1 Second exercise set

E 1.1. Show that the simple random walk is recurrent on $\mathbb{Z}$, but transient on the d-regular tree T_d for $d \geq 3$.

E 1.2. Let H be a subgroup of the free group $F(S)$. Show that the fundamental group of the coset Schreier graph $\mathrm{Sch}(F/H, S)$ is isomorphic to H. As a consequence deduce that H is itself free. In case H has finite index derive a formula for its rank.

E 1.3. Show that $\mathrm{Alt}_f(\mathbb{N})$ is simple.

E 1.4. Show that two translations along disjoint axes on the d-regular tree generate a free group.

E 1.5. Assume $\langle R \mid S \rangle$ is a finite presentation of the group G. Show that any other presentation $\langle R' \mid S' \rangle$ contains a finite presentation of G.

E 1.6. Show that a maximal filter is an ultrafilter. Show that as a consequence, Zorn's lemma implies the existence of a non-principal ultrafilter.

E 1.7. Let Γ be a transitive permutation group. Show that Γ is primitive if and only if the point stabilizer Γ_α is a maximal subgroup in Γ.

E 1.8. Pyramid scheme. Let G be an infinite, locally finite, connected graph. Define the *value* of a pyramid scheme $f : \vec{E} \to \mathbb{R}$ to be the infimum of gains, that is

$$\mathtt{value}(f) = \inf_{v \in V(G)} \sum_{e^+ = v} f(e) - \sum_{e^- = v} f(e).$$

How large can the value of a bounded ($||f||_\infty \leq 1$) pyramid scheme be?

Definitions

Definition 1.9. (Filter, ultrafilter) Let Ω be a set. A family of subsets $F \subseteq 2^\omega$ is a *filter*, if it is nontrivial ($\emptyset \notin F,\ \Omega \in F$), upward closed ($A \in F,\ A \subseteq B \Rightarrow B \in F$), and closed under finite intersection ($A, B \in F \Rightarrow A \cap B \in F$). A filter U is an *ultrafilter*, if for any $A \subseteq \Omega$ either $A \in U$ or $\Omega \setminus A \in U$. An ultrafilter U is *principal* if it consists exactly of the sets containing a fixed element $\omega \in \Omega$, i.e. $U = \{A \subseteq \Omega \mid \omega \in A\}$.

Definition 1.10 (Primitive permutation group)**.** For a transitive permutation group $\Gamma \subseteq \mathrm{Sym}(\nleq)$ a partition $P = \{P_i\}_{i \in I}$ of Ω is Γ-*invariant*, if for every $g \in \Gamma$, and $\omega_1, \omega_2 \in \Omega$ that are in the same part P_i, the images $g.\omega_1$ and $g.\omega_2$ are also in the same part P_j (but maybe $i \neq j$). The group Γ is *primitive*, if there is no Γ-invariant partition of $\mathbb{N}$.

Definition 1.11 (Schreier graph)**.** Given a (finitely generated) group $\Gamma = \langle S \rangle$ witha generating set, and an action $\Gamma \curvearrowright X$ the *Schreier graph* $\mathrm{Sch}(\Gamma, X, S)$ has vertex set X, and we connect every $x \in X$ to $s.x$ for every generator $s \in S$. (The Cayley graph is a special case, with $\Gamma \curvearrowright \Gamma$ the left or right multiplication.) For a subgroup $H \leq \Gamma$ the coset Schreier graph $\mathrm{Sch}(\Gamma/H, S)$ is the one corresponding to the action $\Gamma \curvearrowright \Gamma/H$.