6 Sixth exercise set

E 6.1. Let $\Gamma \curvearrowright (X, \mu)$ be a p.m.p. action, and $x \in X$ a μ-random point. Show that $H = \mathrm{Stab}_\Gamma(x)$ is an Invariant Random Subgroup (IRS) of Γ. For $g \in \Gamma$, what is the probability of $g \in H$?

E 6.2. Show that an infinite index nontrivial normal subgroup of the free group has infinite rank.

E 6.3. Show that if a finitely generated group Γ is amenable then the Markov operator (for the simple random walk) on its Cayley graph has norm 1.

E 6.4. Let B denote the infinite rooted binary tree. Show that a random element of $\mathrm{Aut}(B)$ acts freely on the boundary of B with probability 1.

E 6.5. Condition the simple random walk on T_d to converge to a given boundary point. Describe this new random walk directly!

E 6.6. Show that for $A = \mathbb{Z}$, $10^{10^{10}}$ horizontal and vertical sheers suffice to get any permutation of $A \times A$ as their product.

E 6.7. Show an example of an IRS in $\mathrm{Sym}_f(\mathbb{N})$ that has infinite index almost surely.

E 6.8. Let G be a d-regular graph, and let $\varepsilon = 1 - \rho/d$ be the spectral gap of its Markov operator. Show that for an arbitrary subset $A \subseteq V(G)$ the probability that the simple random walk started from a random vertex of G does not meet A in n steps is at most

$$(1 - \varepsilon a)^n (1 - a),$$

where $a = |A|/|V(G)|$.

Definitions

Definition 6.9 (Invariant Random Subgroup)**.** Let Γ be a countable group. An *Invariant Random Subgroup* (IRS) is a random subgroup H of Γ whose distribution is invariant under all conjugations by elements of Γ. That is, let $\mathrm{Sub}(\Gamma) = \{H \mid H \leq \Gamma\}$ denote the space of its subgroups, endowed with the subspace topology inherited from 2^Γ and the conjugation action of Γ. An IRS is a Γ-invariant probability measure on $\mathrm{Sub}(\Gamma)$.