\overline{J}} \cdot M_{\overline{J}} = 0$$

$$\sum_{\overline{J}} n_{\overline{J}} H_{\overline{J}} \cdot (M_{\overline{J}})^2 = 0$$

$$\vdots$$

$$\sum_{\overline{J}} n_{\overline{J}} H_{\overline{J}} \cdot (M_{\overline{J}})^N = 0.$$

This is a Vandermonde system which has no nontrivial solution if $N$ is at least the number of multisets $\overline{J}$ with the given properties. This completes the proof of (21).

($\Longleftarrow$) Since $\phi_j$ is an automorphism, $\phi_j(b_i x + c_i h) = \phi_j(b_i)\phi_j(x) + \phi_j(c_i)\phi_j(h)$. Thus,

$$\sum_{i=1}^{n} a_i \prod_{j=1}^{k} \phi_j(b_i x + c_i h) = \sum_{i=1}^{n} a_i \prod_{j=1}^{k} (\phi_j(b_i)\phi_j(x) + \phi_j(c_i)\phi_j(h)) =$$

$$= \sum_{i=1}^{n} a_i \sum_{J \subseteq \{1,\dots,k\}} \prod_{j \in J} (\phi_j(b_i)\phi_j(x)) \prod_{j' \notin J} (\phi_{j'}(c_i)\phi_{j'}(h)) =$$

$$= \sum_{J \subseteq \{1,\dots,k\}} \left( \sum_{i=1}^{n} a_i \cdot \prod_{j \in J} \phi_j(b_i) \prod_{j' \notin J} \phi_{j'}(c_i) \right) \prod_{j \in J} \phi_j(x) \prod_{j' \notin J} \phi_{j'}(h) = 0,$$

because every term is 0. Thus the product of $\phi_1 \cdot \ldots \cdot \phi_k$ is a solution of (20). □

**Theorem 3.18.** *For every positive integer $k$ the following are equivalent.*

(i) *There exists a generalized polynomial of degree $k$ which is a solution of* (20)*.*

(ii) *There exist field automorphisms $\phi_1, \dots, \phi_k$ of $\mathbb{C}$ such that $\phi_1 \cdot \ldots \cdot \phi_k$ is a solution of* (20)*.*

(iii) *There exist field automorphisms $\phi_1, \dots, \phi_k$ of $\mathbb{C}$ such that*

$$\sum_{i=1}^{n} a_i \prod_{j \in J} \phi_j(b_i) \prod_{j' \notin J} \phi_{j'}(c_i) = 0$$

*for every $J \subseteq \{1, \dots, k\}$.*

**Proof.** (ii)$\Longrightarrow$(i) is trivial with $F_k(x_1, \dots, x_k) = \phi_1(x_1) \cdot \ldots \cdot \phi_k(x_k)$ and $f_k(x) = \operatorname{diag} F_k$. (i)$\Longrightarrow$(ii): Suppose that there is a generalized polynomial solution of (20) of degree $k$. By Lemma 2.30, there is a solution which is a nonzero (generalized) monomial of degree $k$.

We recall that $S_k$ denotes the set of solutions of (20) defined on $K$ which are monomials of degree $k$. Thus $S_k \neq \{0\}$. Let $f_k(x) = F_k(x, \dots, x)$ be a solution, where $F_k$ is nonzero, symmetric and $k$-additive. Let $d_1, \dots, d_k \in \mathbb{C}$ be such that $F_k(d_1, \dots, d_k) \neq 0$. We put

$$K = \mathbb{Q}(b_1, \dots, b_n, c_1, \dots, c_n, d_1, \dots, d_k).$$

we recall that $(K^*)^k$ is an Abelian group under multiplication by coordinates. Since $(K^*)^k$ is countable, spectral analysis holds on it according to Theorem 2.18.

Since $S_k \neq \{0\}$, it follows from Lemma 3.3 that $M_k^* \neq \{0\}$ (for definitions see subsection 3.1).

By Lemma 3.2, $M_k^*$ is a variety on $(K^*)^k$.

Since spectral analysis holds on $M_k^*$, there is an element $m(x_1, x_2, \dots, x_k) \in M_k^*$ which is multiplicative in each coordinate. That is,

$$m(x_1y_1, x_2y_2, \dots, x_ky_k) = m(x_1, x_2, \dots, x_k) \cdot m(y_1, y_2, \dots, y_k) \tag{24}$$

for every $x_i, y_i \in K^*$. By Lemma 3.6,

$$m(x_1, x_2, \dots, x_k) = \phi_1(x_1) \cdot \phi_2(x_2) \cdot \dots \cdot \phi_k(x_k)$$

for every $x_1, \dots, x_k, \in K^*$, where $\phi_1, \dots, \phi_k$ are field automorphism of $\mathbb{C}$.
(iii)$\Longleftrightarrow$(ii): It is clear by using Lemma 3.17. □

**Corollary 3.19.** *Suppose that equation* (20) *satisfies condition* (5). *Then* (20) *has a nonconstant solution if and only if there is a* $k \in \mathbb{N}$ *and there are automorphisms* $\phi_1, \dots, \phi_k$ *such that* $\phi_1 \cdot \dots \cdot \phi_k$ *is a solution.*

**Proof.** If $f$ is a nonconstant solution then, by Theorem 2.24, it must be a generalized polynomial of degree $k > 0$. Then we can apply Lemma 2.30 and Theorem 3.18. □

**Remark 3.20.** Condition (5) is necessary in Theorem 2.24 and Corollary 3.19. We showed in Proposition 2.28 that the functional equation $f(x) - f(2x) = 0$ has a nonconstant solution although every generalized polynomial solution is zero. Thus, there are no automorphisms $\phi_1, \dots, \phi_k$ such that $\phi_1 \cdot \dots \cdot \phi_k$ is a solution, because, clearly, the product $\phi_1 \cdot \dots \cdot \phi_k$ is a generalized polynomial.

As we noted after Theorem 2.27, condition (5) is automatically satisfied by the functional equations of the form

$$\sum_{i=1}^{n} a_i f(b_i x + y) = 0 \tag{25}$$

and

$$\sum_{i=1}^{n} a_i f(b_i x + (1 - b_i)y) = 0. \tag{26}$$

Therefore, Corollary 3.19 is true for these types of equations. For these equations, we supplement Theorem 3.18 as follows.

**Theorem 3.21.** *If $\phi_1 \cdot \ldots \cdot \phi_k$ is a solution of the functional equation* (25) *or* (26)*, where $\phi_i$ is an automorphism for every $i = 1, \ldots, k$, then every subproduct of $\phi_1 \cdot \ldots \cdot \phi_k$ is also a solution of* (25) *or* (26)*.*

**Proof.** The equations (25) and (26) can be transformed into each other by an inner linear substitution, and thus they have the same solutions. Therefore, it is enough to consider (25).

By Lemma 3.17, $\phi_1 \cdot \ldots \cdot \phi_k$ is a solution of (25) if and only if

$$\sum_{i=1}^{n} a_i \phi_{j_1}(b_i) \ldots \phi_{j_s}(b_i) = 0 \tag{27}$$

for every choice of the integers $0 \le j_1 < j_2 < \ldots < j_s \le k$ $(0 \le s \le k)$. Therefore, the conditions are automatically satisfied for any subproduct, thus the statement is clear. □

Theorem 3.18 motivates the question whether or not the existence of a solution of degree $k$ implies the existence of solutions of the form $\phi^k$, where $\phi$ is an automorphism of $\mathbb{C}$. The next proposition shows that the answer is negative.

**Proposition 3.22.** *There is a linear functional equation which has a solution of degree two, but has no solution of the form $\phi^2$ where $\phi$ is an isomorphism.*

**Proof.** Put $K = \mathbb{Q}(i)$. This field has only 2 isomorphism, $\phi_1(z) = z$ and $\phi_2(z) = \overline{z}$ . Using Theorem 3.18. it is enough to guarantee that:

1. $\sum_{i=1}^{n} a_i = 0$, otherwise there is no non-trivial solution.

2. $\sum_{i=1}^{n} a_i b_i = 0$ and $\sum_{i=1}^{n} a_i \overline{b_i} = 0$ which imply that $\phi_1$ and $\phi_2$ are solutions of the functional equation.

3. $\sum_{i=1}^{n} a_i b_i^2 \neq 0$, $\sum_{i=1}^{n} a_i \overline{b_i^2} \neq 0$ which means that neither $\phi_1^2$ nor $\phi_2^2$ is a solution.

4. $\sum_{i=1}^{n} a_i |b_i|^2 = 0$ which implies that $\phi_1\phi_2$ is a solution.

It can be easily shown that

$$f(z+1) + f(z-1) - f(z+i) - f(z-i) = 0$$

is such an equation. □

As we promised in the last chapter, now we show that the translation invariance of the space of solutions does not imply that the functional equation can be reduced to the form (25).

**Proposition 3.23.** *There is a functional equation of the type* (20) *which only has constant solutions, and the points* $(b_i, c_i) \in \mathbb{C}^2$ *are not collinear.*

**Proof.** It is easy to find, for every $n \geq 3$, integers $a_i, b_i$ and $c_i$ $(i = 1, \ldots, n)$ with the following properties:

1. $\sum_{i=1}^{n} a_i = 0$,
2. $\sum_{i=1}^{n} a_i b_i^r \neq 0$ for any $r = 1, 2, \ldots$,
3. $a_i, b_i, c_i$ satisfy condition (5), and
4. the points $(b_i, c_i)$ are not collinear.

(For example, $f(x) + f(x + y) - 2 \cdot f(2y) = 0$ is such an equation.) Since the identity is the only injective homomorphism of $\mathbb{Q}$, it follows from Theorem 3.18 that if there is a nonconstant solution, then there is a solution which is the $m^{\text{th}}$ power of the identity on $\mathbb{Q}$ for a suitable $m \in \mathbb{N}$. Substituting $x = 1$, $y = 0$ we get $\sum_{i=1}^{n} a_i b_i^m$ which is nonzero. Thus any solution of this equation must be constant. $\square$

# 4 Linear functional equations with algebraic parameters

## 4.1 The space of additive solutions in the algebraic case

**Definition 4.1.** Let $K$ denote a subfield of $\mathbb{C}$. We say that a function $d : K \to K$ is a *derivation* if it has the following two properties:

$$d(x+y) = d(x) + d(y) \quad \text{and} \quad d(xy) = d(x)y + xd(y) \tag{28}$$

for every $x, y \in K$. The function $d : K \to \mathbb{C}$ is a *derivation in the wide sense*, if it satisfies (28) for every $x, y \in K$.

**Lemma 4.2.** *Let $K$ be a subfield of $\mathbb{C}$. If $d$ is derivation (in the wide sense) on $K$, then it can be extended to $\mathbb{C}$ as a derivation (in the wide sense).*

The proof can be found in [16, Theorem 14.2.2].

Our main result in this subsection is the following theorem.

**Theorem 4.3.** *Let $b_1, \dots, b_n, c_1, \dots, c_n$ be algebraic numbers, and put $K = \mathbb{Q}(b_1, \dots, b_n, c_1, \dots, c_n)$. Then every additive solution of*

$$\sum_{i=1}^{n} a_i f(b_i x + c_i y) = 0 \qquad (x, y \in K) \tag{29}$$

*is of the form*

$$d_1\phi_1 + \cdots + d_k\phi_k,$$

*where $d_1, \dots, d_k$ are complex numbers and $\phi_1, \dots, \phi_k : K \to \mathbb{C}$ are injective homomorphisms satisfying*

$$\sum_{i=1}^{n} a_i \phi_j(b_i) = 0 \text{ and } \sum_{i=1}^{n} a_i \phi_j(c_i) = 0 \tag{30}$$

*for every $j \in \{1, \dots, k\}$.*

**Proof.** If $f$ is an additive solution of (29) on $K$, then

$$0 = \sum_{i=1}^{n} a_i f(b_i x + c_i y) = \sum_{i=1}^{n} a_i f(b_i x) + \sum_{i=1}^{n} a_i f(c_i y)$$

for every $x, y \in K$. By substituting $x = 0$ and $y = 0$ we get $\sum_{i=1}^{n} a_i f(c_i y) = 0$ and $\sum_{i=1}^{n} a_i f(b_i x) = 0$, respectively. We recall that $S_1$ denotes the set of additive solutions of (29) on $K$. By Lemma 3.2, the set $S_1^* = \{f|_{K^*} : f \in S_1\}$ is a variety on the Abelian group $K^*$.

Since the $b_i$'s are algebraic numbers, the field $K$ is a finite algebraic extension of the field $\mathbb{Q}$. If $\beta_1, \dots, \beta_N$ is a basis of $K$ as a linear space over $\mathbb{Q}$ and if $f : K \to \mathbb{C}$ is additive, then

$$f(r_1\beta_1 + \ldots + r_N\beta_N) = r_1 f(\beta_1) + \ldots + r_N f(\beta_N)$$

for every $r_1, \dots, r_N \in \mathbb{Q}$. Therefore, the values of $f$ at any point of $K$ are determined by the values $f(\beta_1), \dots, f(\beta_N)$. Since the set of all functions mapping $\{\beta_1, \dots, \beta_N\}$ into $\mathbb{C}$ has dimension $N$ it follows that the set $\mathcal{A}$ of all additive functions defined on $K$ forms a finite dimensional linear space over $\mathbb{C}$. This implies that $S_1$, as a linear subspace of $\mathcal{A}$ is also finite dimensional. Consequently, $S_1^*$ is a finite dimensional translation invariant linear space of functions defined on $K^*$. By Theorem 2.14, it follows that every element of $S_1^*$ is a polynomial-exponential function.

In other words, every element $f \in S_1^*$ can be written as a finite sum $f = \sum_{j=1}^{M} p_j \cdot m_j$, where $p_1, \dots, p_M$ are nonzero polynomials and $m_1, \dots, m_M$ are distinct exponentials on $K^*$. By Lemma 2.15, $m_1, \dots, m_M \in S_1^*$. Therefore, by Lemma 3.5, $m_i$ can be extended to $\mathbb{C}$ as an automorphism. We shall denote the extension also by $m_i$. Clearly, the restriction of $m_i$ to $K$ is an injective homomorphism.

We prove that each $p_j$ is constant. Suppose that $p_1$ is not constant. Then, by Lemma 2.15, there is a nonzero additive function $A$ on $K^*$ such that $A \cdot m_1 \in S_1^*$.

We put $d = m_1^{-1} \circ (A \cdot m_1)$ and $d(0) = 0$. We show that $d$ is a derivation in the wide sense. Since $m_1$ is an automorphism, it is an additive function, and then $m_1^{-1}$ is also additive. Since $A \cdot m_1 \in S_1^*$, thus $A \cdot m_1$ also additive, and the composition $d$ is also additive; that is, $d(x + y) = d(x) + d(y)$. Since $A$ is additive on $K^*$ with respect to the multiplication, we have $A(xy) = A(x) + A(y)$ for every $x, y \in K^*$. Therefore,

$$\begin{aligned} d(xy) &= (m_1^{-1} \circ (A \cdot m_1))(xy) = m_1^{-1}(A(xy) \cdot m_1(xy)) = \\ &= m_1^{-1}\left((A(x) + A(y)) \cdot m_1(x) m_1(y)\right) = \\ &= m_1^{-1}(A(x) \cdot m_1(x)) \cdot y + m_1^{-1}(A(y) \cdot m_1(y)) \cdot x = \\ &= d(x)y + d(y)x \end{aligned} \tag{31}$$

for every $x, y \in K^*$. Clearly, $d(xy) = d(x)y + d(y)x$ holds in the cases $x = 0$ or $y = 0$ as well. Therefore, $d$ is a derivation in the wide sense.

On the other hand, it is well known [16, Lemma 14.1.3] (and it is easy to check) that on algebraic extensions of $\mathbb{Q}$ the only derivation in the wide sense is the identically zero function. However, the function $A$ is not identically zero and $m_1$ is an automorphism, thus $d$ cannot be identically zero. This contradiction shows that $p_1$ must be constant, which completes the proof. □

## 4.2 The space of solutions of higher degree

Now we are interested in the solutions of (29) of arbitrary degree. The following theorem generalizes Theorem 4.3.

**Theorem 4.4.** *Let $b_i, c_i$ $(i = 1, \dots, n)$ be algebraic numbers, and let $f$ be a solution of (29) defined on $K = \mathbb{Q}(b_1, \dots, b_n, c_1, \dots, c_n)$. If $f$ is a generalized polynomial of degree at most $k$, then $f$ is the linear combination of products of at most $k$ injective homomorphisms of $K$ which products are also solutions of* (29).

**Proof.** By Lemma 2.30, $f$ is a sum of monomials of degree at most $k$, each is a solution. Therefore, it is enough to prove that if $f_k(x) = F_k(x, \dots, x)$ is a solution, where $F_k$ is nonzero, symmetric and $k$-additive, then $F_k$ is the linear combination of functions of the form $\phi_1 \cdots \phi_k$, where $\phi_1, \dots, \phi_k : K \to \mathbb{C}$ are injective homomorphisms such that the terms of the linear combination are solutions of (29) as well.

Let $M_k$ and $M_k^*$ be as in Lemmas 3.2 and 3.3.

By Lemma 3.2, $M_k^* = \{F|_{(K^*)^k} : F \in M_k\}$ is a variety. Since $f_k$ is a solution, by Lemma 3.3, $F_k \in M_k$. Therefore, $M_k^* \neq \{0\}$.

The parameters $b_i$'s and $c_i$'s are algebraic numbers, thus the field $K$ is a finite algebraic extension of the field $\mathbb{Q}$. Let $\beta_1, \dots, \beta_N$ be a basis of $K$ as a linear space over $\mathbb{Q}$. If $E_k : (K^*)^k \to \mathbb{C}$ is $k$-additive, then

$$\begin{aligned} E_k(x, \dots, x) &= E_k(r_1\beta_1 + \ldots + r_N\beta_N, \dots, r_1\beta_1 + \ldots + r_N\beta_N) = \\ &= (r_1)^k E_k(\beta_1, \dots, \beta_1) + \ldots + (r_N)^k E_k(\beta_N, \dots, \beta_N) = \\ &= \sum_{(i_1, \dots, i_k) \in I^k} \left( \prod_{s=1}^{k} r_{i_s} \right) E_k(\beta_{i_1}, \dots, \beta_{i_k}) \end{aligned}$$

for every $r_1, \dots, r_N \in \mathbb{Q}$ and $I = \{1, \dots, N\}$.

