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# ON THE NUMBER OF ANTIPODAL OR STRICTLY ANTIPODAL PAIRS OF POINTS IN FINITE SUBSETS OF $\mathbf{R}^d$, II.

E. MAKAI, Jr. (Budapest)[1] and H. MARTINI (Dresden)

**Abstract**

The paper is a continuation of [MM], namely containing several statements related to the concept of antipodal and strictly antipodal pairs of points in a subset $X$ of $\mathbf{R}^d$, which has cardinality $n$. The points $x_i, x_j \in X$ are called antipodal if each of them is contained in one of two different parallel supporting hyperplanes of the convex hull of $X$. If such hyperplanes contain no further point of $X$, then $x_i, x_j$ are even strictly antipodal. We shall prove some lower bounds on the number of strictly antipodal pairs for given $d$ and $n$. Furthermore, this concept leads us to a statement on the quotient of the lengths of longest and shortest edges of special $d$-simplices, and finally a generalization (concerning strictly antipodal segments) is proved.

## § 1. Introduction

For basic notation and usual abbreviations the reader is referred to [G2].

Let $X = \{x_1, \ldots, x_n\} \subset \mathbf{R}^d$ ($x_i \neq x_j$ for $i \neq j$), and let $P$ denote its convex hull. We always suppose that $X$ is not contained in a hyperplane of $\mathbf{R}^d$. The points $x_i, x_j \in X$ are called *antipodal* (*strictly antipodal*), if there are different parallel supporting hyperplanes $H', H''$ of $P$ with $x_i \in H'$, $x_j \in H''$ (with $\{x_i\} = P \cap H'$, $\{x_j\} = P \cap H''$). These concepts are due to V. Klee (cf. [G2], p. 420) and to B. Grünbaum (see [G1]), respectively.

We denote by $a(X)$ ($sa(X)$) the number of antipodal (strictly antipodal) pairs $x_i, x_j$ of $X$. When investigating $a(X)$ ($sa(X)$), we may and will suppose that each $x_i$ is a boundary point (a vertex) of the polytope $P$. By [G1], $sa(X)$ is half the vertex number of the *difference body* $P + (-P)$ of $P$.

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Lower estimates for $sa(X)$, and for $a(X)$ (supposing $X = \mathrm{vert}P$), in terms of $|X|$ and for given $d$, have been given by [N] and [SN1], respectively (cf. also [SN2]). In particular, for $d = 3$, [N] determined the number $c(n) = \min\{sa(X) \big| |X| = n\}$ for every $n$, namely

$$
\begin{aligned}
c(4) = c(5) = c(7) = c(9) = 6,\\
c(n) = \lceil \frac{n}{2} \rceil \text{ otherwise.}
\end{aligned} \tag{1}
$$

Partially based on results of [SN1], [N] derived for $d > 3$, with $c_d(n) = \min\{sa(X) \big| |X| = n\}$,

$$(2_1) \qquad c_d(d+1) = \frac{d(d+1)}{2},$$

$$(2_2) \qquad c_d(n) \geq 3(d-1), \quad d+2 \leq n \leq 2d-2,$$

$$(2_3) \qquad c_d(2d-1) = 3(d-1),$$

$$(2_4,) \qquad c_d(n) = 2d, \quad n \text{ odd with } 2d+1 \leq n \leq 4d-1,$$

$$(2_5) \qquad c_d(n) = \lceil \frac{n}{2} \rceil \text{ otherwise.}$$

Upper estimates for $a(X)$, $sa(X)$, or for $a(X)$ supposing $X = \mathrm{vert}\, P$, have been given by [G1] and [MM]. For a more detailed overview on these results cf. e.g. [MM], §§ 1, 2.

We recall that a centrally symmetric convex polytope $Q$ is irreducible, if it is not of the form $\frac{1}{2}(P + (-P))$, where $P$ is a convex polytope that is no translate of $Q$. Then (1) implies that a centrally symmetric convex polyhedron $Q$ in $\mathbf{R}^3$, of $\leq 10$ vertices is irreducible, at least w.r.t. convex polyhedra $P$ with an odd number of vertices. About the concept of irreducibility cf. also the very recent paper [Y], which shows e.g. that for $d \geq 3$ each centrosymmetric convex polytope in $\mathbf{R}^d$, with $< 4d$ vertices, is irreducible (that implies (1) and $(2_4)$).

