1 Third exercise set

E 1. Let ∂F_2 denote the boundary of the free group, i.e. the Cantor space of infinite reduced words in the letters $\{a, b\}$. Let ν denote the product measure on ∂F_2: the first letter is chosen uniformly from $\{a, a^{-1}, b, b^{-1}\}$, while every later letter is chosen independently and uniformly from the 3 letters that do not cancel the previous one.

Show that $F_2 \curvearrowright \partial(F_2, \nu)$ by precomposition, but the action does not preserve ν. On the other hand, show that ν is stationary with respect to the simple random walk on F_2.

E 2. Consider a random walk μ on a group G, and assume $H \leq G$ is a *recurrent subgroup*, i.e. it is visited infinitely often with probability 1. Define μ_H as the hitting measure on H. Prove that $H^\infty(G, \mu) = H^\infty(H, \mu_H)$.

E 3. Let Γ be a group, and μ the step-distribution of some random walk on Γ. Consider the wreath product $G = C_2 \wr \Gamma$, and equip it with the random walk that has step-distribution $\nu * \mu * \nu$, where ν is the uniform distribution on $\{\mathrm{id}_G, \delta_{\mathrm{id}_\Gamma}\}$. Show that this random walk on G is Liouville (i.e. all bounded harmonic functions are constant) if and only if the μ-random walk on Γ is recurrent.

(In $\bigoplus_\Gamma C_2$ the element $\delta_{\mathrm{id}_\Gamma}$ is the vector with value 1 at id_Γ, and 0 everywhere else. Taking a step on G with respect to $\nu * \mu * \nu$ means first flipping a fair coin to decide if we switch the lamp or not, then taking a μ-random step with the lamplighter, and finally flipping a fair coin again to decide if we switch the lamp at the new position.)