1 First exercise set

E 1.1. Show that the 3-cycles generate $\mathrm{Alt}_f(\mathbb{N})$, the group of finitely supported, even permutations of $\mathbb{N}$.

E 1.2. Show that recurrence/transience of a random walk on an infinite, connected graph does not depend on the choice of starting vertex.

E 1.3. Show that transience is equivalent to the expected number of returns to the root being finite.

E 1.4. Show that an automorphism of a d-regular (infinite) tree fixes either a vertex, an edge, or a biinfinite geodesic, on which it acts by translation.

E 1.5. Show that the fundamental group of a graph is a free group of rank equal to the number of edges outside an arbitrary spanning tree.

E 1.6. (Ping-pong lemma) Let a and b be two bijections of the set X. Assume there exist disjoint subsets $A, B \subseteq X$ such that for all $n \in \mathbb{Z} \setminus \{0\}$ we have $a^n B \subseteq A$ and $b^n A \subseteq B$. Show that a and b generate a free group.

E 1.7. Consider the simple random walk on $\mathbb{Z}$ between a and b ($a < b$), stopped at the endpoints. Condition the walk on ending up at b.

a) Show that after the conditioning the random walk is still a Markov chain.

b) Determine the transition probabilites.

E 1.8. Show that every group has $0, 1, 2$ or infinitely many ends, and the number of ends does not depend on the choice of generating set.

Definitions

Definition 1.9 (Recurrence, transience)**.** Let G be a connected, infinite graph and $o \in V(G)$ a starting vertex. We say that a random walk on G is transient, if (started from o) with positive probability the walk never returns to o again. Otherwise it is called recurrent, i.e. when the walk returns to o with probability 1.

Definition 1.10 (Number of ends)**.** Let G be an infinite, locally finite, connected graph, and for a finite subset of vertices F denote by $c_\infty(F)$ the number of *infinite* components of $G \setminus F$. The *number of ends* of G is defined as $\sup\{c_\infty(F) \mid F \subseteq V \text{ finite}\}$. (It is infinite if $c_\infty(F)$ is not bounded.) For a finitely generated group $\Gamma = \langle S \rangle$ the number of ends is computed for the Cayley graph $\mathrm{Cay}(\Gamma, S)$.