6 Sixth exercise set

E 6.1. Exhibit a Følner sequence in the lamplighter group $C_2 \wr \mathbb{Z}$, but also show that it has exponential growth. (That is, the size of the r-ball around the identity grows exponentially in r.)

E 6.2. Let $\Gamma = \langle S \rangle$, $|S| < \infty$ be a finitely generated group, and assume $\mathrm{Cay}(\Gamma, S)$ has a positive Cheeger costant. Show that Γ admits a paradoxical decomposition.

E 6.3. Let B denote the infinite rooted binary tree. Show that a random element of $\mathrm{Aut}(B)$ produces an orbit-tree (the factor of B by the automorphism) that is a Galton-Watson tree.

E 6.4. Let T_d° denote the d-ary rooted tree, and $\Gamma = \mathrm{Alt}_f(T_d^\circ)$ the group of finitary alternating automorphisms of T_d°. Show that for $d \geq 5$ any nontrivial normal subgroup of Γ contains the pointwise stabilizer of some finite level of the tree. Is the same true for $\mathrm{Aut}_f(T_d^\circ)$?

E 6.5. Let $\Gamma \curvearrowright (X, \mu)$ be a p.m.p. action, and $x \in X$ a μ-random point. Show that $H = \mathrm{Stab}_\Gamma(x)$ is an Invariant Random Subgroup (IRS) of Γ. For $g \in \Gamma$, what is the probability of $g \in H$?

E 6.6. Let Γ be a group, and μ the step-distribution of some random walk on Γ. Consider the wreath product $G = C_2 \wr \Gamma$, and equip it with the random walk that has step-distribution $\nu * \mu * \nu$, where ν is the uniform distribution on $\{\mathrm{id}_G, \delta_{\mathrm{id}_\Gamma}\}$. Show that this random walk on G satisfies Liouville's theorem (i.e. all bounded harmonic functions are constant) if and only if the μ-random walk on Γ is recurrent.

(In $\bigoplus_\Gamma C_2$ the element $\delta_{\mathrm{id}_\Gamma}$ is the vector with value 1 at id_Γ, and 0 everywhere else. Taking a step on G with respect to $\nu * \mu * \nu$ means first flipping a fair coin to decide if we switch the lamp or not, then taking a μ-random step with the lamplighter, and finally flipping a fair coin again to decide if we switch the lamp at the new position.)

E 6.7. Let B denote the infinite rooted binary tree. Show that a random element of $\mathrm{Aut}(B)$ acts freely on the boundary of B with probability 1.

E 6.8. Show that for $A = \mathbb{Z}$, $10^{10^{10}}$ horizontal and vertical sheers suffice to get any permutation of $A \times A$ as their product.

Definitions

Definition 6.9 (Galton-Watson tree)**.** Fix a probability distribution (p_n) on $\mathbb{N}$. The Galton-Watson tree with descendant distribution (p_n) is obtained by starting from a root, adding a (p_n)-random number of edges connecting it to its "descendants", and then iterating the process by connecting a (p_n)-random number of descendants to every new vertex, and so on. (All the random choices are made independently.)

Definition 6.10 (Paradoxical decomposition)**.** In a group Γ a paradoxical decomposition is a collection of disjoint subsets $A_1, \dots, A_k, B_1, \dots, B_l$, and corresponding group elements $g_1, \dots, g_k, h_1, \dots h_l$ such that

$$\bigcup_{i=1}^{k} g_i A_i = \Gamma = \bigcup_{j=1}^{l} h_j B_j.$$

Definition 6.11 (Finitary alternating automorphism group of a rooted tree.)**.** $\mathrm{Alt}_f(T_d^\circ)$ is the group of automorphisms that

- permutes the d children of every vertex using elements from $\mathrm{Alt}(d)$;
- and only performs finitely many nontrivial such permutations. (In other words, every element moves the subtrees hanging off a finite level rigidly, but the depth depends on the element.)

The group $\mathrm{Aut}_f(T_d^\circ)$ is defined similarly, but without the first condition.

Definition 6.12 (Invariant Random Subgroup)**.** Let Γ be a countable group. An *Invariant Random Subgroup* (IRS) is a random subgroup H of Γ whose distribution is invariant under all conjugations by elements of Γ. That is, let $\mathrm{Sub}(\Gamma) = \{H \mid H \leq \Gamma\}$ denote the space of its subgroups, endowed with the subspace topology inherited from 2^Γ and the conjugation action of Γ. An IRS is a Γ-invariant probability measure on $\mathrm{Sub}(\Gamma)$.