5 Fifth exercise set

E 5.1. Let B_n denote the rooted binary tree of depth n. Show that $\mathrm{Aut}(B_n)$ is exactly the Sylow 2-subgroup of $\mathrm{Sym}(2^n)$. How does this generalize to other primes?

E 5.2. We call a vertex of an infinite Cayley graph a *trap*, if all its neighbors are at a strictly smaller distance to the identity than itself. Give an example of an infinite Cayley graph with a trap.

E 5.3. Let A be an Abelian group. A *vertical sheer* of $A \times A$ is a bijection $F(x, y) = (x, y + f(x))$ where $f : A \to A$ is an arbitrary function. We define horizontal sheers accordingly. Show that for $A = C_p$, horizontal and vertical sheers generate the full alternating group $\mathrm{Alt}(A \times A)$. On the other hand, show that you need to allow at least $c|A|$ terms in the products, if you want to express all elements of $\mathrm{Alt}(A \times A)$ as products of sheers.

E 5.4. Let G be a bounded degree, infinite, connected graph with nonzero Cheeger constant. Show that there exists a $C > 0$ and $f_1, f_2 : V(G) \to V(G)$ injective functions such that $d(x, f_i(x)) < C$ for all $x \in V(G), i = 1, 2$ and $f_1(V(G)) \cap f_2(V(G)) = \emptyset$.

E 5.5. Give an interesting example of a $\mathrm{Sym}_f(\mathbb{N})$-invariant random partition of $\mathbb{N}$.

E 5.6. Let Γ be a countable group, and μ a probability measure on it. Assume that the two μ-random walks started from two arbitrary group elements can be coupled such that they eventually coincide with probability one. Prove that Γ does not admit any non-constant bounded harmonic functions.

E 5.7. Show that no Abelian group admits non-constant bounded harmonic functions with respect to any μ-random walk if the support of μ generates the group.

E 5.8. Show that the graph of an irrational rotation of the circle is not measurably 2-colorable. (Let r be the irrational rotation of S^1, and consider the graph G with $V(G) = S^1$, and connect every $x \in S^1$ to $r(x)$. We claim there is no $c : S^1 \to \{0, 1\}$ measurable map that is a proper vertex-2-coloring of G.)

Definitions

Definition 5.9. For a locally finite graph G define its Cheeger constant by

$$c(G) = \inf_{F \subset V(G) \text{ finite}} \frac{|\partial_E F|}{|F|},$$

where $\partial_E F$ stands for the edge boundary of F, i.e. set of edges with exactly one endpoint in F.

Definition 5.10 (Invariant random partition)**.** Let $\Gamma \curvearrowright X$, and let $\texttt{Part}(X) = \mathbb{N}^X$ stand for the space of (countable) partitions of X. The group Γ acts on $\texttt{Part}(X)$ in a natural way. A Γ-*invariant random partition* is an invariant probability measure μ on $\texttt{Part}(X)$.

Definition 5.11. Given a probability measure μ on a group Γ, one generates the μ-random walk from an element $g \in \Gamma$ by generating independent, identically distributed (abbreviated iid) elements $(g_i)_{i \in \mathbb{N}}$ from Γ according to μ, and considering the sequence $(X_n)_{n \in \mathbb{N}}$, where $X_n = g \cdot g_1 \cdot \ldots \cdot g_n$.

Definition 5.12. A coupling of two random walks (X_n) and (Y_n) is a random sequence $(\tilde{X}_n, \tilde{Y}_n)$ in the product space such that the two marginals, that is $(\tilde{X}_n)$ and $(\tilde{Y}_n)$ are the same as (X_n) and

(Y_n) in distribution. Morally, we generate the two walks together, using some shared randomness, in such a way that if one observes only one of the walks, they see the right random process.