Therefore, the values of $E_k$ at any point of $(K^*)^k$ are determined by the values $E_k(\beta_1, \dots, \beta_1), \dots, E_k(\beta_N, \dots, \beta_N)$. Since the set of all functions mapping $\{(\beta_1, \dots, \beta_1), \dots, (\beta_N, \dots, \beta_N)\}$ into $\mathbb{C}$ has dimension $N^k$, it follows that the set $\mathcal{E}_k$ of all $k$-additive functions defined on $(K^*)^k$ forms a finite dimensional vector space over $\mathbb{C}$. This implies that $V^*$, as a linear subspace of $\mathcal{E}_k$ is also finite dimensional.

Consequently, $M_k^*$ is a finite dimensional translation invariant, closed linear space of functions defined on $(K^*)^k$. By Theorem 2.14, it follows that every element of $M_k^*$ is a polynomial-exponential function.

In particular, $F_k = \sum_{j=1}^{L} P_j \cdot N_j$, where $P_1, \dots, P_L$ are nonzero polynomials and $N_1, \dots, N_L$ are distinct exponentials on $(K^*)^k$ with respect to the multiplication.

We shall prove that every exponential element of $M_k^*$ is of the form $\phi_1(x_1)\dots\phi_k(x_k)|_{(K^*)^k}$ where $\phi_j$ is an injective homomorphism for every $j\in\{1,\dots,k\}$. We shall also prove that every exponential monomial element of $M_k^*$ is a constant multiple of an exponential. This will imply

$$F_k(x_1,\dots,x_k)=\sum_{j=1}^{L}c_j\cdot\phi_1(x_1)\dots\phi_k(x_k)$$

for every $x_1,\dots,x_k\in K^*$. Putting $x_1=\dots=x_k=x$ we obtain

$$f_k(x)=F_k(x,\dots,x)=\sum_{j=1}^{L}c_j\cdot\phi_1(x)\dots\phi_k(x) \tag{32}$$

for every $x\in K^*$. Since (32) is also true for $x=0$, this will complete the proof.

Let $N(x_1,x_2,\dots,x_k)$ be an exponential element of $M_k^*$. Then $N$ is multiplicative in each coordinate, thus by Lemma 3.6,

$$\begin{aligned}N(x_1,x_2,\dots,x_k)&=N(x_1,1,\dots,1)N(1,x_2,\dots,1)\cdots N(1,1,\dots,x_k)=\\&=m_1(x_1)m_2(x_2)\cdots m_k(x_k)\end{aligned}$$

for every $x_1,\dots,x_k\in K^*$. Using Lemma 3.5, $m_j$ can be extended as an automorphism of $\mathbb{C}$.

Now we prove that each $P_j$ is constant.

Suppose, e.g. that $P_1$ is not constant. Then, by Lemma 2.15, there is a nonzero additive function $A$ on $(K^*)^k$ such that

$$d(x_1,\dots,x_k)=A(x_1,\dots,x_k)\cdot m_1(x_1)\cdots m_k(x_k)\in V^*.$$

The additivity of $A$ means that

$$A(x_1\cdot y_1,x_2\cdot y_2,\dots,x_k\cdot y_k)=A(x_1,x_2,\dots,x_k)+A(y_1,y_2,\dots,y_k) \tag{33}$$

for every $x_1,\dots,x_n,y_1,\dots,y_n\in K^*$. We prove that in this case $d\equiv 0$ which is a contradiction.

Then $d(x_1,1,\dots,1)=A(x_1,1,\dots,1)m_1(x_1)$ is a polynomial exponential of the case of Theorem 4.3, therefore $d(x_1,1,\dots,1)=0$ and since $m_1(x_1)\neq 0$, $A(x_1,1,\dots,1)=0$. This is true for every coordinate. Using (33), we obtain that

$$A(x_1,x_2,\dots,x_k)=A(x_1,1,\dots,1)+\dots+A(1,1,\dots,x_k)=0$$

for every $x_1,x_2,\dots,x_k\in K^*$, therefore $d$ must be zero. □

**Remark 4.5.** In Lemma 3.17 we showed that if $\phi_1, \ldots, \phi_k : K \to \mathbb{C}$ are injective homomorphisms then the product $\phi_1 \cdots \phi_k$ is a solution of (29) if and only if

$$\sum_{i=1}^{n} a_i \prod_{j \in J} \phi_j(b_i) \prod_{j' \notin J} \phi_{j'}(c_i) = 0 \tag{34}$$

for every $J \subseteq \{1, \ldots, k\}$.

In the special case when the functional equation can be written of the form

$$\sum_{i=1}^{n} a_i f(b_i x + y) = 0 \tag{35}$$

the situation is slightly simpler. By Theorem 3.21, if we take the injective homomorphism solutions of (35), then forming the products at most $n-2$ of them gives a basis of the space of the solutions. Furthermore, we showed in Theorem 3.21 that the fact that $f$ is a solution of (35) and it is a product of injective homomorphisms implies that every subproduct of $f$ is also solution of (35).

## 4.3 Trivial functional equations

In this section we present another class of equations such that the set of its additive solutions is spanned by those solutions which are injective homomorphisms.

We say that the equation (29) is trivial if every additive function is a solution of the functional equation. Examples of trivial equations are

$$3f(x+\sqrt{2}y) - f(6x + (3\sqrt{2}+5\sqrt{3})y) + 5f(x+\sqrt{3}y) = 0$$

and

$$2f(x+y) - f(x+2y) - 2f(x+\pi y) + f(x+2\pi y) = 0.$$

We prove that the trivial equations also have the property that the set of their additive solutions is spanned by those solutions which are injective homomorphisms. This amounts to show that the set of all additive functions is spanned by injective homomorphisms.

**Theorem 4.6.** *The variety $V_{\mathbb{C}}$ generated by the automorphisms of $\mathbb{C}$ contains every additive function.*

**Proof.** The variety $V_{\mathbb{C}}$ contains the closure of the set of linear combinations of the automorphisms of $\mathbb{C}$. This means that whenever $f : \mathbb{C} \to \mathbb{C}$ is such that, for every $y_1, y_2, \ldots, y_n \in \mathbb{C}$ and $\varepsilon > 0$ there are automorphisms $\phi_1, \phi_2, \ldots, \phi_M$ satisfying

$$|f(y_j) - \sum_{m=1}^{M} c_i \cdot \phi_m(y_j)| < \varepsilon$$

for every $j = \{1, \ldots, n\}$, then $f \in V_{\mathbb{C}}$. We show that every additive function on $\mathbb{C}$ has this property.

It is enough to prove that if $K$ is finitely generated, then the variety $V_K$, generated by the injective homomorphisms of $K$ into $\mathbb{C}$ contains every additive function on $K$. Indeed, for given points $y_1, y_2, \ldots, y_n \in \mathbb{C}$, we may take the subfield $K$ generated by these points. If we can guarantee that the restriction of a given additive function $f$ to $K$ is in the variety $V_K$ for every finitely generated subfield $K$ of $\mathbb{C}$, then, by Proposition 3.4, $V_{\mathbb{C}}$ contains the function $f$.

We will prove the following stronger statement by induction on the number of generators of $K$ : if a function $f \in V_K$ and the points $y_1, y_2, \ldots, y_n \in K$ are given, then there are injective homomorphisms $\phi_1, \phi_2, \ldots, \phi_M$ and there are complex numbers $c_1, \ldots, c_M$ such that

$$f(y_j) = \sum_{m=1}^{M} c_m \cdot \phi_m(y_j)$$

for every $j = \{1, \ldots, n\}$. It is clear that the statement is true for $K = \mathbb{Q}$, since every additive function on $\mathbb{Q}$ is of the form $c \cdot \mathrm{Id}$, where Id denotes the identity function of $\mathbb{Q}$. Let us assume that the statement is true for a finitely generated $K$. We prove the statement for $K(\beta)$ and for $K(t)$, where $\beta$ is an algebraic number, and $t$ is a transcendental number.

*Case 1:* $\beta$ is an algebraic number, and $|K(\beta) : K| = n$. Then

$$K(\beta) = \{a_0 + a_1\beta + \cdots + a_{n-1}\beta^{n-1} : a_i \in K \ (i = 0, \ldots, n-1)\}.$$

Let the additive function $f : K(\beta) \to \mathbb{C}$ and the numbers $y_1, \ldots, y_N \in K(\beta)$ be given.

Since every $y_j$ is a sum of terms of the form $b \cdot \beta^i$ $(b \in K)$, therefore, by additivity, it is enough to represent $f$ at the points $b_{i,j} \cdot \beta^i$ for every $b_{i,j} \in K$ $(j = 1, \ldots, N,\ i = 0, \ldots, n-1)$. For every fix $i$, the function $f(x \cdot \beta^i)$ is an additive function on $K$. Thus, by the induction hypothesis, there are injective homomorphisms $\phi_1, \ldots, \phi_M$ of $K$ into $\mathbb{C}$, and there are complex numbers $c_{i,m}$ such that

$$f(b_{i,j} \cdot \beta^i) = \sum_{m=1}^{M} c_{i,m} \cdot \phi_m(b_{i,j}) \tag{36}$$

for every $i = 0, \ldots, n-1$ and $j = 1, \ldots, N$. We need to represent every term on the right hand side of (39) by some linear combination of values of injective homomorphisms of $K(\beta)$ at the point $b_{i,j} \cdot \beta^i$. Let the roots of the minimal polynomial of $\beta$ be $\beta = \beta_0, \ldots, \beta_{n-1}$. We put

$$\psi_{m,k}(a_0 + a_1\beta + \cdots + a_{n-1}\beta^{n-1}) = \phi_m(a_0) + \phi_m(a_1)\beta_k + \cdots + \phi_m(a_{n-1})\beta_k^{n-1}$$

for every $a_0, \dots, a_{n-1} \in K$ and $k = 0, \dots, n-1$. Then $\psi_{m,k}$ is an injective homomorphism of $K(\beta)$ into $\mathbb{C}$ for every $m$ and $k$. We show that there are numbers $x_{m,k}$ such that

$$f(b_{i,j}\beta^i) = \sum_{m=1}^{M} \sum_{k=0}^{n-1} x_{m,k}\psi_{m,k}(b_{i,j}\beta^i); \tag{37}$$

that is,

$$f(b_{i,j}\beta^i) = \sum_{m=1}^{M} \sum_{k=0}^{n-1} x_{m,k}\phi_m(b_{i,j}) \cdot \beta_k^i.$$

By (36), it is enough to find $x_{m,k}$ satisfying the equations

$$c_{i,m} \cdot \phi_m(b_{i,j}) = \sum_{k=0}^{n-1} x_{m,k}\phi_m(b_{i,j}) \cdot \beta_k^i$$

for every $i = 0, \dots, n-1$, $j = 1, \dots, N$ and $m = 1, \dots, M$. That is, $x_{m,k}$ has to satisfy

$$c_{i,m} = \sum_{k=0}^{n-1} x_{m,k} \cdot \beta_k^i \tag{38}$$

for every $i = 0, \dots, n-1$ and $m = 1, \dots, M$. Since the determinant of the system of equations (38) is nonzero (Vandermonde) for every fixed $m$, these systems are solvable. If $x_{m,k}$ is a solution, (37) shows that $f$ is represented on the set $\{b_{i,j}\beta^i\}$ as a linear combination of injective homomorphisms.

*Case 2:* If $t$ is a transcendental number, then every element of $K(t)$ is a rational function of the variable $t$ with coefficients from $K$.

Let $f : K(t) \to \mathbb{C}$ be an additive function. Let the rational functions $y_1, \dots, y_n \in K(t)$ be given. Let $q(t)$ be a common multiple of the denominators of the rational functions $y_i$.

Since every $y_j$ is a sum of terms of the form $b \cdot t^i/q(t)$ ($b \in K$), therefore, by additivity, it is enough to represent $f$ at the points $b_{i,j} \cdot t^i/q(t)$ for every $b_{i,j} \in K$ ($j = 1, \dots, N,\ i = 0, \dots, n-1$). For every fix $i$, the function $f(x \cdot t^i/q(t))$ is an additive function on $K$. Thus, by the induction hypothesis, there are injective homomorphisms $\phi_1, \dots, \phi_M$ of $K$ into $\mathbb{C}$, and there are complex numbers $c_{i,m}$ such that

$$f(b_{i,j} \cdot t^i/q(t)) = \sum_{m=1}^{M} c_{i,m} \cdot \phi_m(b_{i,j}) \tag{39}$$

for every $i = 0, \dots, n-1$ and $j = 1, \dots, N$. We need to represent every term on the right hand side of (36) by some linear combination of values of injective homomorphisms of $K(t)$ at the points $b_{i,j} \cdot t^i/q(t)$. Let $t_0, t_2, \dots, t_{n-1}$ be algebraically independent elements over $K$, and put

$$\psi_{m,k}(r(t)) = (\phi_m \circ r)(t_k)$$

for every $r \in K(t)$, $m = 1, \dots, M$ and $k = 0, \dots, n-1$. Then $\psi_{m,k}$ is an injective homomorphism of $K(t)$ into $\mathbb{C}$ for every $m$ and $k$. We show that there are numbers $x_{m,k}$ such that

$$f(b_{i,j}t^i/q(t)) = \sum_{m=1}^{M}\sum_{k=0}^{n-1} x_{m,k}\psi_{m,k}(b_{i,j}t^i/q(t)); \tag{40}$$

that is,

$$f(b_{i,j}t^i/q(t)) = \sum_{m=1}^{M}\sum_{k=0}^{n-1} x_{m,k}\phi_m(b_{i,j}) \cdot \frac{t_k^i}{\phi_m(q(t_k))}.$$

By (36), it is enough to find $x_{m,k}$ satisfying the equations

$$c_{i,m} \cdot \phi_m(b_{i,j}) = \sum_{k=0}^{n-1} x_{m,k}\phi_m(b_{i,j}) \cdot \frac{t_k^i}{\phi_m(q(t_k))} \tag{41}$$

for every $i = 0, \dots, n-1$, $j = 1, \dots, N$ and $m = 1, \dots, M$. Let $z_{m,k}$ satisfy

$$c_{i,m} = \sum_{k=0}^{n-1} z_{m,k} \cdot t_k^i \tag{42}$$

for every $i = 0, \dots, n-1$ and $m = 1, \dots, M$. Since the determinant of the system of equations (42) is nonzero (Vandermonde) for every fixed $m$, these systems are solvable. If $z_{m,k}$ is a solution, then put $x_{m,k} = z_{m,k} \cdot \phi_m(q(t_k))$. Then (40) shows that $f$ is represented on the set $\{b_{i,j}t^i/q(t)\}$ as a linear combination of injective homomorphisms.

This completes the proof. □

## 4.4 Further developments

Theorem 4.6 might suggest that if there are infinitely many injective homomorphisms which are solutions of (29), then the variety generated by these injective homomorphisms contains every additive solution as well. Now we show that this is not true in general as Theorem 4.7 shows below.

**Theorem 4.7.** *Let $K \subset \mathbb{C}$ be a field which contains a transcendental number. Then there exists a linear functional equation*

$$\sum_{i=1}^{n} a_i f(b_i x + y) = 0$$

*such that $b_i \in K$ for every $i = 1, \dots, n$, and there exists an additive solution on $K$ which is not contained by the variety generated by the injective homomorphisms which are solutions.*

**Proof.** Let us suppose that $t \in K$ is a transcendental number and $a_1, \ldots, a_n$ are nonzero complex numbers satisfying

$$\sum_{i=1}^{n} a_i = 0,\ \sum_{i=1}^{n} a_i t^i = 0,\ \sum_{i=1}^{n} a_i i t^{i-1} = 0. \tag{43}$$

We will prove that in this case the functional equation

$$\sum_{i=1}^{n} a_i f(t^i x + y) = 0 \tag{44}$$

satisfies the requirements.

Every element of $\mathbb{Q}(t)$ is a rational function with rational coefficients of the variable $t$. The operation $d_0 = \frac{\partial}{\partial t}$ is a well-defined derivation on $\mathbb{Q}(t)$. It follows from Lemma 4.2 that we can extend $d_0$ from $\mathbb{Q}(t)$ to $K$ as a derivation. Then $d_0$ is a solution of (44). Indeed,

$$\begin{aligned}\sum_{i=1}^{n} a_i(d_0(t^i x + y)) &= \sum_{i=1}^{n} a_i d_0(t^i)x + \sum_{i=1}^{n} a_i t^i d_0(x) + \sum_{i=1}^{n} a_i d_0(y) = \\ &= \sum_{i=1}^{n} a_i i t^{i-1} d_0(t) x + \sum_{i=1}^{n} a_i t^i d_0(x) + \sum_{i=1}^{n} a_i d_0(y) = 0\end{aligned} \tag{45}$$

by (43).

Let $V$ denote the variety generated by those injective homomorphisms which are solutions of (44) on $K$.

Suppose $\phi \in V$ is an injective homomorphism. Then, by (43), we have $\sum_{i=1}^{n} a_i \phi(t)^i = 0$, and thus $\phi(t)$ is a root of the polynomial $p(x) = \sum_{i=1}^{n} a_i x^i$. Let $u_1, \ldots, u_k$ be the distinct roots of $p$. Suppose $\phi(t) = u_i$. If $\psi \in V$ is another injective homomorphism with $\psi(t) = u_i$, then restrictions of $\phi$ and $\psi$ to $\mathbb{Q}(t)$ coincide. This easily implies that the family of restrictions $W = \{g|_{\mathbb{Q}(t)} : g \in V\}$ is finite dimensional. It is clear that if $\phi_1, \ldots, \phi_k$ are fixed injective homomorphisms such that $\phi_j(t) = u_j$ $(j = 1, \ldots, k)$, then the restrictions $\phi_j|_{\mathbb{Q}(t)}$ form a basis of $W$ as a linear space.