## § 2. Results

The aim of our paper is first to give alternative proofs to (1) and $(2_5)$. Other results from [N], namely $(2_1)$, $(2_2)$ and $(2_3)$, were also independently rediscovered by the authors, before learning of [N], and presented at the Oberwolfach conference on Combinatorial Convexity and Algebraic Geometry in 1989 (see the corresponding research report 35/1989). But since here the authors used quite the same methods as [N] (only in other terms), namely induction over $d$ with starting point $d = 2$ and $n = 3$, we will not repeat these proofs. Second, we will prove a proposition, used in the first part of our paper [MM], but not proved there. It is actually independent of antipodality, and is a variant of K. Schütte's theorem (cf. [Sch]) on the minimal quotient of the maximal and minimal distances among $d + 1$ points in $\mathbf{R}^{d-1}$, see also [DW]. (Actually, Schütte's theorem is the case $\phi = \pi$ of our proposition, which is however for $d > 3$ not covered by our proposition.)

PROPOSITION 1. *Let in* $\mathbf{R}^d$, $d \geq 2$, *a simplex* $a_1 \dots a_{d-1} b_1 b_2$ *have two facets* ($a_1 \dots a_{d-1} b_1$ *and* $a_1 \dots a_{d-1} b_2$, *say), enclosing an angle* $\geq \phi$, *where* $1/d \geq \cos\phi \geq -1/d^2 - 3d + 1$ *(the lower inequality holding for* $d \geq 3$*). Then the quotient of the lengths of the longest and shortest edges is*

$$\geq \sqrt{\frac{2 - 2\cos\phi}{1 + \frac{d-2}{d-1}(1 - \cos\phi)}}.$$

*Equality holds if and only if the angle of the mentioned two facets is* $\phi$, *both facets are right pyramids over a regular* $(d-2)$*-simplex as base (namely over* $a_1 \dots a_{d-1}$*), and*

$$a_{i_1} a_{i_2} = b_1 b_2 = a_i b_j \cdot \sqrt{\frac{2 - 2\cos\phi}{1 + \frac{d-2}{d-1}(1 - \cos\phi)}},$$

*for each* $i, i_1, i_2 \leq d-1$, $j \leq 2$.

Third we will prove a proposition dealing with a variant of antipodality. Let $S^k = \{s_1^k, \dots, s_n^k\}$ be a finite set of $k$-simplices in $\mathbf{R}^d$. We call $S^k$ (strictly) $k$-antipodal if for any $i \neq j$ there are different parallel supporting hyperplanes $H', H''$ of the convex hull $P$ of the union of the simplices belonging to $S^k$, such that $s_i^k \subset H', s_j^k \subset H''$ (resp. $s_i^k = P \cap H', s_j^k = P \cap H''$). One can conjecture that, analogously to the Danzer–Grünbaum theorem holding for the case $k = 0$ (cf. [DG]), a $k$-antipodal set of $k$-simplices has at most $2^{d-k}$ elements, which is attained for $2^{d-k}$ $k$-simplices on $2^{d-k}$ parallel $k$-faces of a cube. (Formerly, also I. Bárány and V. Soltan have formulated this conjecture.)

For the strictly antipodal case we have the following

PROPOSITION 2. *If* $S^1 = \{s_1^1, \dots, s_n^1\}$ *is a strictly antipodal set of* $n$ *1-simplices in* $\mathbf{R}^3$, *which are pairwise skew, then* $n \leq 3$. *This bound is sharp.*

All these results have been announced in [MM].

## § 3. Proofs to some lower bounds on $sa(X)$

1. Obviously, $(2_1)$ is trivial, since then $P$ has to be a $d$-simplex having $\frac{d(d+1)}{2}$ edges, each connecting a strictly antipodal pair from vert $P$. The case when $c_d(n) = \frac{n}{2}$ ($n$ even) is also clear. Namely, every vertex of $P$ belongs to at least one strictly antipodal pair. On the other hand, a vertex pair $x, y$ forms precisely one strictly antipodal pair (i.e. neither $x$ nor $y$ belongs to another such pair) if and only if the edge systems around $x$ and $y$ consist of only pairwise parallel and oppositely directed edges (directed starting from $x$, resp. $y$). For seeing this, one uses the concept of half the vertex number of the difference body $P + (-P)$, which is equal to $sa(X)$ with $X$ as vertex set of $P$ (cf. [G1] and § 1). More precisely, since faces of convex bodies in parallel supporting hyperplanes at the same sides of these bodies behave additively under Minkowski addition, for a pair $H, \bar{H}$ of different parallel supporting hyperplanes of $P$ with $H \cap P$ an edge and $\bar{H} \cap P$ not containing an edge

parallel to $H \cap P$, $P + (-P)$ has translates of $H \cap P$ as edges (and not as parts of edges) in both supporting hyperplanes parallel to $H$. Thus, $c_d(n) = \frac{n}{2}$ ($n$ even) characterizes all convex polytopes having pairwise parallel and oppositely directed edge systems as described above. (Central symmetry is sufficient, but in general not necessary.)

Further on, we shall repeatedly use that concept, whereas [N] considered outer normal cones of vertices of the polytope $P$.

