5 Fifth exercise set

E 5.1. Let B_n denote the rooted binary tree of depth n. Show that $\mathrm{Aut}(B_n)$ is exactly the Sylow 2-subgroup of $\mathrm{Sym}(2^n)$. How does this generalize to other primes?

E 5.2. Give an example of a $\mathrm{Sym}_f(\mathbb{N})$-invariant random partition of $\mathbb{N}$. (How many points is this problem worth in total?)

E 5.3. Show that if G is a transitive graph, then

$$p_{x,x,2n} \geq p_{x,y,2n}$$

for any pair $x, y \in V(G)$.

E 5.4. (Goursat lemma, special case) Let G be a simple group, and $H < G \times G$ a proper subgroup such that both coordinate projections are surjective onto G. Show that $H = \{(g, \varphi(g)) \mid g \in G\}$ where φ is a fixed automorphism of G.

E 5.5. Exhibit a Følner sequence in the lamplighter group $C_2 \wr \mathbb{Z}$, but also show that it has exponential growth. (That is, the size of the r-ball around the identity grows exponentially in r.)

E 5.6. Let B denote the infinite rooted binary tree. Show that a random element of $\mathrm{Aut}(B)$ produces an orbit-tree (the factor of B by the automorphism) that is a Galton-Watson tree.

E 5.7. Let G be a bounded degree, infinite, connected graph with nonzero Cheeger constant. Show that there exists a $C > 0$ and $f_1, f_2 : V(G) \to V(G)$ injective functions such that $d(x, f_i(x)) < C$ for all $x \in V(G), i = 1, 2$ and $f_1(V(G)) \cap f_2(V(G)) = \emptyset$.

E 5.8. Let G be a finite d-regular graph, and let ρ be the spectral radius of its adjacency matrix. Let $C \subseteq V(G)$, and let A' stand for the adjacency matrix of the induced graph on C. Show that every eigenvalue of A' is at most

$$cd + (1 - c)\rho,$$

where $c = |C| \, / \, |V(G)|$.

Definitions

Definition 5.9 (Invariant random partition)**.** Let $\Gamma \curvearrowright X$, and let $\texttt{Part}(X) = \mathbb{N}^X$ stand for the space of (countable) partitions of X. The group Γ acts on $\texttt{Part}(X)$ in a natural way. A Γ-*invariant random partition* is an invariant probability measure μ on $\texttt{Part}(X)$.

Definition 5.10 (Spectral radius)**.** For a finite d-regular graph G let $d = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the eigenvalues of its adjacency matrix. The spectral radius of G is $\rho(G) = \max\{|\lambda_2|, |\lambda_n|\}$.

Definition 5.11 (Galton-Watson tree)**.** Fix a probability distribution (p_n) on $\mathbb{N}$. The Galton-Watson tree with descendant distribution (p_n) is obtained by starting from a root, adding a (p_n)-random number of edges connecting it to its "descendants", and then iterating the process by connecting a (p_n)-random number of descendants to every new vertex, and so on. (All the random choices are made independently.)