Suppose $d_0 \in V$. Then there are complex numbers $c_1, \ldots, c_k$ such that

$$\sum_{j=1}^{k} c_j \phi_j(w) = d_0(w) \tag{46}$$

for every $w \in \mathbb{Q}(t)$. Applying (46) with $w = t^i$ and using $d_0(t^i) = i \cdot t^{i-1}$, we find

$$\sum_{j=1}^{k} c_j \phi_j(t^i) = \sum_{j=1}^{k} c_j u_j^i = i \cdot t^{i-1}$$

for every $i = 1, 2, \ldots$. Multiplying by $z^i$ and summing for $i = 1, 2, \ldots$ we obtain

$$\sum_{j=1}^{k} c_j \sum_{i=1}^{\infty} u_j^i z^i = \sum_{i=1}^{\infty} i \cdot t^{i-1} z^i,$$

as an equation between formal power series. However, for $z$ small enough both power series are convergent, and we obtain

$$\sum_{j=1}^{k} c_j \frac{u_j z}{1 - u_j z} = \frac{\partial}{\partial t} \cdot \frac{1}{1 - tz} = -\frac{z}{(1 - tz)^2} \tag{47}$$

for every $|z| < \delta$. The functions was defined with power series and the intersection of domains of convergence contain a set that has a non-isolated point. These implies that the functions are equal in the union of the domains using Uniqueness Principle. This implies that equation (47) is an identity on the whole complex plane, since the last term of equation (47) is a meromorphic function on the whole space with a pole singularity in $z = 1/t$. Therefore,

$$\sum_{j=1}^{k} c_j \frac{(1 - tz)^2 u_j z}{1 - u_j z} = -z$$

holds for every $z \neq 1/t$ and if $z \to \frac{1}{t}$ the left side tends to 0, but the right is not.

This contradiction shows that $d_0 \notin V$, which completes the proof. □

**Example 4.8.** Consider the function equation

$$f(t^2 x + y) - 2tf(tx + y) + t^2 f(x + y) - (t - 1)^2 f(y) = 0, \tag{48}$$

where $t$ is transcendental. There exists only one injective homomorphism solution on $K = \mathbb{Q}(t)$. Indeed, substituting $\phi$ to (48), we obtain $(\phi(t) - t)^2 = 0$ which implies that $\phi$ must be the identity on $K$. On the other hand, $\partial/\partial t$ is a solution on $K$.

# 5 Spectral synthesis in varieties of additive functions

## 5.1 Differential operators

First we recall the definition of a derivation.

For every field $K$, a *derivation (on $K$)* is a map $d : K \to K$ such that $d(x+y) = d(x) + d(y)$ and $d(xy) = d(x) \cdot y + d(y) \cdot x$ for every $x, y \in K$. It is well-known that if $d$ is a derivation (on $K$) and $L$ is a field containing $K$, then $d$ can be extended to $L$ as a derivation (on $L$). (See [16, Theorem 14.2.1])

Suppose that the complex numbers $t_1, \ldots, t_n$ are algebraically independent over $\mathbb{Q}$. The elements of the field $\mathbb{Q}(t_1, \ldots, t_n)$ are the rational functions of $t_1, \ldots, t_n$ with rational coefficients. By a *differential operator on* $\mathbb{Q}(t_1, \ldots, t_n)$ we mean an operator of the form

$$D = \sum c_{i_1,\ldots,i_n} \cdot \frac{\partial^{i_1+\cdots+i_n}}{\partial t_1^{i_1} \cdots \partial t_n^{i_n}}, \tag{49}$$

where $\partial/\partial t_i$ are the usual partial derivatives, the sum is finite, in each term the coefficient is a complex number, and the exponents $i_1, \ldots, i_n$ are nonnegative integers. If $i_1 = \ldots = i_n = 0$, then by $\partial^{i_1+\cdots+i_n}/\partial t_1^{i_1} \cdots \partial t_n^{i_n}$ we mean the identity operator on $\mathbb{Q}(t_1, \ldots, t_n)$. The degree of the differential operator $D$ is the maximum of the numbers $i_1 + \ldots + i_n$ such that $c_{i_1,\ldots,i_n} \neq 0$.

It is obvious that $\partial/\partial t_i$ is a derivation on $\mathbb{Q}(t_1, \ldots, t_n)$ for every $i = 1, \ldots, n$. Therefore, every differential operator on $\mathbb{Q}(t_1, \ldots, t_n)$ is the linear combination with complex coefficients of finitely many maps of the form $d_1 \circ \ldots \circ d_k$, where $d_1, \ldots, d_k$ are derivations on $\mathbb{Q}(t_1, \ldots, t_n)$. This observation motivates the following definition.

**Definition 5.1.** Let $K$ be a subfield of $\mathbb{C}$. We say that the map $D : K \to \mathbb{C}$ is a *differential operator* on $K$, if $D$ is the linear combination, with complex coefficients, of finitely many maps of the form $d_1 \circ \ldots \circ d_k$, where $d_1, \ldots, d_k$ are derivations on $K$. If $k = 0$ then we interpret $d_1 \circ \ldots \circ d_k$ as the identity function on $K$.

Note that if $K \subset L \subset \mathbb{C}$ are fields and $D$ is a differential operator on $K$, then $D$ can be extended to $L$ as a differential operator. This is clear from the fact that every derivation can be extended from $K$ to $L$.

We show that if $K = \mathbb{Q}(t_1, \ldots, t_n)$, then the two definitions of differential operators coincide. Actually, more is true.

**Proposition 5.2.** *Let $K$ be a subfield of $\mathbb{C}$, and suppose that the elements $t_1, \ldots, t_n \in K$ are algebraically independent over $\mathbb{Q}$. If $D$ is a differential operator on $K$ according to Definition 5.1, then the restriction of $D$ to $\mathbb{Q}(t_1, \ldots, t_n)$ is of the form* (49).

**Proof.** Put $\mathbb{Q}(t_1, \dots, t_n) = F$, and let $\mathcal{D}$ denote the set of all functions defined on $F$ that can be represented in the form (49). It is enough to show that if $d_1, \dots, d_k$ are derivations (on $K$), then the restriction of $d_1 \circ \ldots \circ d_k$ to $F$ belongs to $\mathcal{D}$. We prove this by induction on $k$. First we note that if $d$ is a derivation and $d(t_i) = \alpha_i$ $(i = 1, \dots, n)$, then

$$d(x) = \sum_{i=1}^{n} \alpha_i \cdot \frac{\partial x}{\partial t_i} \tag{50}$$

for every $x \in F$. Indeed, (50) can be easily checked first for every $x \in \mathbb{Q}[t_1, \dots, t_n]$ and then for every $x \in F$. Therefore, the statement is true for $k = 1$.

Let $k > 1$, and suppose the statement is true for $k-1$. Let $d_1, \dots, d_k$ be derivations (on $K$). Then, by the induction hypothesis, the map $g = d_2 \circ \ldots \circ d_k$ restricted to $F$ belongs to $\mathcal{D}$. Let $g = \sum_{j=1}^{N} c_j g_j$, where each $g_j$ is of the form $\partial^{i_1+\dots+i_n}/\partial t_1^{i_1} \cdots \partial t_n^{i_n}$. Extend $d_1$ to $\mathbb{C}$ as a derivation. Then

$$d_1 \circ g = d_1 \circ \left( \sum_{j=1}^{N} c_j g_j \right) = \sum_{j=1}^{N} (d_1(c_j) \cdot g_j + c_j \cdot (d_1 \circ g_j)).$$

Now the statement $(d_1 \circ g)|_F \in \mathcal{D}$ follows from (50) when applied to $d = d_1$. $\square$

In the sequel we shall denote by $j$ the identity function defined on $\mathbb{C}$.

**Theorem 5.3.** *Let $K$ be a subfield of $\mathbb{C}$, and let $D$ be a differential operator on $K$. Then $D/j$ is a polynomial on $K^*$.*

**Proof.** It is enough to show that if $d_1, \dots, d_n$ are derivations on $K$, then $(d_1 \circ \ldots \circ d_n)/j$ is a polynomial on $K^*$. We prove by induction on $n$. It is easy to check that if $d$ is a derivation, then $d/j$ is additive on $K^*$. Since every additive function is a polynomial, the statement is true for $n = 1$.

Suppose that $n > 1$, and the statement is true for $n - 1$. Let $d_1, \dots, d_n$ be derivations on $K$. By the induction hypothesis, $(d_2 \circ \ldots \circ d_n)/j = p$ is a polynomial on $K^*$. Extend $p$ to $K$ by putting $p(0) = 0$. We have to show that $(d_1 \circ (p \cdot j))/j$ is a polynomial on $K^*$. Since $d_1$ is a derivation, we have

$$d_1(p(x) \cdot x) = d_1(p(x)) \cdot x + p(x) \cdot d_1(x)$$

for every $x \in K$. Thus $(d_1 \circ (p \cdot j))/j = (d_1 \circ p) + p \cdot (d_1/j)$ on $K^*$. Since $p$ is a polynomial and $d_1/j$ is additive on $K^*$, it follows that $p \cdot (d_1/j)$ is a polynomial on $K^*$. Therefore, it is enough to show that $d_1 \circ p$ is a polynomial on $K^*$.

Extend $d_1$ to $\mathbb{C}$ as a derivation. Let $d_1/j = a$; then $a$ is additive on $\mathbb{C}^*$, and $d_1 = a \cdot j$, where we extended $a$ to $\mathbb{C}$ by putting $a(0) = 0$. (The additivity of $a$ on $\mathbb{C}^*$ means

$a(xy) = a(x) + a(y)$ for every $x, y \in \mathbb{C}^*$.) Now $p$ is a sum of functions of the form $a_1 \cdots a_k$, where each of $a_1, \ldots, a_k$ is either additive on $K^*$ or constant. Since $d_1$ is additive on $\mathbb{C}$, it is enough to show that $d_1 \circ (a_1 \cdots a_k)$ is a polynomial on $K^*$. We have

$$\begin{aligned} d_1 \circ (a_1 \cdots a_k) = (a \cdot j) \circ (a_1 \cdots a_k) = (a \circ (a_1 \cdots a_k)) \cdot a_1 \cdots a_k = \\ = [(a \circ a_1) + \ldots + (a \circ a_k)] \cdot a_1 \cdots a_k \end{aligned} \tag{51}$$

everywhere on $K$. Since $a \circ a_i$ is either constant or additive on $K^*$, it follows that the right hand side of (51) is a polynomial on $K^*$. □

Our next aim is to prove the following result.

**Theorem 5.4.** *Suppose that the transcendence degree of the field $K$ over $\mathbb{Q}$ is finite, and let the map $D : K \to \mathbb{C}$ be additive. Then the following are equivalent.*

(i) *$D$ is a differential operator on $K$.*

(ii) *$D/j$ is a polynomial on $K^*$.*

(iii) *$D/j$ is a generalized polynomial on $K^*$.*

(iv) *$D/j$ is a local polynomial on $K^*$.*

We remark that the torsion free rank of the Abelian group $K^*$ is infinite for any $K$. Indeed, the set of rational primes constitutes an independent family of elements of infinite order in $K^*$. Therefore, for any field $K \subset \mathbb{C}$, the families of polynomials, generalized polynomials, and local polynomials defined on $K^*$ are different.

In the next two lemmas we shall use the following notation. Let $K$ be a field as in Theorem 5.4, and let $T \subset K$ be a maximal set of algebraically independent elements over $\mathbb{Q}$. By assumption, $T$ is finite; let $T = \{t_1, \ldots, t_n\}$. We shall denote by $G$ the subgroup of $K^*$ generated by $t_1, \ldots, t_n$. Then $G$ is a finitely generated subgroup of $K^*$.

**Lemma 5.5.** *Let $f : K \to \mathbb{C}$ be an additive function with respect to addition. Let $H$ be a subgroup of $K^*$ such that $G \subset H \subset K^*$. Suppose that $p = f/j$ is a generalized polynomial on $H$. If $f \equiv 0$ on $G$, then $f \equiv 0$ on $H$.*

**Proof.** We prove by induction on the degree of the generalized polynomial $p$ on $H$. If $\deg p = 0$, then $p$ is constant. Since $f \equiv 0$ on $G$, we have $p \equiv 0$ on $H$, and $f \equiv 0$ on $H$.

Suppose $m = \deg p > 0$, and that the statement is true for degrees less than $m$. Let $g \in G$ be fixed. Then

$$\begin{aligned} \Delta_g p(x) = p(gx) - p(x) = \frac{f(gx)}{gx} - \frac{f(x)}{x} = \\ = \frac{g^{-1} f(gx) - f(x)}{x} = \frac{f_1(x)}{x}, \end{aligned} \tag{52}$$

where $f_1(x) = g^{-1}f(gx) - f(x)$ for every $x \in K$. Then $f_1$ is additive on $K$, and $f_1/j$ is a generalized polynomial on $H$ by (52). Moreover, we have $f_1/j = \Delta_g p$, and thus $\deg((f_1/j)|_H) \le m-1$. Since $f_1 \equiv 0$ on $G$, it follows from the induction hypothesis that $f_1 \equiv 0$ on $H$. Thus $f(gx) = g \cdot f(x)$ for every $g \in G$ and $x \in H$. By the additivity of $f$ we obtain

$$f(cx) = c \cdot f(x) \qquad (c \in \mathbb{Q}[T],\ x \in H). \tag{53}$$

Let $\alpha \in H$ be arbitrary. Then, by $\alpha \in K$, $\alpha$ is algebraic over the field $\mathbb{Q}(T)$. Let $c_0, \ldots, c_k \in \mathbb{Q}[T]$ be such that

$$c_k\alpha^k + \ldots + c_1\alpha + c_0 = 0, \tag{54}$$

$c_k \neq 0$ and $k$ is minimal. Let $f(\alpha^i) = a_i$ ($i \in \mathbb{Z}$). Multiplying (54) by $\alpha^{n-k}$ for every $n \in \mathbb{Z}$ we obtain

$$c_k\alpha^n + \ldots + c_1\alpha^{n-k+1} + c_0\alpha^{n-k} = 0.$$

By (53) and by the additivity of $f$, this implies

$$c_k a_n + \ldots + c_1 a_{n-k+1} + c_0 a_{n-k} = 0$$

for every $n$. Therefore, the sequence $(a_n)$ satisfies a linear recurrence relation. It is well-known that $a_n$ can be uniquely represented in the form $a_n = \sum_{\lambda\in\Lambda} q_\lambda(n)\cdot\lambda^n$, where $\lambda$ runs through $\Lambda$, the set of roots of the characteristic polynomial $\chi(x) = c_k x^k + \ldots + c_0$, and for every root $\lambda \in \Lambda$, $q_\lambda \in \mathbb{C}[x]$ is a polynomial of the degree less than the multiplicity of $\lambda$.

By the minimality of $k$, the polynomial $\chi$ is irreducible over $\mathbb{Q}(T)$. Therefore, every $\lambda$ is a simple root of $\chi$, and thus

$$a_n = \sum_{\lambda\in\Lambda} d_\lambda \cdot \lambda^n \tag{55}$$

for every $n$, where $d_\lambda$ is a constant for every $\lambda \in \Lambda$.

Since $p$ is a generalized polynomial on $H$ and $\{\alpha^n\}$ is a finitely generated subgroup of $H$, it follows that $p$ is a polynomial on $\{\alpha^n\}$ (see Theorem 2.10). Therefore, the map $n \mapsto p(\alpha^n)$ ($n \in \mathbb{Z}$) is a polynomial on $\mathbb{Z}$. Now, we have $a_n = f(\alpha^n) = p(\alpha^n)\cdot\alpha^n$ for every $n$. The uniqueness of the representation (55) implies that $\alpha \in \Lambda$ and the function $n \mapsto p(\alpha^n)$ ($n \in \mathbb{Z}$) is constant. Since $p(1) = f(1) = 0$ by $1 \in G$, it follows that $p(\alpha^n) = 0$ for every $n$. In particular, $p(\alpha) = 0$ and $f(\alpha) = 0$. Since this is true for every $\alpha \in H$, we obtain $f \equiv 0$ on $H$. □

**Lemma 5.6.** *Let $f : K \to \mathbb{C}$ be an additive function with respect to addition such that $p = f/j$ is a local polynomial on $K^*$. If $f \equiv 0$ on $G$, then $f \equiv 0$ on $K$.*

**Proof.** The additivity of $f$ implies $f(0) = 0$. Let $\alpha \in K^*$ be arbitrary, and let $H$ be the multiplicative group generated by $T = \{t_1, \dots, t_n\}$ and $\alpha$. Since $H$ is a finitely generated subgroup of $K^*$, it follows that $p$ is a polynomial on $H$. By the previous lemma we obtain that $f \equiv 0$ on $H$. In particular, $f(\alpha) = 0$. Since this is true for every $\alpha \in K^*$, we obtain $f \equiv 0$ on $K^*$. □

**Proof of Theorem 5.4.** The implication (i)$\Longrightarrow$(ii) was proved in Theorem 5.3 (for every field). The implications (ii)$\Longrightarrow$(iii)$\Longrightarrow$(iv) are obvious.

Now we prove (iv)$\Longrightarrow$(i). Let $p = D/j$, then $p$ is a local polynomial on $K^*$. Since $G$ is a finitely generated subgroup of $K^*$, it follows by Proposition 2.13 that $p$ is a generalized polynomial and also a polynomial on $G$. By Theorem 2.4, $p$ has the $P_{loc}$ property on $G$; that is, the map

$$(k_1, \dots, k_n) \mapsto p\left(t_1^{k_1} \cdots t_n^{k_n}\right)$$

is a polynomial on $\mathbb{Z}^n$.