On this base, we will construct suitable convex $d$-polytopes $P$ yielding the sharp lower bound for odd $n \geq 4d - 1$: Choose two $(d-1)$-simplices $S_1, S_2$ in different parallel hyperplanes such that their convex hull $\mathrm{co}(S_1, S_2)$ is centrally symmetric. Consider now the polytope $Q$ having the same set of facet hyperplanes as the polytope $\mathrm{co}(S_i)$, except the affine hulls $\mathrm{aff}(S_i)$, $i = 1, 2$. Intersect $Q$ by two different parallel hyperplanes $H_1, H_2$ with $\mathrm{co}(S_1, S_2) \subset \mathrm{co}(H_1, H_2)$ and $H_i \cap \mathrm{int}\, Q \neq \emptyset$ $(i = 1, 2)$. If $H_1$ contains no vertex and $H_2$ contains precisely one vertex of $\mathrm{co}(S_1, S_2)$, then $\mathrm{co}(H_1, H_2) \cap Q$ is a convex $d$-polytope with exactly $4d - 1$ vertices. This polytope has $2d - 2$ vertex pairs with pairwise parallel and oppositely directed edges in each case, and additionally it has a triple of vertices yielding two strictly antipodal pairs, i.e., altogether there are $2d$ strictly antipodal vertex pairs. Now one can add an arbitrary number of vertex pairs $x_{1i}, x_{2i} \notin \mathrm{co}(H_1, H_2)$ close to the centroids of the facets $F_1, F_2$ of $\mathrm{co}(H_1, H_2) \cap Q$ lying on $H_1, H_2$, so that $\{F_1, x_{1i}\}$ and $\{F_2, x_{2i}\}$ are negative homothetic images of each other. The corresponding polytopes yield $c_d(n) = \lceil \frac{n}{2} \rceil$ for every odd vertex number $n > 4d$. Thus, we have $(2_5)$.

2. For showing the complete solution in 3-space (i.e. (1)), by the arguments above we need only to consider the non-trivial cases $n = 5, 7, 9$.

A. If $n = 5$, then we have only two combinatorial types of convex polyhedra. For $P$ being a pyramid over a 4-gon, this 4-gon yields at least two vertex pairs of $P + (-P)$, and every edge containing the apex of $P$ connects an additional strictly antipodal pair. On the other hand, the combinatorial type of a double pyramid $P$ over a triangle $T$ with apices $x_1, x_2$ has $\{x_1, x_2\}$ and any pair of vertices of $T$ as strictly antipodal vertex pairs. Let us project $P$ along a direction of general position in $\mathrm{aff}(T)$, not parallel to any side of $T$. Since $P$ does not have parallel edges, the projection $Q$ of $P$ will be a convex quadrangle without parallel sides, hence some two of its sides connect strictly antipodal vertex pairs of $Q$. The two vertex pairs of $P$, projecting to these vertex pairs of $Q$, are two more strictly antipodal vertex pairs of $P$.

B. For $P$ with $n = 7$, we first show $|\mathrm{vert}\,(P + (-P))| \geq 12$ if $P$ has no parallel facets. Let $f, f_i, e, v (= 7)$ denote the number of facets, $i$-gonal facets, edges and vertices of $P$, respectively. Euler's theorem gives

$$v - 2 = e - f = \sum_{i \geq 3} \frac{i f_i}{2} - \sum_{i \geq 3} f_i = \sum_{i \geq 3} \frac{(i-2) f_i}{2} = \sum_{j \geq 3} \sum_{i \geq j} \frac{f_i}{2}.$$

Denote by $f', f_i', e', v'$ the analogous numbers for $P + (-P)$. Since each $i$-gonal facet of $P$ is parallel to two $\geq i$-gonal facets of $P + (-P)$, we have $\sum_{i \geq j} f_i' \geq 2 \sum_{i \geq j} f_i$, implying

$$v' - 2 = \sum_{j\geq 3}\sum_{i\geq j} \frac{f_i'}{2} \geq \sum_{j\geq 3}\sum_{i\geq j} f_i = 2(v-2) = 10.$$

Therefore we may suppose $P$ has a pair of facets $F, \bar{F}$ in different parallel supporting planes $H, \bar{H}$. Furthermore, $\rho$ shall denote the total number of vertices of $P+(-P)$ lying in those supporting planes of the difference body which are parallel to $H$. (This notion is useful for the following enumeration of subcases.)

(a) For $F$ a 4-gon and $\bar{F}$ a triangle, $\rho \geq 8$ is assured. If the three sides of $\bar{F}$ are oppositely parallel to three sides of $F$ ($F, \bar{F}$ considered in the same orientation, w.r.t. directly parallel normal vectors), and thus $\rho = 8$, then at least two vertex pairs in $F$ or, otherwise, in $\bar{F}$ are strictly antipodal in $P$, i.e., $P + (-P)$ has at least four additional vertices in $\operatorname{aff}(F + (-F))$. In each other case, we have $\rho \geq 10$ and at least one strictly antipodal vertex pair in $F$ or in $\bar{F}$.

