1 First exercise set

(Definitions on the next page.)

E 1.1. Show that the 3-cycles generate $\mathrm{Alt}_f(\mathbb{N})$, the group of finitely supported, even permutations of $\mathbb{N}$.

E 1.2. (Double commutators) Show that $\mathrm{Alt}_f(\mathbb{N})$ is simple.

E 1.3. Show that an automorphism of a d-regular (infinite) tree fixes either a vertex, an edge, or a biinfinite geodesic, on which it acts by translation.

E 1.4. Show that recurrence/transience of a random walk on an infinite, connected graph does not depend on the choice of starting vertex.

E 1.5. Show that transience is equivalent to the expected number of returns to the root being finite.

E 1.6. Show that the fundamental group of a graph is a free group of rank equal to the number of edges of outside an arbitrary spanning tree.

E 1.7. Show that two translations along disjoint axes on the d-regular tree generate a free group.

E 1.8. Show that the lamplighter group $C_2 \wr Z$ is not finitely presented.

E 1.9. Consider the simple random walk on $\mathbb{Z}$ between a and b ($a < b$), stopped at the endpoints. Condition the walk on ending up at b.

a) Show that after the conditioning the random walk is still a Markov chain.

b) Determine the transition probabilites.

E 1.10. Show that on the d-regular tree the Markov operator has no eigenfunctions in $\ell^2(V)$ ($d \geq 2$).

E 1.11. Show that every group has $0, 1, 2$ or infinitely many ends, and the number of ends does not depend on the choice of generating set.

2 Definitions

Definition 2.1 (Recurrence, transience)**.** Let G be a connected, infinite graph and $o \in V(G)$ a starting vertex. We say that a random walk on G is transient, if (started from o) with positive probability the walk never returns to o again. Otherwise it is called recurrent, i.e. when the walk returns to o with probability 1.

Definition 2.2 (Wreath product and lamplighter group)**.** The (restricted) *wreath product* of two groups L and B, denoted $L \wr B$ is the semidirect product $\bigoplus_B L \rtimes B$, where B acts on $\bigoplus_B L$ by translation of coordinates, i.e. if $\omega \in \bigoplus_B L$ and $b \in B$, then $b.\omega(a) = \omega(b^{-1}a)$. The *lamplighter group* is the wreath product $C_2 \wr \mathbb{Z}$, and it should be thought of as a person walking up and down an infinite street with lamps at every integer, and switching the light at certain lamps.

Definition 2.3 (Finitely presented)**.** A group Γ is *finitely presented* if it has a representation with finitely many generators and relations, i.e. $\Gamma = \langle S \mid R \rangle$ with $|S|, |R| < \infty$.

Definition 2.4 (Markov operator)**.** The *Markov operator* on a locally finite graph G is the operator that averages the value at the neighbors of all vertices, i.e.

$$M : \ell^2(V) \to \ell^2(V), \quad (Mf)(u) = \frac{1}{\deg(u)} \sum_{(u,v) \in E} f(v).$$

Definition 2.5 (Number of ends)**.** Let G be an infinite, locally finite, connected graph, and for a finite subset of vertices F denote by $c_\infty(F)$ the number of *infinite* components of $G \setminus F$. The *number of ends* of G is defined as $\sup\{c_\infty(F) \mid F \subseteq V \text{ finite}\}$. (It is infinite if $c_\infty(F)$ is not bounded.) For a finitely generated group $\Gamma = \langle S \rangle$ the number of ends is computed for the Cayley graph $\mathrm{Cay}(\Gamma, S)$.