We shall use the notation $x^{[0]} = 1$ and $x^{[i]} = x(x-1)\cdots(x-i+1)$ for every $i = 1, 2, \dots$ and $x \in \mathbb{Z}$. It is well-known that every polynomial belonging to $\mathbb{C}[x_1, \dots, x_n]$ can be written in the form $\sum c_i \cdot x_1^{[i_1]} \cdots x_n^{[i_n]}$, where $i = (i_1, \dots, i_n)$ runs through a finite set of $n$-tuples of nonnegative integers, and in each term the coefficient $c_i$ is a complex number. Therefore, the map $(k_1, \dots, k_n) \mapsto p\left(t_1^{k_1} \cdots t_n^{k_n}\right)$ has such a representation. Then we have

$$\begin{aligned} D\left(t_1^{k_1} \cdots t_n^{k_n}\right) &= p\left(t_1^{k_1} \cdots t_n^{k_n}\right) \cdot t_1^{k_1} \cdots t_n^{k_n} = \\ &= \sum c_i \cdot k_1^{[i_1]} \cdots k_n^{[i_n]} \cdot t_1^{k_1} \cdots t_n^{k_n} = \\ &= \sum c_i \cdot t_1^{i_1} \cdots t_n^{i_n} \cdot k_1^{[i_1]} \cdots k_n^{[i_n]} \cdot t_1^{k_1 - i_1} \cdots t_n^{k_n - i_n} = \\ &= E\left(t_1^{k_1} \cdots t_n^{k_n}\right) \end{aligned} \tag{56}$$

for every $k_1, \dots, k_n \in \mathbb{Z}$, where $E$ is the differential operator

$$\sum c_i \cdot t_1^{i_1} \cdots t_n^{i_n} \cdot \frac{\partial^{i_1 + \cdots + i_n}}{\partial t_1^{i_1} \cdots \partial t_n^{i_n}}.$$

By extending the derivations $\partial/\partial t_i$ to $K$, we can extend $E$ to $K$ as a differential operator $\overline{E}$. Then $\overline{E}$ is additive on $K$, and $\overline{E}/j$ is a polynomial on $K^*$ by Theorem 5.3. Let $q(0) = 0$, and let $q(x) = p(x) - \overline{E}(x)/x$ for every $x \in K^*$. Then $q \cdot j = D - \overline{E}$ is additive on $K$, and $q$ is a local polynomial on $K^*$. Since $q$ vanishes on $G$ by (56), it follows from Lemma 5.6 that $q \equiv 0$ on $K$. Thus $D = \overline{E}$ on $K$ which completes the proof. □

## 5.2 Proof of spectral synthesis in varieties of additive functions

Let $K$ be a subfield of $\mathbb{C}$, and let $K^*$ denote the Abelian group $\{x \in K : x \neq 0\}$ with respect to multiplication.

In this section we fix a field $K$ such that its transcendence degree over $\mathbb{Q}$ is finite. We recall that $V_1$ denotes the set of additive functions $f : K \to \mathbb{C}$ and $V_1^* = \{f|_{K^*} : f \in V_1\}$. If the function equation is trivial, then Lemma 3.2 shows that $V_1^*$ is a variety on $K^*$.

Our aim is to prove that spectral synthesis holds in every variety on $K^*$ contained by $V_1$ (see Theorem 4.7). Theorem 2.9 shows that as the torsion free rank of $K^*$ is infinite (see the remark after Theorem 5.4), there are varieties on $K^*$ in which spectral synthesis does not hold.

**Theorem 5.7.** *Suppose that the transcendence degree of the field $K$ over $\mathbb{Q}$ is finite. Let $f : K \to \mathbb{C}$ be additive, and let $m$ be an exponential on $K^*$. Let $\phi$ be an extension of $m$ to $\mathbb{C}$ as an automorphism of $\mathbb{C}$. Then the following are equivalent.*

(i) *$f = p \cdot m$ on $K^*$, where $p$ is a local polynomial on $K^*$.*

(ii) *$f = p \cdot m$ on $K^*$, where $p$ is a generalized polynomial on $K^*$.*

(iii) *$f = p \cdot m$ on $K^*$, where $p$ is a polynomial on $K^*$.*

(iv) *There exists a unique differential operator $D$ on $K$ such that $f = \phi \circ D$ on $K$.*

**Proof.** The implications (iii)$\Longrightarrow$(ii)$\Longrightarrow$ (i) are obvious.
(i)$\Longrightarrow$(iv): Let $D = \phi^{-1} \circ f$. Then $D$ is additive on $K$, and

$$D = \phi^{-1} \circ (p \cdot m) = (\phi^{-1} \circ p) \cdot j$$

on $K$. Thus $D/j = \phi^{-1} \circ p$ on $K^*$. Since $p$ is a local polynomial on $K^*$ and $\phi^{-1}$ is an automorphism of $\mathbb{C}$, it follows that $\phi^{-1} \circ p = D/j$ is a local polynomial on $K^*$ as well. Therefore, $D$ is a differential operator on $K$ by Theorem 5.4. Since $f = \phi \circ D$ on $K$, this proves the existence of $D$. The uniqueness is clear: if $\phi \circ D_1 = \phi \circ D_2$ on $K$, then $D_1 = D_2$, since $\phi$ is injective.
(iv)$\Longrightarrow$(iii): Suppose $f = \phi \circ D$ on $K$, where $D$ is a differential operator on $K$. Then $D/j$ is a polynomial on $K^*$, and so is

$$p = \phi \circ (D/j) = (\phi \circ D)/\phi.$$

Thus $f = \phi \circ D = p \cdot \phi$. Since $m = \phi$ on $K$, the proof is complete. □

**Theorem 5.8.** *Suppose that the transcendence degree of the field $K$ over $\mathbb{Q}$ is finite. Then spectral synthesis holds in every variety on $K^*$ consisting of additive functions (with respect to addition).*

**Proof.** Since $K$ is countable, so is the Abelian group $K^*$. Let $V \subset V_1^*$ be a variety on $K^*$ consisting of additive functions. By Theorem 2.22, local spectral synthesis holds on $K^*$, and thus $V$ is spanned by local polynomial-exponential functions. Since, by Theorem 5.7, every local polynomial-exponential function contained by $V$ is a polynomial-exponential function, it follows that $V$ is spanned by polynomial-exponential functions. $\square$

## 5.3 The space of additive solutions

As an application of Theorems 5.7 and 5.8 we describe the additive solutions of the linear functional equation

$$\sum_{i=1}^{n} a_i f(b_i x + c_i y) = 0, \tag{57}$$

where $a_i, b_i, c_i$ are given complex numbers and $f : \mathbb{C} \to \mathbb{C}$ is the unknown function. Let $K = \mathbb{Q}(b_1, \dots, b_n, c_1, \dots, c_n)$. We recall that $S_1$ denotes the set of additive solutions of (57) defined on $K$. We showed that $f : K \to \mathbb{C}$ belongs to $S_1$ if and only if

$$\sum_{i=1}^{n} a_i f(b_i x) = 0, \qquad \sum_{i=1}^{n} a_i f(c_i x) = 0 \tag{58}$$

holds for every $x \in K$. By Lemma 3.2,

$$S_1^* = \{f|_{K^*} : f \in S_1\}$$

is a variety on $K^*$.

The next theorem is our main result concerning the additive solutions of linear functional equations.

**Theorem 5.9.** (i) *For every function $f \in S_1^*$, $f$ is an exponential monomial on $K^*$ if and only if $f = \phi \circ D$ on $K^*$, where $\phi$ is an automorphism of $\mathbb{C}$ and is a solution of* (57)*, and $D$ is a differential operator on $K$.*

(ii) *The variety $S_1^*$ is spanned by the functions $(\phi \circ D)|_{K^*} \in S_1^*$, where $\phi$ and $D$ are as above.*

(iii) *The linear space $S_1$ is spanned by the functions $\phi \circ D$, where $\phi$ and $D$ are as above.*

**Proof.** (i) Suppose that $f : K \to \mathbb{C}$ is an additive solution of (57), and $f$ is an exponential monomial function on $K^*$. Let $f = p \cdot m$, where $p$ is a polynomial, and $m$ is an exponential on $K^*$. Since $S_1^*$ is a variety and $p \cdot m \in S_1^*$, it follows that $m \in S_1^*$ (see Lemma 2.15). This means that defining $m(0) = 0$, the function $m$ is a solution of (57) on $K$. By Lemma

3.5, $m$ can be extended as an automorphism of $\mathbb{C}$. Let $\phi$ denote such an extension. As $m$ is a solution of (58) as well, we have

$$\sum_{i=1}^{n} a_i m(b_i) = 0, \qquad \sum_{i=1}^{n} a_i m(c_i) = 0. \tag{59}$$

Then, by (59), $\phi$ is a solution of (57) on $\mathbb{C}$. The rest of the statement (i) follows from Theorem 5.7.

Statement (ii) is an immediate consequence of Theorem 5.8 and of (i). The statement (iii) is clear from (ii). □

The description of the additive solutions becomes especially simple if the coefficients $a_i$ are algebraic.

**Theorem 5.10.** *Suppose that $a_1, \ldots, a_n$ are algebraic numbers. If $\phi$ is an automorphism of $\mathbb{C}$ and $\phi$ is a solution of* (57)*, then $\phi \circ D \in S_1$ for every differential operator $D$ on $K$. Therefore, $S_1^*$ is spanned by the functions $(\phi \circ D)|_{K^*} \in S_1^*$, where $\phi$ is an automorphism of $\mathbb{C}$ and is a solution of* (57)*, and $D$ is an arbitrary differential operator on $K$.*

**Proof.** Since we are only interested in the additive solutions of (57), it is enough to deal with the additive solutions of the system (58). It is enough to show that $\phi \circ D$ is a solution of (58) on $K$ for any differential operator $D = d_1 \circ \cdots \circ d_k$, where $d_1, \ldots, d_k$ are derivations. We will prove this by induction on $k$.

If $k = 0$; that is, if $D$ is the identity, then $\phi \circ D = \phi$ is a solution by assumption.

Let $k > 0$, and suppose the statement is true for $k - 1$. We have to prove that if $d_1, \ldots, d_k$ are derivations on $K$, then $\phi \circ (d_1 \circ \cdots \circ d_k)$ is a solution on $K$. We have

$$\phi \circ (d_1 \circ \cdots \circ d_k) = d \circ f,$$

where $d = \phi \circ d_1 \circ \phi^{-1}$ and $f = \phi \circ (d_2 \circ \cdots \circ d_k)$. Then $f : K \to \phi(K)$ is a solution of (58) by the induction hypothesis.

Let $K_1 = K(\phi^{-1}(a_1), \ldots, \phi^{-1}(a_n))$, and let $d_1$ be extended to $K_1$ as a derivation. We denote the extended derivation by $\overline{d}_1$. Note that $\overline{d}_1(a) = 0$ for every algebraic element of $K_1$.

Let $\overline{d} = \phi \circ \overline{d}_1 \circ \phi^{-1}$. It is easy to check that $\overline{d}$ is a derivation on $\phi(K_1)$. If $a \in \phi(K_1)$ is algebraic, then so is $\phi^{-1}(a)$, and thus $\overline{d}_1(\phi^{-1}(a)) = 0$. Therefore, $\overline{d}(a) = 0$ for every algebraic element of $\phi(K_1)$. In particular, $\overline{d}(a_i) = 0$ for every $i = 1, \ldots, n$. Since $f$ is a solution of (58) we have, for every $x \in K$,

$$0 = \overline{d}(0) = \overline{d}\left(\sum_{i=1}^{n} a_i \cdot f(b_i x)\right) = \sum_{i=1}^{n} \overline{d}(a_i \cdot f(b_i x))) =$$

$$= \sum_{i=1}^{n} \overline{d}(a_i) \cdot f(b_i x) + \sum_{i=1}^{n} a_i \cdot d(f(b_i x)) = \sum_{i=1}^{n} a_i \cdot (d \circ f)(b_i x).$$

The same argument shows that $\sum_{i=1}^{n} a_i \cdot (d \circ f)(c_i x) = 0$ for every $x \in K$. Thus $\phi \circ d$ is a solution of (58) on $K$ which completes the proof. □

By Theorems 5.9 and 5.10 we have the following corollary.

**Corollary 5.11.** *If the coefficients $a_1, \ldots, a_n$ are algebraic, then the variety of additive solutions of* (57) *defined on $K$ is spanned by the functions $\phi \circ D$, where $\phi$ is an isomorphism solution and $D$ is an arbitrary differential operator.* □

**Theorem 5.12.** *Suppose that $a_1, \ldots, a_n$ are algebraic and that* (57) *has a nonzero additive solution on $K$. Then $S_1$ is of finite dimension over $\mathbb{C}$ if and only if each of the numbers $b_1, \ldots, b_n, c_1, \ldots, c_n$ is algebraic.*

**Proof.** If $b_i$ and $c_i$ are algebraic numbers, then the field

$$K = \mathbb{Q}(b_1, \ldots, b_n, c_1, \ldots, c_n)$$

is a finite dimensional linear space over $\mathbb{Q}$ (see the proof of Theorem 4.3). Consequently, the linear space of additive functions defined on $K$ is also of finite dimension. This implies that the space of additive solutions of (57) is, a fortiori, of finite dimension.

Next suppose that at least one of the numbers $b_1, \ldots,\ b_n, c_1, \ldots, c_n$ is transcendental. We show that if (57) has a nonzero additive solution on $K$, then the set $S_1$ of all nonzero additive solution defined on $K$ has infinite dimension over $\mathbb{Q}$.

By Theorem 3.7 (or, by (iii) of Theorem 5.9), there is an automorphism $\phi$ of $\mathbb{C}$ which is a solution of (57) on $K$. Let $T$ be a maximal subset of $K$ consisting of algebraically independent elements over $\mathbb{Q}$. Since the degree of transcendence of $K$ is at least 1, $T \neq \emptyset$. Let $t \in T$ be selected. For every $n$ there is a differential operator $D_n$ on $K$ which is an extension of $\frac{\partial^n}{\partial t^n}$ from $\mathbb{Q}(T)$. Clearly, the operators $D_n$ are linearly independent over $\mathbb{Q}(T)$. Then so are the maps $\phi \circ D_n$. Since, by Theorem 5.10, the maps $\phi \circ D_n$ are additive solutions of (57) on $K$, the proof is complete. □

If $b_i$ and $c_i$ are algebraic numbers, then every differential operator on $K$ is a constant multiple of the identity. Therefore, in this case a finite basis of the linear space $S_1$ consists of the injective homomorphisms satisfying

$$\sum_{i=1}^{n} a_i \phi_j(b_i) = 0 \text{ and } \sum_{i=1}^{n} a_i \phi_j(c_i) = 0 \tag{60}$$

for every $j \in \{1, \ldots, k\}$. We deal with this case in Theorems 4.3 and 4.4 (See also [13]).

Example 4.8 shows that if the numbers $b_i$ and $c_i$ are not all algebraic, then the injective homomorphism solutions do not necessarily span $S_1$; that is, we may need nontrivial differential operators in order to generate $S_1$. One can show that $\partial/\partial t^k$ is not a solution of (48) if $k \geq 2$, and thus $S_1$ is of finite dimension over $\mathbb{Q}(t)$. In the next theorem we show that this behaviour is typical, supposing that $b_i, c_i$ generate a purely transcendental field of transcendence degree 1.

We remind that a functional equation of the form (57) is called *trivial* if every additive function $f : \mathbb{C} \to \mathbb{C}$ is a solution.

**Theorem 5.13.** *Suppose that $b_1, \dots, b_n, c_1, \dots, c_n \in \mathbb{Q}(t)$, where $t$ is transcendental over $\mathbb{Q}$. Then the equation* (57) *is either trivial or $S_1$ is of finite dimension over $\mathbb{C}$.*

**Proof.** The additive solutions of (57) are the same as the solutions of the system (58). If both of the equations of (58) are trivial, then so is (57). Also, if the space of additive solutions of any of the equations of (58) is of finite dimension, then the same is true for the space of additive solutions of (57). Therefore, it is enough to show that if the equation $\sum_{i=1}^{n} a_i f(b_i x) = 0$ is nontrivial, then the linear space of its additive solutions defined on $\mathbb{Q}(t)$ is of finite dimension.

For every $\gamma \neq 0$, the equations $\sum_{i=1}^{n} a_i f(b_i x) = 0$ and

$$\sum_{i=1}^{n} a_i f(b_i \gamma x) = 0 \tag{61}$$

are equivalent in the sense that if one of the equations is trivial then so is the other, and if the space of additive solutions of one of them is of finite dimension, then the same is true for the other.

By assumption, $b_1, \dots, b_n$ are rational functions of $t$ with rational coefficients. Let $\gamma$ denote the common denominator of $b_1, \dots, b_n$. Then $b_i \gamma \in \mathbb{Q}[t]$. Let $b_i \gamma = \sum_{j=0}^{m} \alpha_{i,j} t^j$, where $\alpha_{i,j} \in \mathbb{Q}$ for every $i, j$. Since we are interested in the additive solutions only, we may replace each term $f(b_i \gamma x)$ by $\sum \alpha_{i,j} f(t^j x)$ (here we used the fact that $f(\alpha x) = \alpha \cdot f(x)$ for every rational $\alpha$). Collecting the terms $a_i \cdot \alpha_{i,j} f(t^j x)$ in the sum $\sum_{i=1}^{n} a_i f(b_i \gamma x)$ for every $j$, we find that there is an equation

$$\sum_{j=0}^{m} A_j f(t^j x) = 0 \tag{62}$$

such that the additive solutions of (61) and those of (62) coincide. If $A_j = 0$ for every $j = 0, \dots, m$, then the equations are trivial. Therefore, we may assume that $A_j \neq 0$ for some $j$. We prove that in this case the set of additive solutions of (62) defined on $\mathbb{Q}(t)$ is of finite dimension.

Let $\Phi$ denote the set of functions $\phi \circ D$, where $\phi$ is an automorphism of $\mathbb{C}$ and is a solution of (62), $D$ is a differential operator of $\mathbb{Q}(t)$, and $\phi \circ D$ is a solution of (62). By

Theorem 5.9, applied to the equation $\sum_{j=0}^{m} A_j f(t^j x + 0 \cdot y) = 0$, we find that the linear space of additive solutions of (62) is spanned by $\Phi$ and thus it is enough to show that $\Phi$ generates a finite dimensional linear space over $\mathbb{C}$.

