4 Fourth exercise set

E 4.1. Let a and b be two bijections of the set X. Assume there exist disjoint subsets $A, B \subseteq X$ such that $\forall n \neq 0$ we have $a^n.B \subseteq A$ and $b^n.A \subseteq B$. Show that a and b generate a free group.

E 4.2. We call a subset of the integers *absolutely small*, if it has zero measure with respect to any invariant finitely additive measure. Show that a set is absolutely small if and only if the closure of its translates (in the product topology) contains only zero-density subsets (in the classical sense).

E 4.3. Show that the trajectory of the simple random walk on the d-regular tree T_d ($d \geq 3$) converges to a point on the boundary ∂T with probability 1.

E 4.4. We call a vertex of a Cayley graph a *trap*, if it has no neighbour that is at a greater distance to the identity than itself. Give an example of an infinite Cayley graph of a finitely generated group with a trap.

E 4.5. Let A be an Abelian group. A *vertical sheer* of $A \times A$ is a bijection $F(x, y) = (x, y + f(x))$ where $f : A \to A$ is an arbitrary function. We define horizontal sheers accordingly. Show that for $A = C_p$, horizontal and vertical sheers generate the full alternating group $\mathrm{Alt}(A \times A)$ but you need at least $C|A|$ sheers to get all elements of $\mathrm{Alt}(A \times A)$ as their product.

E 4.6. Show that for a d-regular graph G the Markov operator is self adjoint on $\ell^2(V(G))$ and

$$||M||_2 = \lim_{n\to\infty} \left(p_{o,o,2n}\right)^{1/2n}.$$

E 4.7. (Ornstein-Weiss example) Show that $F_2 \curvearrowright (\{0,1,2,3\}^{F_2}, u_4^{F_2})$ is a factor of $F_2 \curvearrowright (\{0,1\}^{F_2}, u_2^{F_2})$. (As before, u_n stands for the uniform distribution on n elements.)

E 4.8. Let ∂T denote the boundary of the regular tree T_d ($d \geq 3$) as before. Find a σ-finite Borel measure on the product $\partial T \times \partial T$ that is $\mathrm{Aut}(T)$-invariant.

Definitions

Definition 4.9 (Density)**.** The *density* of a set $A \subseteq \mathbb{Z}$ is

$$d(A) = \lim_{n\to\infty} \frac{|A \cap [-n, n]|}{2n+1},$$

whenever the limit exists.

Definition 4.10 (Factor)**.** Let α and β be two p.m.p. actions of the group Γ on the spaces (X, μ) and (Y, ν) respectively. We say β is a *factor* of α, if (up to discarding nullsets) there exists a surjective measure preserving Γ-equivariant map $\Phi : X \to Y$. (Γ-equivariant means $\Phi(\gamma.x) = \gamma.\Phi(x)$ for all x and γ.)