3 Third exercise set

E 3.1. Show that the lamplighter group $C_2 \wr \mathbb{Z}$ is solvable, but not nilpotent.

E 3.2. Let G be a connected, locally finite graph, and let $p_{x,y,n}$ denote the probability that a simple random walk started at x is at y after n steps. Show that

$$\lim_{n\to\infty} (p_{o,o,2n})^{1/2n}$$

does not depend on the choice of o.

E 3.3. Consider a $2 \times n$ matrix, filled by the integers from 1 to $2n$. A permutation of the entries is *vertical* if it fixes the first coordinate of every entry and it is *horizontal* if it fixes their second coordinates. Prove that every permutation of the entries can be obtained as the product of a vertical, a horizontal and a vertical permutation.

E 3.4. Let H be a subgroup of the free group $F(S)$. Show that the fundamental group of the coset Schreier graph $\mathrm{Sch}(F/H, S)$ is isomorphic to H. As a consequence deduce that H is itself free. In case H has finite index derive a formula for its rank.

E 3.5. Let ∂T denote the boundary of the regular tree T_d ($d \geq 3$), i.e. the space of infinite geodesics (starting from a fixed root) endowed with the product topology. Show that $\mathrm{Aut}(T)$ acts on ∂T by homeomorphisms, but there is no $\mathrm{Aut}(T)$-invariant Borel probability measure on ∂T.

E 3.6. Show that a finitely generated Følner amenable group Γ admits a normalized left-invariant finitely additive measure (on the whole of 2^Γ). (Finitely generated is not necessary.)

E 3.7. Let Γ be a countably infinite group, and let

$$\Omega = \{0,1\}^\Gamma = \{\omega : \Gamma \to \{0,1\}\}$$

be the space of all 0-1-colorings with the product topology. Let u_2 stand for the uniform $\{1/2, 1/2\}$ measure on $\{0,1\}$ and endow Ω with the product measure $\mu = u_2^\Gamma$. Γ acts on (Ω, μ) by shifting the coloring: $g.\omega(h) = \omega(g^{-1}h)$.

Show that this action is ergodic.

E 3.8. (Jordan-Wielandt theorem) Show that the only primitive (transitive) subgroups of $\mathrm{Sym}_f(\mathbb{N})$ are $\mathrm{Alt}_f(\mathbb{N})$ and $\mathrm{Sym}_f(\mathbb{N})$.

Definitions

Definition 3.9 (Solvable and nilpotent group)**.** A group Γ is *solvable* if it can be constructed from abelian groups using extensions. Equivalently, its derived series terminates in the trivial subgroup. (The derived series is defined by $\Gamma_{i+1} = [\Gamma_i, \Gamma_i]$ starting from $\Gamma_0 = \Gamma$.)

On the other hand, Γ is *nilpontent* if its lower central series terminates in the trivial subgroup after finitely many steps. (The lower central series is defined by $\Gamma_{i+1} = [\Gamma_i, \Gamma]$ starting from $\Gamma_0 = \Gamma$.)

Definition 3.10 (Schreier graph)**.** Given a (finitely generated) group $\Gamma = \langle S \rangle$ with a generating set, and an action $\Gamma \curvearrowright X$ the *Schreier graph* $\mathrm{Sch}(\Gamma, X, S)$ has vertex set X, and we connect every $x \in X$ to $s.x$ for every generator $s \in S$. (The Cayley graph is a special case, with $\Gamma \curvearrowright \Gamma$ the left or right multiplication.) For a subgroup $H \leq \Gamma$ the coset Schreier graph $\mathrm{Sch}(\Gamma/H, S)$ is the one corresponding to the action $\Gamma \curvearrowright \Gamma/H$.

Definition 3.11 (Følner condition of amenability)**.** Let $\Gamma =< S >$, $\mathrm{id} \in S$ and $|S| < \infty$. We say Γ is *Følner amenable*, if $\mathrm{Cay}(\Gamma, S)$ contains finite subsets with arbitrarily small boundaries compared to their size, i.e. $\forall \varepsilon > 0\, \exists F \subset \Gamma$ finite such that $|S \cdot F \setminus F| < \varepsilon |F|$. This does not depend on the choice of S. If Γ is not finitely generated, one can define the same by asking that such Følner sets F exist for any finite subset S and $\varepsilon > 0$.

Definition 3.12 (Invariant finitely additive measure)**.** Given a group Γ, a *normalized (left) invariant finitely additive measure* on the group is a set function $m : 2^\Gamma \to [0, 1]$ that is finitely additive, $m(\Gamma) = 1$ and $m(gA) = m(A)$ for all $A \subseteq \Gamma$ and $g \in \Gamma$.

Definition 3.13 (Ergodic action)**.** A probability measure preserving (p.m.p.) action $\Gamma \curvearrowright (X, \mu)$ (i.e. and action of Γ on the probability space (X, μ) by measurable bijections that preserve μ) is called *ergodic*, if for every Γ-invariant measurable subset $A \subseteq X$ we have $\mu(A) = 0$ or $\mu(A) = 1$. (A is Γ-*invariant, if* $\Gamma.A = A$.)