If $\phi$ is an automorphism of $\mathbb{C}$ and a solution of (62), then we have

$$\sum_{j=0}^{m} A_j \phi(t^j x) = 0$$

for every $x \in \mathbb{Q}(t)$. Putting $x = 1$ and using $\phi(t^j) = (\phi(t))^j$ we find that $\phi(t)$ is a root of the polynomial $P(X) = \sum_{j=0}^{m} A_j \cdot X^j$. Since $P$ only has a finite number of roots and $\phi$ is determined by the value of $\phi(t)$, we obtain that the number of possible $\phi$'s is finite.

Consequently, it is enough to show that if $\phi$ is fixed, then those differential operators $D$ for which $\phi \circ D \in S_1$ constitute a finite dimensional space over $\mathbb{C}$.

Fix $\phi$, and let $D = \sum_{k=0}^{s} d_k \frac{\partial^k}{\partial t^k}$ be a differential operator such that $d_s \neq 0$ and $\phi \circ D \in S_1$. We prove that $s \leq m$. Since $\phi \circ D$ is a solution, we have

$$\sum_{j=0}^{m} A_j \sum_{k=0}^{s} \delta_k \cdot \phi \left( \frac{\partial^k}{\partial t^k} (t^j x) \right) = 0$$

for every $x \in \mathbb{Q}(t)$, where $\delta_k = \phi(d_k)$. Since

$$\frac{\partial^k}{\partial t^k} (t^j x) = \sum_{i=0}^{k} \binom{k}{i} \cdot \frac{\partial^{k-i}}{\partial t^{k-i}} (t^j) \cdot \frac{\partial^i}{\partial t^i} x,$$

we obtain

$$\sum_{i=0}^{s} B_i \cdot \phi \left( \frac{\partial^i}{\partial t^i} x \right) = 0, \tag{63}$$

where

$$\begin{aligned} B_i &= \sum_{j=0}^{m} \sum_{k=i}^{s} A_j \cdot \binom{k}{i} \cdot \delta_k \cdot \phi \left( \frac{\partial^{k-i}}{\partial t^{k-i}} t^j \right) = \\ &= \sum_{j=0}^{m} \sum_{\nu=0}^{s-i} \binom{\nu + i}{i} \cdot \delta_{\nu+i} \cdot A_j \cdot \phi \left( \frac{\partial^\nu}{\partial t^\nu} t^j \right) = \qquad (64) \\ &= \sum_{\nu=0}^{s-i} \binom{\nu + i}{i} \cdot \delta_{\nu+i} \cdot \Gamma_\nu ; \end{aligned}$$

here we used the notation

$$\Gamma_\nu = \sum_{j=0}^{m} A_j \cdot \phi \left( \frac{\partial^\nu}{\partial t^\nu} t^j \right). \tag{65}$$

Applying (63) with $x = 1$ we obtain $B_0 = 0$. Then putting $x = t$ into (63) we obtain $B_1 = 0$. We continue, by substituting $t^2, t^3, \ldots$ into (63), and find that $B_i = 0$ for every $i = 0, \ldots, s$.

Now the equation $B_s = 0$ gives $\Gamma_0 = 0$ by $d_s \neq 0$. Then, from $B_{s-1} = 0$ we obtain $\Gamma_1 = 0$. Continuing this way we find that $\Gamma_\nu = 0$ for every $\nu = 0, \ldots, s$.

It is easy to check that

$$\phi\left(\frac{\partial^\nu}{\partial t^\nu} t^j\right) = \left(\frac{\partial^\nu}{\partial X^\nu} X^j\right)_{X=\phi(t)}$$

for every $\nu, j = 0, 1, \ldots$. Therefore, by (65), $\Gamma_\nu = 0$ gives

$$\begin{aligned} 0 = \Gamma_\nu &= \sum_{j=0}^{m} A_j \cdot \phi\left(\frac{\partial^\nu}{\partial t^\nu} t^j\right) = \\ &= \sum_{j=0}^{m} A_j \cdot \left(\frac{\partial^\nu}{\partial X^\nu} X^j\right)_{X=\phi(t)} = \\ &= \frac{\partial^\nu}{\partial X^\nu}\left(\sum_{j=0}^{m} A_j X^j\right)_{X=\phi(t)} = \\ &= P^{(\nu)}(\phi(t)). \end{aligned}$$

Since this is true for every $\nu = 0, \ldots, s$, we obtain that $\phi(t)$ is a root of $P$ of multiplicity at least $s$. However, $P$ is a nonzero polynomial of degree at most $m$, which gives $s \leq m$.

We have proved that if $D$ is a differential operator on $\mathbb{Q}(t)$ such that $\phi \circ D$ is a solution of (57) on $\mathbb{Q}(t)$, then the degree of $D$ is at most $m$. This implies that the set of these functions $\phi \circ D$ generate a linear space of finite dimension, which completes the proof. □

Suppose that $a_1, \ldots, a_n$ are algebraic and $b_i, c_i$ generate a purely transcendental field of transcendence degree 1. Then it follows from Theorems 5.12 and 5.13 that if (57) has a not identically zero additive solution on $K$, then the equation is trivial.

The following result generalizes this observation and it is also a generalization of the Theorems 2.32 (i) and Theorem 3.14 (i).

**Theorem 5.14.** *Suppose that $a_1, \ldots, a_n$ are algebraic and the field $K$ is purely transcendental where $K = \mathbb{Q}(b_1, \ldots, b_n, c_1, \ldots, c_n)$. If (57) has a not identically zero additive solution on $K$, then the equation is trivial.*

**Proof.** Let $K = \mathbb{Q}(t_1, \ldots, t_k)$, where $t_1, \ldots, t_k$ are algebraically independent over $\mathbb{Q}$. Applying the argument of the proof of Theorem 5.13, it is enough to prove the following statement. Consider the equation

$$\sum_{i_1 \ldots i_k = 0}^{m} A_{i_1 \ldots i_k} \cdot f\left(t_1^{i_1} \ldots t_k^{i_k} \cdot x\right) = 0, \tag{66}$$

where the coefficients $A_{i_1 \ldots i_k}$ are algebraic. If (66) has an additive solution on $K$ which is not identically zero, then the equation is trivial.

It is easy to check that every additive solution of (66) defined on $K$ can be extended to $\mathbb{C}$ as an additive solution on $\mathbb{C}$. Therefore, if (66) has an additive solution on $K$ which is not identically zero, then there is such a solution on $\mathbb{C}$. By Theorem 3.7 (or, by (iii) of Theorem 5.9), it follows that there is an automorphism $\phi$ of $\mathbb{C}$ which is a solution. This means that

$$\sum_{i_1\dots i_k=0}^{m} A_{i_1\dots i_k}\cdot(\phi(t_1))^{i_1}\dots(\phi(t_k))^{i_k}=0.$$

Since $\phi(t_1),\dots,\phi(t_k)$ are algebraically independent over $\mathbb{Q}$ and the numbers $A_{i_1\dots i_k}$ are algebraic, it follows that each $A_{i_1\dots i_k}$ equals zero. Then the equation (66) is obviously trivial. □

# 6 Spectral synthesis of higher degree

## 6.1 Proof of spectral synthesis in varieties of $k$-additive functions

In this section our aim is to prove the higher degree analogue of Theorem 5.7. We shall need the following notation.

If $\phi_1, \ldots, \phi_k$ are automorphisms of $\mathbb{C}$, then $\mathcal{M}_{(\phi_1,\ldots,\phi_k)}$ denotes the set of those functions which are finite sums of functions of the form

$$(x_1, \ldots, x_k) \mapsto \prod_{i=1}^{k} (\phi_i \circ D_i)(x_i) \qquad (x_1, \ldots, x_n \in K), \tag{67}$$

where $D_1, \ldots, D_k$ are differential operators on $K$. By $\mathcal{M}_\emptyset$ we mean the class of constant functions.

We shall denote by $\mathcal{A}_{(\phi_1,\ldots,\phi_k)}$ the set of functions $\prod_{i=1}^{k}(\phi_i \circ D_i)$ defined on $K$, where $D_1, \ldots, D_k$ are as above. That is,

$$\mathcal{A}_{(\phi_1,\ldots,\phi_k)} = \{\operatorname{diag} F : F \in \mathcal{M}_{(\phi_1,\ldots,\phi_k)}\}, \tag{68}$$

where $\operatorname{diag} F$ denotes the diagonal of the function $F : K^k \to \mathbb{C}$ defined by $f(x) = F(x, \ldots, x)$ $(x \in K)$.

Recall that $K^*$ denotes the Abelian group $\{x \in K : x \neq 0\}$ with respect to multiplication.

**Theorem 6.1.** *Suppose that the transcendence degree of the field $K$ over $\mathbb{Q}$ is finite. Let $F : K^k \to \mathbb{C}$ be a $k$-additive function. Let $m_1, \ldots, m_k$ be injective homomorphisms of $K$, let*

$$m(x) = m(x_1, \ldots, x_k) = m_1(x_1) \cdots m_k(x_k)$$

*for every $x_1, \ldots, x_k \in K$, and let $\phi_i$ be an extension of $m_i$ to $\mathbb{C}$ as an automorphism of $\mathbb{C}$ for every $i = 1, \ldots, k$. Then the following are equivalent.*

(i) *$F = p \cdot m$ on $(K^*)^k$, where $p$ is a local polynomial on $(K^*)^k$.*

(ii) *$F = p \cdot m$ on $(K^*)^k$, where $p$ is a generalized polynomial on $(K^*)^k$.*

(iii) *$F = p \cdot m$ on $(K^*)^k$, where $p$ is a polynomial on $(K^*)^k$.*

(iv) *$F \in \mathcal{M}_{(\phi_1,\ldots,\phi_k)}$.*

**Proof.** The implications (iii)$\Longrightarrow$(ii)$\Longrightarrow$ (i) are obvious.
(i)$\Longrightarrow$(iv): We prove by induction on $k$. The case $k = 1$ is covered by Theorem 5.7. Let $k > 1$, and suppose the statement is true for $k - 1$. In the proof of the induction step we

shall assume that $k=2$. It is easy to check that the same argument works in the general case.

Suppose $F$ is biadditive on $K^2$, and $F=p\cdot m$ on $(K^*)^2$, where $p$ is a local polynomial on $(K^*)^2$, and

$$m(x,y)=m_1(x)\cdot m_2(y)$$

for every $x,y\in K$, where $m_1$ and $m_2$ are given injective homomorphisms of $K$. Let $\phi$ be an extension of $m_1$ and $\psi$ be an extension of $m_2$ to $\mathbb{C}$ as automorphisms.

Let $T$ be a maximal subset of $K$ consisting of algebraically independent elements over $\mathbb{Q}$. Then $T$ is finite; let $T=\{t_1,\dots,t_N\}$. Let $G$ denote the subgroup of $K^*$ generated by $T$. Then $G^2$ is a finitely generated subgroup of $(K^*)^2$, and thus $p$ is a polynomial on $G^2$. Therefore, $p$ is a finite sum of terms of the form $a_1\cdots a_s$, where each factor $a_i$ is either additive on $(K^*)^2$ or is a constant. Note that the additivity of the function $a:(K^*)^2\to\mathbb{C}$ means

$$a(uv)=a(u)+a(v)\qquad (u,v\in (K^*)^2).$$

Let $x=t_1^{j_1}\cdots t_N^{j_N}$, $y=t_1^{k_1}\cdots t_N^{k_N}$ be arbitrary elements of $G$, where $j_1,\dots,j_N,k_1,\dots,k_N\in\mathbb{Z}$. Let $a:(K^*)^2\to\mathbb{C}$ be additive. If $a(t_i,1)=\alpha_i$ and $a(1,t_i)=\beta_i$, then

$$a(x,y)=\alpha_1 j_1+\ldots+\alpha_N j_N+\beta_1 k_1+\ldots+\beta_N k_N$$

and

$$m(x,y)=\phi(t_1)^{j_1}\cdots\phi(t_N)^{j_N}\cdot\psi(t_1)^{k_1}\cdots\psi(t_N)^{k_N}$$

for every $j_1,\dots,j_N,k_1,\dots,k_N\in\mathbb{Z}$. Therefore, the value of the function $a_1\cdots a_s\cdot m$ at the point $(x,y)$ is a linear combination, with complex coefficients, of terms of the form

$$\begin{aligned} &j_1^{c_1}\cdots j_N^{c_N}\cdot k_1^{d_1}\cdots k_N^{d_N}\cdot\phi(t_1)^{j_1}\cdots\phi(t_N)^{j_N}\cdot\psi(t_1)^{k_1}\cdots\psi(t_N)^{k_N}=\\ &=\phi\left(j_1^{c_1}\cdots j_N^{c_N}\cdot t_1^{j_1}\cdots t_N^{j_N}\right)\cdot\psi\left(k_1^{d_1}\cdots k_N^{d_N}\cdot t_1^{k_1}\cdots t_N^{k_N}\right), \end{aligned}\tag{69}$$

where $c_i,d_i$ are nonnegative integers. Then the value of $p\cdot m$ at the point $(x,y)$ is also a linear combination of terms of the same form.

It is easy to see that for every choice of the nonnegative integers $c_i,d_i$ there are differential operators $D$ and $E$ on $\mathbb{Q}(T)$ such that

$$D\left(t_1^{j_1}\cdots t_N^{j_N}\right)=j_1^{c_1}\cdots j_N^{c_N}\cdot t_1^{j_1}\cdots t_N^{j_N},$$

and

$$E\left(t_1^{k_1}\cdots t_N^{k_N}\right)=k_1^{d_1}\cdots k_N^{d_N}\cdot t_1^{k_1}\cdots t_N^{k_N}$$

for every $j_1,\dots,j_N,k_1,\dots,k_N\in\mathbb{Z}$ (see the proof of Theorem 5.4). Now it follows from (69) that the map $p\cdot m$, restricted to $G^2$, is the finite sum of the form

$$(\phi\circ D)(x)\cdot(\psi\circ E)(y)\qquad (x,y\in G),$$

where $D$ and $E$ are differential operators. Let

$$(p \cdot m)(x, y) = \sum_{\nu=1}^{S} (\phi \circ D_\nu)(x) \cdot (\psi \circ E_\nu)(y) \tag{70}$$

for every $x, y \in G$. The maps $D_\nu, E_\nu$ can be extended to $K$ as differential operators. Then the extended maps (denoted by the same letter) are additive on $K$ and $D_\nu/j, E_\nu/j$ are polynomials on $K^*$. The extended differential operators make the right hand side of (70) well-defined on $(K^*)^2$. We prove that (70) holds everywhere on $(K^*)^2$.

Let $x \in G$ be fixed. Then the left hand side of (70) equals $q(y) \cdot m_2(y)$, where $q(y) = p(x, y) \cdot m_1(x)$ for every $y \in K$. It is easy to check that the function $y \mapsto p(x, y)$ is a local polynomial on $K^*$, and thus so is $q$.

Let $(\phi \circ D_\nu)(x) = \gamma_\nu$ and $\psi^{-1}(\gamma_\nu) = \delta_\nu$. Then the right hand side of (70) equals $\psi \circ E$, where $E = \sum_{\nu=1}^{S} \delta_\nu E_\nu$ is a differential operator on $K$. By (70),

$$q(y) \cdot m_2(y) = (\psi \circ E)(y)$$

on $G$. By the equivalence of the statements (i) and (iv) of Theorem 5.7, there is a unique differential operator $\overline{E}$ on $K$ such that

$$q(y) \cdot m_2(y) = (\psi \circ \overline{E})(y)$$

for every $y \in K$. Then $\psi \circ E = \psi \circ \overline{E}$ on $G$, since both sides equal $q \cdot m_2$ on $G$. Since $\psi$ is injective, this implies $E = \overline{E}$ on $G$.

Since $E$ and $\overline{E}$ are differential operators on $K$, they are additive on $K$, and $E/j$ and $\overline{E}/j$ are polynomials on $K^*$ by definition. Since $E = \overline{E}$ on $G$, it follows from Lemma 5.6 that $E = \overline{E}$ on $K$, and thus (70) holds for every $y \in K$.

Now let $y \in K$ be fixed. Repeating the argument above we can see that (70) holds for every $x \in K$. Therefore, we have $p \cdot m \in \mathcal{M}_{(\phi,\psi)}$. This proves the implication (i)$\Longrightarrow$(iv).

(iv)$\Longrightarrow$(iii): It is enough to show that the map $(x_1, \dots, x_k) \mapsto \prod_{i=1}^{k} (\phi_i \circ D_i)(x_i)$ is of the form $p \cdot m$ on $(K^*)^k$. By the equivalence of the statements (iii) and (iv) of Theorem 5.7, there are polynomials $p_i$ on $K^*$ such that $\phi_i \circ D_i = p_i \cdot m_i$ on $K$. Then

$$\prod_{i=1}^{k} (\phi_i \circ D_i)(x_i) = \prod_{i=1}^{k} p_i(x_i) \cdot m_i(x_i) = \left( \prod_{i=1}^{k} p_i(x_i) \right) \cdot m(x_1, \dots, x_k)$$

on $K^*$. It is clear that $(x_1, \dots, x_k) \mapsto \prod_{i=1}^{k} p_i(x_i)$ is a polynomial on $(K^*)^k$, which completes the proof. □

**Remark 6.2.** The proof of the implication (i)$\Longrightarrow$(iv) gives the following: in the representation of $p \cdot m$ as a sum of functions of the form (67), the sum of the degrees of the differential operators equals the degree of $p$ in every term. (The degree of a differential operator was defined in Section 5.1.)

