2 Second exercise set

E 2.1. Show that a maximal filter is an ultrafilter. As a consequence, Zorn's lemma implies the existence of a non-principal ultrafilter.

E 2.2. Given an ultrafilter U on $\mathbb{N}$ show that all bounded sequences admit a unique ultralimit.

E 2.3. Let Γ be a transitive permutation group. Show that Γ is primitive if and only if the point stabilizer Γ_α is a maximal subgroup in Γ.

E 2.4. Let M be a symmetric real matrix. Show (without using the spectral theorem or characteristic polynomials) that all vectors maximizing the Rayleigh-quotient

$$R(v) = \frac{\langle vM, v\rangle}{\langle v, v\rangle}$$

are eigenvectors of M.

E 2.5. Show that all symmetric real matrices have an orthonormal eigenbasis.

E 2.6. (Neumann's lemma.) Assume $\Gamma \leq \mathrm{Sym}_f(\mathbb{N})$ has infinite orbits, and $A, B \subseteq \mathbb{N}$ finite. Show that there is some $g \in \Gamma$ such that $gA \cap B = \emptyset$. Here $\mathrm{Sym}_f(\mathbb{N})$ is the group of finitary permutations of $\mathbb{N}$.

E 2.7. Show that the graph of an irrational rotation of the circle is not measurably 2-colorable. (Let r be the irrational rotation of S^1, and consider the graph G with $V(G) = S^1$, and connect every $x \in S^1$ to $r(x)$. We claim there is no $c : S^1 \to \{0, 1\}$ measurable map that is a proper vertex-2-coloring of G.)

E 2.8. Show that $\mathrm{Sym}_f(\mathbb{N})$ can not be obtained as the product of finitely many Abelian subgroups.

E 2.9. (Pyramid scheme.) Let G be an infinite, locally finite, connected graph. Define the *value* of a pyramid scheme $f : \vec{E} \to \mathbb{R}$ to be the infimum of gains, that is

$$\texttt{value}(f) = \inf_{v \in V(G)} \sum_{e^+ = v} f(e) - \sum_{e^- = v} f(e).$$

How large can the value of a bounded ($||f||_\infty \leq 1$) pyramid scheme be?

E 2.10. Show that on $\mathbb{Z}$ every bounded harmonic function is constant. On the other hand, construct a non-constant bounded harmonic function on the 3-regular tree T_3.

E 2.11. Let Γ be a group, and μ the step-distribution of some random walk on Γ. Consider the wreath product $G = C_2 \wr \Gamma$, and equip it with the random walk that has step-distribution $\nu * \mu * \nu$, where ν is the uniform distribution on $\{\mathrm{id}_G, \delta_{\mathrm{id}_\Gamma}\}$. Show that this random walk on G satisfies Liouville's theorem (i.e. all bounded harmonic functions are constant) if and only if the μ-random walk on Γ is recurrent.

Remark 2.12. In $\bigoplus_\Gamma C_2$ the element $\delta_{\mathrm{id}_\Gamma}$ is the vector with value 1 at id_Γ, and 0 everywhere else. Taking a step on G with respect to the convolution $\nu * \mu * \nu$ means first flipping a fair coin to decide if we switch the lamp or not, then taking a μ-random step with the lamplighter, and finally flipping a fair coin again to decide if we switch the lamp at the new position.

Definitions

Definition 2.13. (Filter, ultrafilter) Let Ω be a set. A family of subsets $F \subseteq 2^\omega$ is a *filter*, if it is nontrivial ($\emptyset \notin F,\ \Omega \in F$), upward closed ($A \in F,\ A \subseteq B \Rightarrow B \in F$), and closed under finite intersection ($A, B \in F \Rightarrow A \cap B \in F$). A filter U is an *ultrafilter*, if for any $A \subseteq \Omega$ either $A \in U$ or $\Omega \setminus A \in U$. An ultrafilter U is *principal* if it consists exactly of the sets containing a fixed element $\omega \in \Omega$, i.e. $U = \{A \subseteq \Omega \mid \omega \in A\}$.

Definition 2.14 (Ultralimit)**.** Given an ultrafilter U and a sequence (a_n) of real numbers we say that A is the *ultralimit* of (a_n) if $\forall \varepsilon > 0,\ \{n \in \mathbb{N} \mid |a_n - A| < \varepsilon\} \in U$. We denote this by $\lim_U a_n = A$.

Definition 2.15 (Primitive permutation group)**.** For a transitive permutation group $\Gamma \subseteq \mathrm{Sym}(\Omega)$ a partition $P = \{P_i\}_{i \in I}$ of Ω is Γ-*invariant*, if for every $g \in \Gamma$, and $\omega_1, \omega_2 \in \Omega$ that are in the same part P_i, the images $g.\omega_1$ and $g.\omega_2$ are also in the same part P_j (but maybe $i \neq j$). The group Γ is *primitive*, if there is no Γ-invariant partition of $\mathbb{N}$.

Definition 2.16 (Harmonic function)**.** Given a random walk on a graph (or a group) G, a function $u : V(G) \to \mathbb{R}$ is *harmonic*, if it is invariant under the Markov operator, i.e. $Mu = u$. That is, the value at every point is equal to the average of the values at the neighbors, weighted with the transition probabilities.