7 Seventh exercise set

E 7.1. Condition the simple random walk on T_d to converge to a given boundary point. Describe this new random walk directly!

E 7.2. Let $\Gamma \leq \mathrm{Aut}(T_d^\circ)$. When is the action $\Gamma \curvearrowright (\partial T_d^\circ, \mu)$ ergodic? (Here ∂T_d° is the boundary, and μ is the standard $\mathrm{Aut}(T_d^\circ)$-invariant measure, one can pick a random infinite non-backtracking path from the root by making a uniform random choice among the d continuations at every step.)

E 7.3. Show an example of an IRS in $\mathrm{Sym}_f(\mathbb{N})$ that has infinite index almost surely.

E 7.4. Let $\Gamma = \mathrm{Alt}_f(T_d)$ (or $\mathrm{Aut}_f(T_d)$). Pick two independent μ-random points x and y from ∂T_d°. Show that $H = \mathrm{Stab}_\Gamma(\{x, y\})$ is an IRS in Γ. Is it ergodic?

E 7.5. Show that an ergodic, infinite index nontrivial IRS of the free group has infinite rank almost surely.

E 7.6. Let Γ be a countably infinite group, and

$$\Omega = \{0,1\}^\Gamma = \{\omega : \Gamma \to \{0,1\}\}$$

be the space of all 0-1-colorings with the product topology. Let u_2 stand for the uniform $\{1/2, 1/2\}$ measure on $\{0,1\}$ and endow Ω with the product measure $\mu = u_2^\Gamma$. Γ acts on (Ω, μ) by shifting the coloring:

$$g.\omega(h) = \omega(g^{-1}h).$$

Show that this action is ergodic.

E 7.7. (Ornstein-Weiss example) Show that $F_2 \curvearrowright (\{0,1,2,3\}^{F_2}, u_4^{F_2})$ is a factor of $F_2 \curvearrowright (\{0,1\}^{F_2}, u_2^{F_2})$.

E 7.8. Let B denote the infinite rooted binary tree. Show that two independent random elements of $\mathrm{Aut}(B)$ generate a free group with probability 1 that acts freely on the boundary.

Definitions

Definition 7.9 (Ergodic action)**.** A probability measure preserving (p.m.p.) action $\Gamma \curvearrowright (X, \mu)$ (i.e. an action of Γ on the probability space (X, μ) by measurable bijections that preserve μ) is called *ergodic*, if for every Γ-invariant measurable subset $A \subseteq X$ we have $\mu(A) = 0$ or $\mu(A) = 1$. (A is Γ*-invariant, if* $\Gamma.A = A$.)

Definition 7.10 (Factor)**.** Let α and β be two p.m.p. actions of the group Γ on the spaces (X, μ) and (Y, ν) respectively. We say β is a *factor* of α, if (up to discarding nullsets) there exists a surjective measure preserving Γ-equivariant map $\Phi : X \to Y$. (Γ-equivariant means $\Phi(\gamma.x) = \gamma.\Phi(x)$ for all x and γ.)