**Theorem 6.3.** *Suppose that the transcendence degree of the field $K$ over $\mathbb{Q}$ is finite. Then spectral synthesis holds in every variety on $(K^*)^k$ consisting of $k$-additive functions (with respect to addition).*

**Proof.** Since $K$ is countable, so is the Abelian group $(K^*)^k$. Let $V$ be a variety on $(K^*)^k$ consisting of $k$-additive functions. By Theorem 2.22, local spectral synthesis holds on $K^*$, and thus $V$ is spanned by local polynomial-exponential functions. Since, by Theorem 6.1, every local polynomial-exponential function contained by $V$ is a polynomial-exponential function, it follows that $V$ is spanned by polynomial-exponential functions. □

## 6.2 The space of solutions of linear functional equations

We continue the description of the solutions of

$$\sum_{i=1}^{n} a_i f(b_i x + c_i y) = 0, \tag{71}$$

where $a_i, b_i, c_i$ are given complex numbers and $f : \mathbb{C} \to \mathbb{C}$ is the unknown function. Let $K = \mathbb{Q}(b_1, \ldots, b_n, c_1, \ldots, c_n)$.

Our aim is to generalize Theorem 5.9 to the case of $k > 1$. Recall that $S_k$ denotes the set of those solutions of (71) defined on $K$ which are generalized monomials of degree $k$. Also, $M_k$ denotes the set of the functions $F\colon K^k \to \mathbb{C}$ such that $F$ is $k$-additive, and the function $x \mapsto F(s_1 x, s_2 x, \ldots, s_k x)$ is a solution of (71) on $K$ for every $s_1, s_2, \ldots s_k \in K^*$. In addition,

$$M_k^* = \{F|_{(K^*)^k} : F \in M_k\}.$$

By Lemmas 3.2 and 3.3, $M_k^*$ is a variety on $(K^*)^k$ and $S_k = \{\operatorname{diag} F : F \in M_k\}$.

In Theorem 3.18 it was proved that if $S_k$ contains a nonzero function, then there are field automorphisms $\phi_1, \ldots, \phi_k$ of $\mathbb{C}$ such that $\phi_1 \cdots \phi_k \in S_k$. The proof depends on the fact that spectral analysis holds in a certain variety. Our proof of Theorem 6.4 is based on the observation that, by Theorem 6.3, spectral synthesis holds in the same variety.

**Theorem 6.4.** (i) *For every function $F \in M_k$, $F$ is an exponential monomial on $(K^*)^k$ if and only if $F \in \mathcal{M}_{(\phi_1,\ldots,\phi_k)}$, where $\phi_1, \ldots, \phi_k$ are automorphisms of $\mathbb{C}$ and $\prod_{i=1}^{k} \phi_i(x)$ is a solution of* (71).

(ii) *The variety $M_k^*$ is spanned by the classes $M_k^* \cap \mathcal{M}_{(\phi_1,\ldots,\phi_k)}$, where $\phi_1, \ldots, \phi_k$ are as above.*

**Proof.** (i) Suppose that $F \in M_k$ is an exponential monomial on $(K^*)^k$, and let $F = p \cdot m$, where $p$ is a polynomial, and $m$ is an exponential on $(K^*)^k$. Since $M_k^*$ is a variety, $p{\cdot}m \in M_k^*$

implies $m \in M_k^*$ by Lemma 2.15. Note that $M_k \subset V_k$ and $M_k^* \subset V_k^*$. Therefore, by Lemma 3.6, there are injective field homomorphisms $m_1, \ldots, m_k$ from $K$ into $\mathbb{C}$ such that

$$m(x_1, \ldots, x_k) = m_1(x_1) \cdots m_k(x_k) \qquad (x_1, \ldots, x_k \in K^*).$$

Let $\phi_i$ be an extension of $m_i$ to $\mathbb{C}$ as an automorphism of $\mathbb{C}$. Then $m(x, \ldots, x) = \prod_{i=1}^k \phi_i(x)$ is a solution of (71) on $K$. The rest of the statement (i) follows from Theorems 6.1.

Statement (ii) is a consequence of Theorem 6.3. □

The statement of the next theorem follows immediately from Lemma 3.3 and (ii) of Theorem 6.4.

**Theorem 6.5.** *The set $S_k$ is spanned by the classes $S_k \cap \mathcal{A}_{(\phi_1,\ldots,\phi_k)}$, where $\phi_1, \ldots, \phi_k$ are automorphisms of $\mathbb{C}$, and $\prod_{i=1}^k \phi_i$ is a solution of (71).* □

Let $S_{\leq k}$ denote the set of those solutions of (71) defined on $K$ which are generalized polynomials of degree at most $k$. Then $S_{\leq k}$ is a closed linear space over $\mathbb{C}$.

**Theorem 6.6.** *The set $S_{\leq k}$ is spanned by the classes $S_m \cap \mathcal{A}_{(\phi_1,\ldots,\phi_m)}$, where $0 \leq m \leq k$, $\phi_1, \ldots, \phi_m$ are automorphisms of $\mathbb{C}$, and $\prod_{i=1}^m \phi_i$ is a solution of (71).*

**Proof.** Let $f \in S_{\leq k}$ be arbitrary. Then $f = \sum_{m=0}^k f_m$, where $f_m$ is a generalized monomial of degree $m$ for every $m = 1, \ldots, k$, and $f_0$ is constant. By Lemma 2.30, each of the functions $f_0, \ldots, f_k$ is a solution of (71). Therefore, we have $f_m \in S_m$ for every $m = 1, \ldots, k$.

It is enough to show that each $f_m$ is in the closure of the linear space spanned by the classes $\mathcal{A}_{(\phi_1,\ldots,\phi_m)} \cap S_m$, where $\phi_1, \ldots, \phi_m$ are automorphisms of $\mathbb{C}$, and $\prod_{i=1}^m \phi_i(x) \in S_m$. If $m \geq 1$ then this is true by Theorem 6.5.

If $m = 0$, then there are two cases to consider. If $\sum_{i=1}^n a_i \neq 0$, then the only constant solution of (71) is the zero function, so $f_0 = 0$. On the other hand, if $\sum_{i=1}^n a_i = 0$, then all constant functions are solutions of (71). Then the statement is true, since $\mathcal{A}_\emptyset$ is the class of constant functions. □

We shall say that the equation (71) is *normal of degree $k$*, if every solution of (71) is a generalized polynomial of degree at most $k$. Now we give a description the set of solutions of normal equations.

**Corollary 6.7.** *Suppose that the equation (71) is normal of degree $k$. Then the linear space of its solutions defined on $K$ is spanned by the classes $S_m \cap \mathcal{A}_{(\phi_1,\ldots,\phi_m)}$, where $0 \leq m \leq k$, $\phi_1, \ldots, \phi_m$ are automorphisms of $\mathbb{C}$, and $\prod_{i=1}^m \phi_i$ is a solution of (71).*

*In particular, this is true for the equation*

$$\sum_{i=1}^{n} a_i f(b_i x + y) = 0 \qquad (x, y \in \mathbb{C}) \tag{72}$$

*with* $k = n - 2$. □

**Theorem 6.8.** *Let* $S$ *denote the set of solutions of* (71) *defined on* $K$.

(i) *If the equation is normal, then the set of polynomials on the additive group of* $K$ *is dense in* $S$.

(ii) *If the equation is normal and* $\sum_{i=1}^{n} a_i = 0$, *then spectral synthesis holds in* $S$ *considered as a variety on the additive group of* $K$.

(iii) *If the equation is normal and translation invariant, then spectral synthesis holds in* $S$ *considered as a variety on the additive group of* $K$.

**Proof.** If $\phi$ is an automorphism of $\mathbb{C}$, and $D$ is a differential operator on $K$, then the function $\phi \circ D$ is additive on $K$. Therefore, each class $\mathcal{A}_{(\phi_1,\ldots,\phi_k)}$ consists of polynomials. Thus (i) follows from Corollary 6.7.

Let $\mathbf{1}$ denote the identically 1 function defined on $K$. If $\sum_{i=1}^{n} a_i = 0$, then $\mathbf{1}$ is a solution. Since $\mathbf{1}$ is an exponential function, every polynomial in $S$ is, in fact, a polynomial-exponential function. Thus (ii) follows from (i).

Suppose that the equation is normal and translation invariant. If $S = \{0\}$, then spectral synthesis holds in $S$. If $S$ contains a nonzero generalized polynomial $p$, then it follows from the translation invariance of $S$ that $\Delta_{h_1} \ldots \Delta_{h_k} p \in S$ for every $k$ and $h_1, \ldots, h_k \in \mathbb{C}$. If $k = \deg p$, then $\Delta_{h_1} \ldots \Delta_{h_k} p$ is a nonzero constant for suitable $h_1, \ldots, h_k$. Thus $\mathbf{1} \in S$, and thus (iii) follows from (ii). □

## 6.3 An example and concluding remarks

The following example serves as an illustration: it shows how to use our previous results in order to determine the set of solutions of a given equation. We consider the equation

$$\begin{aligned} -2f(y) + f(tx+y) + f((1-t)x+y) + \\ + f((t^2-t)x+y) - f((t^2-t+1)x+y) = 0, \end{aligned} \tag{73}$$

where $t$ is a fixed transcendental number. The equation is of the form (72) where $n = 5$, $(a_1, \ldots, a_5) = (-2, 1, 1, 1, -1)$ and $(b_1, \ldots, b_5) = (0, t, 1-t, t^2-t, t^2-t+1)$. Thus every solution is a generalized polynomial of degree at most three.

In order to simplify notation, we shall write $x'$ instead of $\frac{\partial}{\partial t}x$, $x''$ instead of $\frac{\partial^2}{\partial t^2}x$ etc. It is easy to check that we have

$$\sum_{i=1}^{5} a_i \cdot b_i^n = 0 \ (n = 0, 1, 2), \qquad \sum_{i=1}^{5} a_i \cdot b_i^3 \neq 0. \tag{74}$$

We shall also need

$$\sum_{i=1}^{5} a_i \cdot b_i^{(n)} \cdot b_i^{(m)} = \begin{cases} 2 & \text{if } n = m = 1, \\ -2 & \text{if } n = 2,\ m = 0, \\ 0 & \text{if } n \geq m \geq 0,\ n + m \neq 2. \end{cases} \tag{75}$$

It is easy to see that the equation is trivial; that is, every additive function is a solution. (Since $a_1, \ldots, a_5$ are integers and $K = \mathbb{Q}(t)$ is purely transcendental, it follows from Theorem 5.14 that if (73) were not trivial, it wouldn't have any nonzero solution on $K$.) Since the maps $x \mapsto x^{(n)}$ are additive functions, we have

$$\sum_{i=1}^{5} a_i \cdot b_i^{(n)} = 0 \qquad (n = 1, 2, \ldots). \tag{76}$$

Let $\phi, \psi$ be automorphisms of $\mathbb{C}$, and suppose that $g = \phi \cdot \psi$ is a solution of (73). If we substitute $f = g,\ y = 0$ and $x = 1$ into (73), the left hand side of the equality obtained equals $\phi(t)^2 + \psi(t)^2 - 2\phi(t)\psi(t) = (\phi(t) - \psi(t))^2$. Since $g$ is a solution, we get $\phi(t) = \psi(t)$. This implies that $\phi = \psi$ on $K$.

We prove that there is no nonzero solution which is a generalized monomial of degree 3. Indeed, suppose there is such a solution. Then it follows from Theorem 6.5 that there are automorphisms $\phi, \psi, \chi$ of $\mathbb{C}$ such that $\phi \cdot \psi \cdot \chi$ is a solution. Then, by Theorem 3.21, the functions $\phi \cdot \psi,\ \phi \cdot \chi,\ \psi \cdot \chi$ are also solutions. As we saw above, this implies that $\phi = \psi = \chi$ on $K$, and thus $\phi^3$ is a solution. Then we have

$$0 = \sum_{i=1}^{5} a_i \cdot \phi(b_i)^3 = \phi\left(\sum_{i=1}^{5} a_i \cdot b_i^3\right),$$

which contradicts (74). This proves that there is no nonzero solution which is a generalized monomial of degree 3.

Therefore, in order to determine all solutions of (73), it is enough to describe the set $S_2$ of those solutions which are defined on $K$ and are generalized monomials of degree two.

If $\phi, \psi$ are automorphisms of $\mathbb{C}$, then $\mathcal{A}_{(\phi,\psi)}$ denotes the set of functions of the form $\sum_{j=1}^{N} (\phi \circ D_j) \cdot (\psi \circ E_j)$, where $D_j$ and $E_j$ are differential operators on $K$. By Theorem 6.5, the set $S_2$ is spanned by $S_2 \cap \mathcal{A}_{(\phi,\psi)}$, where $\phi, \psi$ are automorphisms of $\mathbb{C}$ such that

$\phi \cdot \psi$ is a solution of (73). Since this implies $\phi = \psi$, we may confine our attention to the sets $\mathcal{A}_{(\phi,\phi)}$. It is clear that

$$\mathcal{A}_{(\phi,\phi)} = \phi\left(\mathcal{A}_{(j,j)}\right)$$

for every automorphism $\phi$, where $j$ denotes the identity map. Also, $f$ is a solution of (49) if and only if $\phi \circ f$ is, so we only need to describe $S_2 \cap \mathcal{A}_{(j,j)}$.

Since every differential operator on $K = \mathbb{Q}(t)$ is the linear combination of the maps $x \mapsto x^{(n)}$ $(x \in K,\ n = 0, 1, \ldots)$, the elements of $\mathcal{A}_{(j,j)}$ are linear combinations of the maps

$$f_{n,m}(x) = x^{(n)} \cdot x^{(m)} \qquad (x \in K,\ 0 \le m \le n).$$

In order to determine which linear combinations of the maps $f_{n,m}$ are solutions, we have to compute the sums

$$S_{n,m}(x, y) = \sum_{i=1}^{5} a_i \cdot f_{n,m}(b_i x + y),$$

and determine those linear combinations of the functions $S_{n,m}$ which are identically zero on $K = \mathbb{Q}(t)$. A computation, based on (75) and (76), shows that we have $S_{0,0} = S_{1,0} = S_{1,1} + S_{2,0} = 0$ on $K$, and if a linear combinations of the functions $S_{n,m}$ is zero on $K$, then it is also a linear combination of $S_{0,0}$, $S_{1,0}$ and $S_{1,1} + S_{2,0}$.

Indeed, we have

$$(b_i x + y)^{(n)} = y^{(n)} + \sum_{\nu=0}^{n} \binom{n}{\nu} b_i^{(\nu)} x^{(n-\nu)} = y^{(n)} + \sum_{\nu=0}^{2} \binom{n}{\nu} b_i^{(\nu)} x^{(n-\nu)},$$

and

$$(b_i x + y)^{(m)} = y^{(m)} + \sum_{\mu=2}^{n} \binom{m}{\mu} b_i^{(\nu)} x^{(m-\mu)}.$$

Taking the products of the right hand sides, multiplying by $a_i$, summing for $i = 1, \ldots, 5$ and using (74)-(75), we obtain

$$\begin{aligned} S_{n,m}(x) = \sum_{\nu,\mu=0}^{2} \binom{n}{\nu}\binom{m}{\mu} x^{(n-\nu)} x^{(m-\mu)} \cdot \sum_{i=1}^{5} a_i \cdot b_i^{(\nu)} b_i^{(\mu)} = \\ = -2\binom{m}{2} x^{(n)} x^{(m-2)} + 2nm \cdot x^{(n-1)} x^{(m-1)} - 2\binom{n}{2} x^{(n-2)} x^{(m)}. \end{aligned} \tag{77}$$

Suppose $f = \sum_{(n,m)\in I} c_{n,m} \cdot f_{n,m}$ is a solution, where the coefficients $c_{n,m}$ are nonzero for every $(n, m) \in I$. Then

$$\sum_{(n,m)\in I} c_{n,m} \cdot S_{n,m}(x) = 0$$

for every $x \in \mathbb{Q}(t)$. Now we use the fact that for every finite set $I$ of pairs $(n, m)$ there exists a polynomial $p \in \mathbb{Q}[t]$ such that the polynomials $p^{(n)} \cdot p^{(m)}$ $((n, m) \in I)$ are linearly independent over $\mathbb{C}$. (We omit the proof.) This implies that if in the sum $\sum_{(n,m)\in I} c_{n,m} \cdot S_{n,m}(x)$ we replace each $S_{n,m}(x)$ by the right hand side of the equation (77), and replace $x$ by $p(t)$, then we obtain zero only if the equation obtained is an identity; that is, if every term $x^{(i)} \cdot x^{(j)}$ is canceled.

Suppose $I$ contains a pair $(n,m)$ with $m \geq 2$. Let $(n,m)$ be such a pair with the largest $n$ and, if there are more than one such pair then with a largest $m \geq 2$. It is clear that in this case the term $x^{(n)} \cdot x^{(m-2)}$ occurs only once in the sum with a nonzero coefficient, so it is not canceled out.

Therefore, we have $m = 0$ or $m = 1$ for every $(n,m) \in I$. We have

$$S_{(n,0)} = n(n-1) \cdot x^{(n-2)} \cdot x, \qquad S_{(n,1)} = 2n \cdot x^{(n-1)} \cdot x - n(n-1) \cdot x^{(n-2)} \cdot x'$$

for every $n \geq 2$. If $I$ contains a pair $(n,1)$ with $n \geq 2$, then the sum $\sum_{(n,m)\in I} c_{n,m} \cdot S_{n,m}(x)$ will contain a term $x^{(n-2)} \cdot x'$ that occurs only once with a nonzero coefficient, which is impossible. Therefore, the only pair $(n,m) \in I$ with $m = 1$ is $(1,1)$. Then, if $I$ contains a pair $(n,0)$ with $n \geq 3$, then the sum $\sum_{(n,m)\in I} c_{n,m} \cdot S_{n,m}(x)$ will contain a term $x^{(n-2)} \cdot x$ that occurs only once with a nonzero coefficient, which is impossible. Therefore, the only pair $(n,m) \in I$ with $m = 1$ is $(1,1)$.

This means that $S_2 \cap \mathcal{A}_{(j,j)}$ is the linear span of the functions $x^2, x \cdot x'$ and $(x')^2 + x \cdot x''$. Summing up: *The space of solutions of* (73) *defined on* $K$ *is the closed linear hull of all additive functions and the functions*

$$\phi^2,\ \phi \cdot \left(\phi \circ \frac{\partial}{\partial t}\right) \ \textit{and}\ \left(\phi \circ \frac{\partial}{\partial t}\right)^2 + \phi \cdot \left(\phi \circ \frac{\partial^2}{\partial t^2}\right),$$

*where* $\phi$ *is an arbitrary injective homomorphism of* $K$.

