1 First exercise set

Fundamental

E 1. Show that $\mathbb{Z}^d$ is amenable.

Definition. Given a group Γ with a generating set S, and an action $\Gamma \curvearrowright X$ on a set, the associated *Schreier graph* $\mathrm{Sch}(X, \Gamma, S)$ has vertex set X, and (oriented, labeled) edge set $\{(x, s.x) \mid x \in X,\ s \in S\}$.

E 2. Let $\Gamma = \langle S \rangle$ be a group. Show that the following are in 1-to-1 correspondence.

1. Subgroups of Γ;
2. Transitive, pointed actions of Γ;
3. Rooted, connected Schreier graphs of Γ.

E 3. Draw the coset Schreier graph of the subgroup $H = \langle a \rangle$ of the free group $F_2 = \langle a, b \rangle$ (with respect to the generators a, b).

E 4. Let G be a connected, locally finite graph, and let $p_{x,y,n}$ denote the probability that a simple random walk started at x is at y after n steps. Show that

$$\lim_{n\to\infty} \left(p_{o,o,2n}\right)^{1/2n}$$

exists, and does not depend on the choice of o.

On topic

E 5. Let (X, μ) be a standard Borel probability space, and let Γ act on X by probability measure preserving (p.m.p.) Borel bijections.

1. Show that the stabilizer Γ_x of a μ-random point $x \in X$ is an IRS of Γ.
2. Show that this construction is general, i.e. *every* IRS of Γ arises this way.

E 6. Exhibit a Følner sequence in the lamplighter group $C_2 \wr \mathbb{Z}$, but also show that it has exponential growth. (That is, the size of the r-ball around the identity grows exponentially in r.)

Connected

E 7. Show that any $2d$-regular finite graph is a Schreier graph of the free group on d generators.

E 8. Given a Følner sequence in a finitely generated group Γ, construct a *translation invariant*, finitely additive, normalized measure on *all subsets* of Γ.