7 Seventh exercise set

E 7.1. Show that an ergodic, infinite index nontrivial IRS of the free group has infinite rank almost surely.

Hint: how many finitely generated subgroups are there?

E 7.2. Let B denote the infinite rooted binary tree. Show that two independent random elements of $\mathrm{Aut}(B)$ generate a free group with probability 1 that acts freely on the boundary.

Hint: Let $w(a,b) = u_1 \dots u_n$ be a reduced word of length n, and let $w_i = u_i \dots u_n$. Show that with probability 1 there is a level L_k of the tree where x, $w_n.x$, $\dots w_1.x$ are all distinct for every $x \in L_k$.

E 7.3. Show the converse of Exercise 6.6, i.e. that if a group Γ is non-amenable, the the Markov operator on its Cayley graph G has norm strictly less than 1.

a) Note that $||M|| = \sup_{f \in H} \left\{ \frac{\langle Mf, f\rangle}{\langle f, f\rangle} \right\}$, where $H \subseteq \ell^2(\Gamma)$ are the finitely supported functions.

b) Let C stand for the edge Cheeger constant of G. Given a function $g : \Gamma \to [0,\infty)$ consider all level sets $\{x \mid g(x) > t\}$ of g and show that

$$C \cdot \sum_{v \in V(G)} g(v) \le \sum_{(x,y) \in E(G)} |g(y) - g(x)|.$$

c) Show that

$$\langle f, f\rangle^2 \le \frac{1}{C^2} \Big[\langle f, f\rangle - \langle Mf, f\rangle\Big] \Big[\langle f, f\rangle + \langle Mf, f\rangle\Big].$$

E 7.4. (Vershik's theorem) We characterize all ergodic IRS's H of $\mathrm{Sym}_f(\mathbb{N})$.

a) We will repeatedly use the obvious fact that there is no uniform distribution on a countably infinite set.

b) First consider the orbits of H, and use De Finetti's theorem (Exercise 5.2).

c) Let B_0 be the set of fixed points of H, and $B_1, B_2, \dots$ denote the other orbits. Show that the action of H on each B_i $(i > 0)$ is primitive almost surely, and use the Jordan-Wielandt theorem (Exercise 3.8).

d) We aim to show that $H \lhd \Gamma$, where $\Gamma = \mathrm{Sym}_f(B_1) \times \mathrm{Sym}_f(B_2) \times \dots$. Show that it is equivalent to prove $H' = \mathrm{Alt}_f(B_1) \times \mathrm{Alt}_f(B_2) \times \dots = \Gamma'$.

e) By item (c) the projection p_i is surjective from H' to $\mathrm{Alt}_f(B_i)$. Use (a) and Goursat's lemma (Exercise 5.4) to show that for any pair of indices i, j the projection π_{ij} is also surjective from H' to $\mathrm{Alt}_f(B_i) \times \mathrm{Alt}_f(B_j)$. (Fact: $\mathrm{Aut}(\mathrm{Alt}_f(\mathbb{N})) = \mathrm{Sym}(\mathbb{N})$.)

f) Show that for any finite set of coordinates $\mathbf{i} = (i_1, \dots, i_l)$ the projection $\pi_{\mathbf{i}}$ is surjective from H' to $\mathrm{Alt}_f(B_{i_1}) \times \dots \times \mathrm{Alt}_f(B_{i_l})$.

g) Finally show the same as in (f) for all the (possibly infinitely many) coordinates.