The example above shows that some of the results of Section 4.4 cannot be generalized for solutions of degree greater than 1. Theorem 5.10 says that if $a_1, \ldots, a_n$ are algebraic numbers and the injective homomorphism $\phi$ is a solution, then $\phi \circ D$ is also a solution for every differential operator $D$. In the example above, $\phi \cdot \phi$ is a solution for every $\phi$, but $(\phi \circ \frac{\partial}{\partial t})^2$ is not a solution, so the analogy is false for monomials of degree 2. This implies that the analogy of Corollary 6.7 is also false for monomials of degree 2.

Theorem 5.13 says that if $b_i, c_i \in \mathbb{Q}(t)$, where $t$ is transcendental over $\mathbb{Q}$, then the equation is either trivial or $S_1$ is of finite dimension over $\mathbb{C}$. The analogous statement would be that if $b_i, c_i \in \mathbb{Q}(t)$, then either every monomial of degree two is a solution, or $S_2$ is of finite dimensional. The example above shows that this is not true in general. We can see that the analogue of Theorem 5.14 is also false in $S_2$.

We remark, however, that *if the space of additive solutions of an equation* (72) *is of finite dimensional, then so is the space of those solutions which are generalized monomials* *of degree two.* (This follows from the fact that if $A(x, y)$ is symmetric and biadditive, and $x \mapsto A(x, x)$ is a solution, then the functions $y \mapsto A(x, y)$ $(x \in K)$ are additive solutions. If these latter functions span a linear space of finite dimension over $\mathbb{C}$ generated by the additive functions $a_1, \ldots, a_k$, then $A(x, y)$ is the linear combination of the functions $a_i(x) \cdot a_j(y)$ $(i, j = 1, \ldots, k)$.)

Although the description of the set of solutions of a given equation can be difficult, the example above shows that, at least in principle, the description is possible in the case of many equations.

We conclude with some remarks concerning the 'generic' or 'random' equation. By that we mean an equation (71) in which the numbers $a_i, b_i, c_i$ are algebraically independent over $\mathbb{Q}$. Such an equation is normal, but not translation invariant. An injective homomorphism $\phi$ is a solution if and only if

$$\sum_{i=1}^{n} a_i \phi(b_i) = \sum_{i=1}^{n} a_i \phi(c_i) = 0 \tag{78}$$

holds. This implies that the equations is not trivial (not every additive function is a solution), but $S_1$ is of infinite dimensional. One can prove that $S_1$ *is spanned by the injective homomorphisms satisfying* (78). Note that differential operators do not appear in the description of $S_1$.

# 7 The discrete Pompeiu problem

In this section we are concerned with the so-called discrete Pompeiu problem and its connection to linear functional equations. The problem, to be explained shortly, is stemmed from the classical Pompeiu problem and from the following question asked by L. Pósa.

**Question 7.1** (Pósa)**.** *Suppose that the function* $f\colon \mathbb{R}^2 \to \mathbb{R}$ *has the property that the sum of the values of* $f$ *at the vertices of any square of fix size is zero. Is it true that* $f \equiv 0$*?*

As we shall see, the answer to Pósa's question is affirmative. This question can be generalized as follows.

**Question 7.2** (Discrete Pompeiu problem)**.** *Let* $D \subset \mathbb{R}^2$ *be a finite set. Suppose that the function* $f\colon \mathbb{R}^2 \to \mathbb{R}$ *has the property that the sum of the values of* $f$ *at the the points of every congruent copy of* $D$ *is zero. Is it true that* $f \equiv 0$*?*

The classical Pompeiu problem is the following question in integral geometry, named after Dimitrie Pompeiu.

**Question 7.3.** *Let* $f$ *be a continuous function defined on the plane, and let* $K$ *be a closed set of positive Lebesgue measure. Suppose that*

$$\int_{\sigma(K)} f(x,y)dxdy = 0 \tag{79}$$

*for every rigid motion* $\sigma$*. Is it true that* $f \equiv 0$*?*

We say that the set $K$ has the *Pompeiu's property* if the answer to the Question 7.3 is affirmative.

In the case where $K$ is a disc, Pompeiu asserted that the answer was affirmative and even published an erroneous proof. The error was pointed out by Nicolesco, who sought to establish generalizations of Pompeiu's result. Chakalov [4] showed that there are nontrivial solutions to (79); for instance, there are infinitely many linearly independent solutions of the form $\sin(ax + by)$ for appropriately chosen constants $a, b$.

In case $K$ is a square, Pompeiu showed that the only continuous solution to (79) tending to a limit at infinity is the zero function. In fact, his proof is more complicated than necessary, and he actually needs only the fact that (79) holds for all squares of a fixed sized parallel to the coordinate axes. Christov [5], [6] showed that Pompeiu's requirement that $f$ tend to a limit could be dropped and subsequently settled the corresponding problem for parallelograms and for triangles.

In general, L. Brown, B. M. Schreiber and B. A. Taylor [3] showed that the domain $K$ has the Pompeiu's property when $K$ is any polygonal region or any convex set with at least one "corner". (For more details see [3, Corollary 5.12].) Their proof is based on the fact that spectral synthesis holds in those varieties of continuous functions on $\mathbb{R}^n$ which are invariant under translations and rotations.

The problem has many variants. The typical ones use similar copies or translates of $K$ instead of congruent copies. One type of the discrete version of Pompeiu problem appeared in the paper of D. Zeilberger [47]. We denote by $\mathbb{Z}^n$ the $n$-dimensional lattice in $\mathbb{R}^n$. Let $\mathcal{D}$ be a finite family of finite subsets of $\mathbb{Z}^n$, and let $\mathcal{T}$ denote the group of all translations on $\mathbb{Z}^n$. We denote variables $(z_1, z_1^{-1}, \ldots, z_n, z_n^{-1})$ by $\mathbf{z}$, and the set of complex valued function on $\mathbb{Z}^n$ by $\mathcal{F}(\mathbb{Z}^n)$ . We say that the family $\mathcal{D} = \{D_1, \ldots, D_k\}$ has the *discrete Pompeiu's property on* $\mathbb{Z}^n$ if the only function $f : \mathbb{Z}^n \to \mathbb{C}$ such that

$$\sum_{d \in \tau(D)} f(d) = 0 \text{ for all } \tau \in \mathcal{T} \text{ and } D \in \mathcal{D} \tag{80}$$

is identically zero. Zeilberger proved the following theorem:

**Theorem 7.4.** *For $K$ a finite subset of $\mathbb{Z}^n$ let $P_D(z) = \sum_{d \in \mathcal{D}} \mathbf{z}^d$. Then the finite family $\mathcal{D}$ has the discrete Pompeiu's property on $\mathbb{Z}^n$ if and only if the polynomials $\{P_D : D \in \mathcal{D}\}$ have common zeros in $\mathbb{C}^n$.*

The proof is based on the connection between difference operators and the polynomials on $\mathbb{C}$, and uses Hilbert's *Nullstellensatz* [45, p. 157] and other ring theoretical results. Recently, M. J. Puls [26] gave a necessary and sufficient condition for a finite collection of finite subsets of a discrete Abelian groups, whose torsion free rank is less than continuum, to have the Pompeiu's property.

We may generalize the discrete Pompeiu problem as follows. Let $D$ be a finite set of $\mathbb{R}^2$ and let $G$ be a transformation group on $\mathbb{R}^2$. We say that $D$ has the *discrete Pompeiu property with respect to $G$* if for every function $f : \mathbb{R}^2 \to \mathbb{C}$ the equation

$$\sum_{d \in \sigma(D)} f(d) = 0 \tag{81}$$

for all $\sigma \in G$ implies $f \equiv 0$.

Our main concern is the answer to Pósa's question when $D$ is the vertices of a square and $G$ is the group of congruences of $\mathbb{C}$.

However, as a motivation, we show that any finite $D$ has the discrete Pompeiu's property with respect to the similarities of $\mathbb{C}$. We denote by $\Sigma$ the similarity group of $\mathbb{C}$.

**Proposition 7.5.** *Suppose that $D$ is a nonempty finite subset of $\mathbb{C}$. Let $f : \mathbb{R}^2 \to \mathbb{C}$ be a function which satisfies equation* (81) *for every $\sigma \in \Sigma$. Then $f \equiv 0$.*

**Proof.** Let $d_1, d_2, \ldots, d_n \in \mathbb{C}$ be the elements of $D$. Then the equation (81) can be written in the form

$$f(x + d_1 y) + f(x + d_2 y) + \ldots + f(x + d_n y) = 0 \tag{82}$$

holds for every $x, y \in \mathbb{C}$. Thus, by Remark 3.12, the equation (82) has a solution if and only if it has an automorphism solution $\phi$ which satisfies the equations $\sum_i a_i = 0$ and $\sum_i a_i \phi(b_i) = 0$. Since the first equation does not hold, therefore the set $D$ has the discrete Pompeiu's property with respect to the similarities. □

From now on we turn to Pósa's question (see Theorem 7.9).

We will use some well-known results of Euclidean Ramsey theory. The Euclidean Ramsey type question is the following: Let $S$ be a set of points in $\mathbb{R}^n$ and let $G$ be a transformation group of $\mathbb{R}^n$. Color the points of the space $\mathbb{R}^n$ with $k$ colors. Is it true that there is a monochromatic copy $g(S)$ of $S$, where $g \in G$? If it is true for any coloring of $\mathbb{R}^n$ with $k$ colors, then we say that $S$ is Ramsey with respect to $G$ and $k$ colors. If $G$ is the group of congruences of $\mathbb{R}^n$, then we just say $S$ is Ramsey. L. E. Shader in his paper [30] showed the following theorem

**Theorem 7.6.** (i) *For any 2-coloring of the plane all right triangles are Ramsey.*

(ii) *For any 2-coloring of the plane and for every nondegenerate parallelogram $P$, there is a congruent parallelogram with three vertices of the same color.*

We will use the following well-known selection theorem of Rado (for details see [28], [37, Theorem A]).

**Theorem 7.7** (Rado's Selection Principle)**.** *Let $\mathcal{A} = \{A_i : i \in I\}$ be a family of finite sets. For every finite $J \subset I$, let $f_J$ be a function defined on $J$ such that $f_J(i) \in A_i$ for every $i \in J$. Then, there exists a function $f$ defined on $I$ such that for each finite $J \subset I$ there is a finite $K \subset I$ such that $J \subseteq K$ and $f|_J = (f_K)|_J$.*

**Remark 7.8.** We use this theorem for colorings. A coloring of a set $A$ with $k < \infty$ colors is a choice function. Thus, we may use it for Ramsey type or Euclidean Ramsey type theorems. From Theorem 7.7 it follows that if there is a set $S$ which is Ramsey, then there exists a finite set of points $R$ such that every $k$-coloring of $R$ contains a monochromatic copy of $S$. We refer to $R$ as a witness set to $S$.

Pósa's question was answered independently by T. Terpai and M. Laczkovich and the author. The latter proof actually gives more; see Remark 7.10.

**Theorem 7.9.** *Let $D$ be the vertex set of the unit square. Then $D$ has the discrete Pompeiu property with respect to the congruences of $\mathbb{C}$.*

**Proof.** The statement can be written in the following form:

$$f(x) + f(x+y) + f(x+iy) + f(x+(1+i)y) = 0 \tag{83}$$

holds for every $x, y \in \mathbb{C}$ and $|y| = 1$. Let us assume that there exists a nonzero $f$ satisfying (83).

Now we would like to use Theorem 2.18 but unfortunately $r_0(H)$, the torsion free rank of the additive group $H$ of $\mathbb{C}$ is continuum. Therefore we restrict our attention to a subgroup $G$ generated by countable many generators of $H$.

In Theorem 7.6 it was proved that all right triangles are Ramsey. Let $S$ be the equilateral right triangle of unit side. By Remark 7.8, we can fix a finite witness set $R$ of $S$.

Since $f \not\equiv 0$, there is an $a \in \mathbb{C}$ such that $f(a) \neq 0$. Let $G$ denote the additive subgroup of $\mathbb{C}$ generated by the elements of $R$ and $a$. Then $G$ is an Abelian group of finite torsion free rank. Let $V$ denote the set of functions $f\colon G \to \mathbb{C}$ satisfying (83) for every $x, y$ such that $x, x+y, x+iy, x+(1+i)y \in G$ and $|y| = 1$. It is easy to see that $V$ is a variety.

Since $f|_G \in V$ and $a \in G$, it follows that $V_G \neq 0$. Then, by Theorem 2.18, there exists a (nonzero) exponential function in $V$. Now, exponential means that $g$ satisfies

$$g(x+y) = g(x)g(y). \tag{84}$$

We may reformulate equation (83) as follows:

$$g(x)(1 + g(y) + g(iy) + g((1+i)y)) = 0.$$

Since $g(x) \neq 0$, we get

$$(1 + g(y))(1 + g(iy)) = 0. \tag{85}$$

Thus $g(y) = -1$ or $g(iy) = -1$. Therefore, if we put $a = x$, $b = x+y$, $c = x+(1+i)y$, $d = x+iy$, then it follows that either $g(b)/g(a) = g(c)/g(d) = -1$ or $g(d)/g(a) = g(c)/g(b) = -1$. Thus the values of $g$ at the points $a, b, c, d$ are either $g(a), -g(a), -g(d), g(d)$ or $g(a), g(b), -g(b), -g(a)$, respectively. Thus the values of $G$ at the vertices of any unit square contained by $G$ can be decompose into two pairs of the form $(x, -x)$. We decompose $\mathbb{C}^* = \mathbb{C} \setminus 0$ into two parts $A$ and $B$ such that $A = -B$. Let $h(x) = 1$ if $g(x) \in A$, and $h(x) = -1$ if $g(x) \in B$. Then $h\colon \mathbb{C} \to \{1, -1\}$ has the property that for every congruent copy $Q$ of the unit square contained by $G$, two of the values of $h$ at the vertices of $Q$ equal 1 and two of the values of $h$ at the vertices of $Q$ equal $-1$.

Since $h$ gives a 2-coloring of the set $R \subset G$, it follows that there is a square with the property that at least three vertices belong to $R$ and have the same color. Let these points be $a, b, c, d$, where $a, b, c \in R$. Since $d = \pm a \pm b \pm c \in G$, the square $a, b, c, d$ belongs to $G$.

Then, as we saw above two of the values $h(a), h(b), h(c), h(d)$ equal 1 and two equal $-1$. This is a contradiction, since $h(a) = h(b) = h(c)$. $\square$

**Remark 7.10.** A similar argument shows that there every non-degenerate parallelogram has the discrete Pompeiu's property with respect to congruences of $\mathbb{C}$. We need to use the second part of Theorem 7.6.

We close this session with a basic open question, which could be the analogue of the result of Brown, Schreiber and Taylor [3].

**Question 7.11.** *Is that true that every nonempty finite $D$ has the discrete Pompeiu property with respect the congruences of the plane?*

# 8 Summary

In the thesis we are concerned with the linear functional equation

$$\sum_{i=1}^{n} a_i f(b_i x + c_i y) = 0 \qquad (x, y \in \mathbb{C}) \tag{86}$$

where $a_i, b_i, c_i$ are given complex numbers, and $f : \mathbb{C} \to \mathbb{C}$ is the unknown function.

By a well-known result of L. Székelyhidi [36], under some mild conditions on the equation, every solution of equation (86) is a generalized polynomial. But the finer structure of the solutions has been investigated only recently.

In Section 2 we define some basic notation which we use throughout the thesis. We introduce some generalizations of the notion of polynomial mappings from an Abelian group $G$ to $\mathbb{C}$ and outline the relations between them. We collect the main results of the theory of spectral analysis and synthesis on discrete groups on which this work is based. Finally we summarize some important results on linear functional equations which stands in our center of interest.

The main result of Section 3 is that *an equation of form* (86) *has a nonzero generalized polynomial solution of degree $k$ if and only if there are field automorphisms $\phi_1, \dots, \phi_k$ of $\mathbb{C}$ such that $\phi_1 \cdots \phi_k$ is a solution.* (see Theorem 3.7, Theorem 3.18, Corollary 3.19). This result provides a theoretical possibility to decide the existence of non-constant solutions of equation (86) which satisfies a weak condition on the parameters $a_i, b_i, c_i$. The proof uses results of spectral analysis on discrete Abelian groups [20] (see Theorem 2.18).

The idea of applying spectral analysis to these varieties on $K^* = \{x \in K : x \neq 0\}$ and $(K^*)^k$ was introduced in [15] (here $K$ denotes a subfield of $\mathbb{C}$), where it was shown that (86) *has a nonzero additive solution if and only if there is a solution which is a field automorphism of* $\mathbb{C}$. In some special cases this was proved earlier by A. Varga and Cs. Vincze ([42], [41]). Their results, and also the earlier theorems of Z. Daróczy ([7], [42], [41]) initiated the idea of connecting the existence of nonzero additive solutions to field isomorphisms.

In Section 4 we show that *if the parameters $b_i, c_i$ are algebraic, then the products $\phi_1 \cdots \phi_k$ span the linear space of solutions of degree $k$.* (See also Theorems 4.3 and 4.4.) In Subsection 4.3 we deal with the special case when every additive function is the solution of (86). We call such an equation *trivial.* It is proved that the space of additive solutions of a trivial functional equation on $\mathbb{C}$ is spanned by the automorphisms of $\mathbb{C}$ (Theorem 4.6). It is shown, using derivations, that the additive solutions are not spanned by automorphisms in general (Theorem 4.7). These results are based on the article [13].

In Sections 5 and 6 our aim is to present a dense subset of solutions space $S$ consisting of functions of simple structure.

A brief formulation of our main results is the following. Let (86) be an arbitrary equation. Then *the set of additive solutions defined on $K$ is spanned by those solutions which can be written in the form $\phi \circ D$, where $\phi$ is a field automorphism of $\mathbb{C}$ and $D$ is a differential operator on $K$* (see Theorem 5.9).

