4 Fourth exercise set

E 4.1. Show that a finitely generated amenable group Γ admits a normalized left-invariant finitely additive measure (on the whole of 2^Γ). (Finitely generated is not necessary.)

E 4.2. Show that one-ended (locally finite) trees are recurrent.

E 4.3. Show that the trajectory of the simple random walk on the d-regular tree T_d ($d \geq 3$) converges to a point on the boundary ∂T with probability 1.

E 4.4. Consider a $2 \times n$ matrix, filled by the integers from 1 to $2n$. A permutation of the entries is *vertical* if it fixes the first coordinate of every entry and it is *horizontal* if it fixes their second coordinates. Prove that every permutation of the entries can be obtained as the product of a vertical, a horizontal and a vertical permutation.

E 4.5. Let G be a (locally finite, connected) graph, and $h : V(G) \to [0, 1]$ a harmonic function with respect to the simple random walk. Let X_n denote the position of the walker after n steps. Show that $\lim_{n\to\infty} h(X_n)$ exists with probability 1.

E 4.6. Show that an infinite index nontrivial normal subgroup of the free group has infinite rank.

E 4.7. Show that the lamplighter group $C_2 \wr Z$ is not finitely presented.

E 4.8. (Jordan-Wielandt theorem) Show that the only primitive (transitive) subgroups of $\mathrm{Sym}_f(\mathbb{N})$ are $\mathrm{Alt}_f(\mathbb{N})$ and $\mathrm{Sym}_f(\mathbb{N})$.

(a) If $G \leq \mathrm{FSym}(\mathbb{N})$ is primitive and contains a 3-cycle, then $\mathrm{Alt}(\mathbb{N}) \subseteq G$.

(b) Show that G_α-has finitely many orbits.

(c) Let $\alpha \sim \beta$ if the orbit of β under the action of G_α is finite. Show that $\sim$ is a G-invariant equivalence relation.

Definitions

Definition 4.9 (Finitely presented)**.** A group Γ is *finitely presented* if it has a representation with finitely many generators and relations, i.e. $\Gamma = \langle S \mid R \rangle$ with $|S|, |R| < \infty$.

Definition 4.10 (Følner condition of amenability)**.** Let $\Gamma =< S >$, $\mathrm{id} \in S$ and $|S| < \infty$. We say Γ is *(Følner) amenable*, if $\mathrm{Cay}(\Gamma, S)$ contains finite subsets with arbitrarily small boundaries compared to their size, i.e. $\forall \varepsilon > 0\ \exists F \subset \Gamma$ finite such that $|S \cdot F \setminus F| < \varepsilon$. This does not depend on the choice of S. If Γ is not finitely generated, one can define the same by asking that such Følner sets F exist for any finite subset S and $\varepsilon > 0$. (What is the correct definition, if Γ is not finitely generated?)

Definition 4.11 (Invariant finitely additive measure)**.** Given a group Γ, a *normalized (left) invariant finitely additive measure* on the group is a set function $m : 2^\Gamma \to [0, 1]$ that is additive, $m(\Gamma) = 1$ and $m(gA) = m(A)$ for all $A \subseteq \Gamma$ and $g \in \Gamma$.