*The set of solutions which are generalized monomials of order $k$ is spanned by those solutions which can be represented as finite sums of functions of the form $\prod_{i=1}^{k}(\phi_i \circ D_i)$, where $\phi_1, \dots, \phi_k$ are field automorphisms of $\mathbb{C}$ and $D_1, \dots, D_k$ are differential operators on $K$* (see Theorem 6.5).

*If the equation* (86) *is normal of degree $k$, then the set $S$ is spanned by those solutions which can be represented as finite sums of functions of the form $\prod_{i=1}^{m}(\phi_i \circ D_i)$, where $m \leq k$, and $\phi_i$ and $D_i$ are as above* (see Corollary 6.7).

The proof of these results is based on the fact that spectral synthesis holds in some related varieties (see Theorems 5.8 and 6.3). These varieties are defined on the groups $K^*$ and, more generally, on $(K^*)^k$. These groups contain free Abelian groups of rank infinity (see the remark after Theorem 5.4), and it is well-known that on such a group there are varieties in which spectral synthesis does not hold (Theorem 2.19). This means that in order to prove Theorems 5.8 and 6.3 we have to use some special properties of the varieties. The crucial observation is that in these varieties every local polynomial-exponential function is a polynomial-exponential function (see Theorems 5.7 and 6.1). Then, using a general theorem stating that local spectral synthesis holds on every countable Abelian group [18] we infer that spectral synthesis holds in these varieties.

In Subsections 5.3 and 6.3 we give several applications of the general theorems concerning the solutions of (86).

Finally in Section 7 we introduce a problem which is called *discrete Pompeiu problem*. We say that the finite set $D \subset \mathbb{R}^2$ has the *discrete Pompeiu property with respect to the group $G$ of transformations* if for every function $f : \mathbb{R}^2 \to \mathbb{C}$ the equation

$$\sum_{d \in \sigma(D)} f(d) = 0$$

for all $\sigma \in G$ implies $f \equiv 0$.

We use linear functional equations to show that every finite set has the discrete Pompeiu property with respect to similarities of the $\mathbb{R}^2$ (see Proposition 7.5). We also show in Theorem 7.9 that the unit square has the discrete Pompeiu property with respect to congruences of $\mathbb{R}^2$. This result is based on an unpublished work of M. Laczkovich and the author. The general problem whether or not every finite set has the discrete Pompeiu property with respect to congruences remains open.

# 9 Összefoglaló

A disszertációban a következő típusú, úgynevezett lineáris függvényegyenletekkel foglalkozunk:

$$\sum_{i=1}^{n} a_i f(b_i x + c_i y) = 0 \qquad (x, y \in \mathbb{C}), \tag{87}$$

ahol $a_i, b_i, c_i$ adott komplex számok, és $f : \mathbb{C} \to \mathbb{C}$ ismeretlen függvény.

Székelyhidi L. [36] egy közismert eredménye szerint, bizonyos enyhe feltételek mellett minden megoldás általánosított polinom. Ám a megoldások finomabb szerkezetét csak az utóbbi években kezdték vizsgálni.

A 2. fejezetben néhány alapvető fogalmat vezetünk be, melyeket kulcsszerephez jutnak az egész disszertáció során. Először néhány lehetséges definíciót mutatunk be, melyek egy Abel-csoportról a komplex számokba menő polinom fogalmát járják körbe, majd összefoglaljuk az ezekkel kapcsolatos eredményeket. Utána a munkánk során alapvető szerepet játszó diszkrét spektrálanalízis és -szintézis elméletét ismertetjük. Végül a lineáris függvényegyenletek elméletének alapvető eredményeit mutatjuk be, melyek a későbbi vizsgálatok központi tárgyát képzik.

A 3. fejezet fő eredményeként bebizonyítjuk, hogy *egy* (87) *típusú egyenletnek pontosan akkor van nem azonosan nulla $k$-ad fokú általánosított polinom megoldása, ha léteznek olyan $\phi_1, \ldots, \phi_k$ testautomorfizmusai a komplex számoknak, melyekre a $\phi_1 \cdots \phi_k$ szorzat megoldás.* (Lásd Theorem 3.7, Theorem 3.18, Corollary 3.19.)

Ez az eredmény legalábbis elméletileg lehetőséget ad arra, hogy eldöntsük egy (87) típusú, és az $a_i, b_i$ és $c_i$ paraméterekre vonatkozó enyhe feltételnek (lásd az (5) feltételt) eleget tevő egyenletről, hogy mikor létezik nem nulla megoldása. A bizonyítás a diszkrét spektrálanalízis eredményein alapszik [20] (lásd Theorem 2.18).

Az ötlet, hogy spektrálanalízist alkalmazhatunk bizonyos, a $K^* = \{x \in K : x \neq 0\}$ multiplikatív csoporton, illetve a $(K^*)^k$-n definiált varietásokra (itt $K$ a $\mathbb{C}$ egy résztestе) a [15] dolgozatból származik. Itt megmutattuk, hogy *a* (87)*-nek akkor és csak akkor van nem-nulla additív megoldása, ha létezik olyan testautomorfizmusa a komplex számoknak, amely maga is megoldás.* Bizonyos speciális esetekre a kérdést Varga A. és Vincze Cs. megválaszolták ([42], [41]). Daróczy Z. és az előbb említett kutatók eredményei ([7], [42], [41]) vetették fel a nem-nulla additív megoldások és a testizomorfizmusok közötti kapcsolatot.

A 4. fejezetben megmutatjuk, hogy *amennyiben a $b_i, c_i$ paraméterek algebraiak $\mathbb{Q}$ felett, akkor a $\phi_1 \cdots \phi_k$ alakú szorzatok kifeszítik a $K$-n értelmezett, pontosan $k$-ad fokú megoldások lineáris terét.* (Lásd Theorem 4.3 és Theorem 4.4.) A 4.3 alfejezetben olyan függvényegyenletekkel foglalkozunk, melyeknek minden additív függvény megoldása. Ezeket nevezzük *triviális* egyenleteknek. Belátjuk, hogy egy triviális függvényegyenlet additív megoldásainak terében sűrűek a komplex számok automorfizmusai. (Itt a sűrűség fogalmához a szorzattopológiát használjuk.) Ugyanakkor a derivációk segítségével mutatunk példát arra, hogy a komplex számoknak azon automorfizmusai, melyek egy adott egyenlet megoldáshalmazába esnek, nem feltétlenül feszítik ki az additív megoldások terét. Ezen eredmények a [13] cikket követik.

A 5. és 6. fejezetekben fő célunk, hogy megadjuk a megoldások halmazának egy olyan sűrű részhalmazát, amely egyszerű szerkezetű függvényekből áll.

Röviden összefoglalva az eredmények a következők. Vegyünk egy tetszőleges (87) típusú egyenletet. Ekkor *a $K$-n értelmezett additív megoldások terét kifeszítik azon megoldások, melyek felírhatóak $\phi \circ D$ alakban, ahol $\phi$ egy testautomorfizmusa a $\mathbb{C}$-nek, mely maga is egy megoldás és $D$ egy differenciáloperátor a $K$-n* (lásd Theorem 5.9).

*A pontosan $k$-ad fokú megoldások terét kifeszítik azon megoldások, amelyek előállnak mint $\prod_{i=1}^{k}(\phi_i \circ D_i)$ alakú szorzatok véges összegei, ahol $\phi_1, \ldots, \phi_k$-k a $\mathbb{C}$ testautomorfizmusai és a $\prod_{i=1}^{k} \phi_i$ szorzat maga is megoldás, valamint $D_1, \ldots, D_k$ differenciáloperátorok $K$-n* (lásd Theorem 6.5).

*Ha a* (87) *típusú egyenlet normális (azaz minden megoldása általánosított polinom), melyben a megoldások foka legfeljebb $k$, akkor a megoldások halmazát kifeszítik azon megoldások, melyek felírhatóak $\prod_{i=1}^{m}(\phi_i \circ D_i)$ alakban, ahol $m \leq k$, és $\phi_i$ illetve $D_i$ mint fent* (lásd Corollary 6.7).

A bizonyítás azon alapszik, hogy a megfelelő varietásokon teljesül a spektrálszintézis (lásd Theorem 5.8 és Theorem 6.3). A vizsgált varietások a $K^*$ multiplikatív csoporton, illetve általánosabban, a $(K^*)^k$-n vannak definiálva. Ezen csoportok tartalmaznak végtelen rangú szabad Abel-csoportot. Ismeretes, hogy ilyen csoportokon vannak olyan varietások, melyekre a spektrálszintézis nem teljesül (lásd Theorem 2.19). Ez esetünkben azt eredményezi, hogy a fent említett tételek bizonyításához a varietásaink bizonyos speciális tulajdonságait kell kihasználnunk. A döntő észrevétel a következő. Az általunk vizsgált varietásokban minden lokális polinom-exponenciális függvény valójában polinom-exponenciális (lásd Theorem 5.7 és Theorem 6.1). Ekkor a lokális polinom-exponenciális függvényekre vonatkozó spektrálelméletet alkalmazva beláthatjuk, hogy a "közönséges" spektrálszintézis is teljesül ezekben a varietásokban.

Az 5.3 és a 6.3 alfejezetekben néhány alkalmazást mutatunk a (87) típusú egyenletekkel kapcsolatos általános eredményekre.

Végül a 7. fejezetben bemutatjuk az úgynevezett *diszkrét Pompeiu-problémát.* Egy $D \subset \mathbb{R}^2$ véges halmazra azt mondjuk, hogy rendelkezik a *diszkrét Pompeiu-tulajdonsággal a $G$ transzformációcsoportra nézve,* ha az $f \equiv 0$ az egyetlen olyan $\mathbb{R}^2 \to \mathbb{C}$ függvény, melyre

$$\sum_{d \in \sigma(D)} f(d) = 0$$

fennáll minden $\sigma \in G$-re.

Lineáris függvényegyenletek segítségével megmutatjuk, hogy minden véges $D$ halmaz rendelkezik a diszkrét Pompeiu-tulajdonsággal a sík hasonlósági transzformációira nézve (lásd Proposition 7.5). Másrészt, a lényegesen nehezebb esetet, amikor $G$ a sík egybevágóságainak csoportja, bizonyos speciális esetben szintén kezelni tudjuk. Megmutatjuk, hogy ha $D$ egy nem elfajuló parallelogramma csúcshalmaza, akkor $D$ rendelkezik a diszkrét Pompeiu-tulajdonsággal a sík egybevágóságaira nézve (lásd Theorem 7.9). Ezen tétel egy Laczkovich Miklóssal közös publikálatlan eredményen alapszik. Nyitott kérdés, hogy minden véges halmaz rendelkezik-e a diszkrét Pompeiu-tulajdonsággal a sík egybevágóságaira nézve.

# References


[1] J. A. Baker, A general functional equation and its stability, *Proc. Am. Math. Soc.* **133** (2004), no. 6, 1657-1664.

[2] K. Baron and W. Jarczyk, Recent results on functional equations in a single variable, perspectives and open problems, *Aequationes Math.* **61** (2001), 1-48.

[3] L. Brown, B. M. Schreiber and B. A. Taylor, Spectral synthesis and the pompeiu problem, *Ann. Inst. Fourier* **23** (1973), 125-154.

[4] Chalakov, Sur un problème de D. Pompeiu, *Annaire Univ. Sofia Fac. Phys. Math.*, Livre 1, **40** (1944), 1-44.

[5] C. Christov, Sur un problème de M. Pompeiu, *Mathematica(Timisoara)* **23** (1947-48), 103-108.

[6] C. Christov, Sur l'équation intégrale généralisée de M. Pompeiu, *Annaire Univ. Sofia Fac. Phys. Math.*, Livre 1, **45** (1948-49), 167-178.

[7] Z. Daróczy, Notwendige und hinreichende Bedingungen für die Existenz von nightkonstanten Lösungen linearer Functionalgleichungen, *Acta. Sci. Math.* (Szeged) **22** (1961), 31-41.

[8] D. Ž. Djoković, A representation theorem for $(X_1 - 1)(X_2 - 1) \dots (X_n - 1)$ and its applications, *Ann. Polon. Math.* **22** (1969), 189–198.

[9] R. J. Elliot, Two notes on spectral synthesis for discrete Abelian groups, *Proc. Camb. Phil. Soc.* **61** (1965), 617-620.

[10] M. Engert, Finite dimensional translation invariant subspaces, *Pacific J. Math.*, **32** (1970), 333-343.

[11] N. Garofalo, The new result on the Pompeiu problem *Rend. Sem. Mat. Univ. Pol. (Torino)* (1988), 25-38.

[12] D. I. Gurevič, Counterexample to a problem of L. Schwartz, *Funkcional. Anal. i Priložen* **9** (1975), no. 2, 29-35.
(English translation: *Funckcional Anal. Appl.* **9** (1975), no. 2, 116-120.)

[13] G. Kiss, Linear functional equations with algebraic parameters, accepted for publication in Publicationes Mathematicae Debrecen.

[14] G. Kiss and M. Laczkovich, Linear functional equations, differential operators and spectral synthesis, accepted for publication in Aequationes Math.

[15] G. Kiss and A. Varga, Existence of nontrivial solutions of linear functional equation, published online *Aequationes Math.* (2013).

[16] M. Kuczma: *An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality.* Państwowe Wydawnictwo Naukowe, Uniw. Śląski, Warszawa-Kraków-Katowice, 1985. Second edition: Birkhäuser, 2009 (A. Gilányi editor).

[17] M. Laczkovich, Polynomial mappings on Abelian groups, *Aequationes Math.* **68** (2004), 177-199.

[18] M. Laczkovich, Local spectral synthesis on Abelian groups, to appear in *Acta Math. Hungar.*

[19] M. Laczkovich, Polinom-exponenciális függvények, *Matematikai Lapok* **10** (1) (2000-2001), 22-31.

[20] M. Laczkovich and G. Székelyhidi, Harmonic analysis on dicrete abelien groups, *Proc. Am. Math. Soc.* **133** (2004), no. 6, 1581-1586.

[21] M. Laczkovich and L. Székelyhidi, Spectral synthesis on dicrete Abelian groups, *Math. Proc. Camb. Phil. Soc.* **143** (2007), 103-120.

[22] M. Lefranc, Analyse spectrale sur $\mathbb{Z}^n$, *C. R. Paris* **246** (1958), 1951-1953

[23] S. Mazur and W. Orlicz, Grundlegende Eigenschaften der polynomischen Operationen, *Fund. Math.* (1939), 42-50.

[24] Zs. Páles, Extension theorems foer functional equations with bisymmetric operators, *Aequationes Math.* **63** (2002), 266-291.

[25] D. Pompeiu, Sur une propriété des fonctions continues dépendent de plusieurs variables *Bull. Sci. Math(2)* **53** (1929), 328-332.

[26] M. J. Puls, The Pompeiu problem and discrete groups, accepted, *Monatsh. Math.* **172** no. 3-4 (2013), 415-429.

[27] A. G. Ramm, The Pompeiu problem, *Applicable Analysis* **64** no. 1-2 (1997), 19-26.

[28] R. Rado, Axiomatic treatment of rank in infinite sets, *Canad. J. Math* **1** (1949), 337-343.

[29] S. Stević, Behaviour of solutions of some linear functional equations at infinity, *Appl. Math. Comput.* **219** (2013), 6134-6141.

[30] L. E. Shader, All right triangles are Ramsey in $E^2$!, *Journ. Comb. Theory (A)* **20** (1976), 385-389.

[31] L. Székelyhidi, Remark on a paper of M. A. McKiernan, *Ann. Polon. Math.* **36** (1979), 245-247.

[32] L. Székelyhidi, Note on exponential polynomials, *Pacific J. Math.* **103** (1982), 583-587.

[33] L. Székelyhidi, On a class of linear functional equations , *Publ. Math.* (Debrecen) **29** (1982), 19-28.

[34] L. Székelyhidi, Note on exponential polynomials, *Pacific J. Math.* **103** (1982), 583-587.

[35] L. Székelyhidi, the failure of spectral synthesis on some types of discrete Abelian groups, *J. Math. Anal. Appl.* **291** (2004), 757-763.

[36] L. Székelyhidi, *Discrete Spectral Synthesis and Its Applications*, Springer-Verlag, 2006.

[37] M. Truszczynski, Zs. Tuza, Rado's Selection Principle: applications to binary relations, graph and hypergraph colorings and partially ordered sets, *Discrete Mathematics***103** (1992), 301-312.

[38] Vinczéné Varga Adrienn: *Középértékeket tartalmazó függvényegyenletek.* PhD thesis, Debrecen, 2012.

[39] A. Varga, On a functional equation containing four weighted arithmetic means, *Banach J. Math. Anal.* **2** (1) (2008), 21-32.

[40] A. Varga, On additive solutions of a linear equation, *Acta Math. Hungar.***128** (1-2), (2010), 15-25.

[41] A. Varga, Cs. Vincze, On Daróczy's problem for additive functions, *Publ. Math.* (Debrecen) **75**(1-2) (2009), 299-310.

[42] A. Varga, Cs. Vincze, On a functional equations containing weighted arithmetic means, *Inter. Ser. Num. Math.* **157** (2009), 305-315.

[43] S. Williams, A partial solutions of Pompeiu problem, *Math. Ann.* **223** (1976), 183-190.

[44] S. Williams, Analyticity of the boundary for Lipschitz domains without Pompeiu property, *Indiana Univ. Math. Journ.***30**(1981) 357-369.

[45] Van der Waerden, *Algebra*, Frederick Ungar Publishing Co., New York, 1970, Vol. 2.

[46] L. Zalcman, A bibliographical survay of the Pompeiu Problem, in the book *Approximation by solutions of partial differential equations.* Edited by B. Fuglede. Kluwer Acad., Dordrecht, 1992, 177-186.

[47] D. Zeilberger, Pompeiu's problem in discrete space, *Proc. Nat. Acad. Sci. USA* **75** (1978), no. 8, 3555-3556.