(b) For $F, \bar{F}$ being triangles, $\rho \geq 6$ is assured. One obtains $\rho = 6$ for $F$ homothetic to $(-\bar{F})$, and $\rho = 8$ for $F$ and $(-\bar{F})$ having exactly two pairs of parallel and similarly oriented sides. (With $\rho \geq 10$, all other cases are trivial, since the vertex outside $F \cup \bar{F}$ belongs to a further strictly antipodal pair.)

In the first case, if $F$ is even congruent to $(-\bar{F})$, the vertex $x \notin F \cup \bar{F}$ lies opposite to at least one 2-face of $P$, i.e. belongs to at least three strictly antipodal pairs of vertices. If $F$ is a greater homothetic copy of $(-\bar{F})$, say, $F = \operatorname{co}\{a, b, c\}, \bar{F} = \operatorname{co}\{\bar{a}, \bar{b}, \bar{c}\}$, with corresponding notation of vertices, then for each side of $F$, say $\operatorname{co}\{a, b\}$, either the third vertex $c$ of $F$ and the vertex $x$, or the vertices $a$ and $b$ form a strictly antipodal vertex pair (depending on whether $x$ lies in the closed outer or in the open inner half-space bounded by $\operatorname{aff}\{a, b, \bar{c}\}$, and with corresponding supporting planes nearly parallel, in the respective cases, to the plane $\operatorname{aff}\{a, b, \bar{c}\}$ or $\operatorname{aff}\{a + b - c, \bar{c}, x\}$).

Clearly, for $F$ and $(-\bar{F})$ having precisely two pairs of parallel and similarly oriented sides we have only to exclude

$$|\operatorname{vert}(P + (-P))| = 10.$$

Here $P = \operatorname{co}(F, \bar{F}, \{x\})$. Assuming $|\operatorname{vert}(P + (-P))| = 10$, $P + (-P)$ has 8 vertices in the planes of its quadrangular facets $G = F + (-\bar{F})$ and $-G$, and two additional vertices $y, -y$ between these facet planes. By a suitable choice of the coordinate system (which we denote by $\xi, \eta, \zeta$), we may assume the vertices of $F$, resp. $\bar{F}$ are $(0, 0, 1), (0, 1, 1), (a, 1, 1)$, resp. $(0, 0, 0), (1, 0, 0), (1, b, 0)$, where $a, b > 0$, and $ab \neq 1$ (since $F, -\bar{F}$ are no homothets).

First suppose $a, b \neq 1$. Then the nine vectors from the vertices of $\bar{F}$ to the vertices of $F$ are all different, thus are pairwise non-parallel (since each of them has $\zeta$-coordinate 1). Denoting $Q = \operatorname{co}(F, \bar{F}) \subset P$, we have $P + (-P) = \operatorname{co}(Q + (-Q), \{y, -y\})$, since the latter convex hull contains all vertices of $P + (-P)$. This convex hull is the union of $Q + (-Q)$ and of all the closed segments with one endpoint at $\pm y$, other endpoint on $\operatorname{bd}(Q + (-Q))$, and disjoint to $\operatorname{int}(Q + (-Q))$. For both $y$ and $-y$ these other endpoints of the mentioned segments form the so called illuminated domain from $\pm y$ (in the not strict sense), which is bounded by a simple closed polygon $\pm J$ on $\mathrm{bd}\,(Q + (-Q))$, consisting of some of its edges. The illuminated domains from $y$ and from $-y$ are, apart from their boundaries, disjoint. The vertices of any of these simple closed polygons $\pm J$ are vertices of $P + (-P)$ as well, and are different from $y$ and $-y$. Thus – by the position of the vertices of $P + (-P)$ – they must lie on the planes of the facets $G$ or $-G$, and $\pm J$ must have edges with one endpoint on $G$ and other endpoint on $-G$. Any such edge $E$ is an edge of $P+(-P)$, hence is the Minkowski sum of two parallel edges, or an edge and a vertex of $P$ (one taken with minus sign), lying in some opposite supporting planes of $P$. The orthogonal projection of $E$ to the $\zeta$-axis has length 2. Since any edge of $P$ has a projection to the $\zeta$-axis of length $\leq 1$, with equality only for edges with one endpoint on $\bar{F}$, other endpoint on $F$, therefore the above mentioned Minkowski summands of $E$ are two different parallel edges of $P$, both with one endpoint on $\bar{F}$, other endpoint on $F$. However, from above, all nine vectors from vertices of $\bar{F}$ to vertices of $F$ are pairwise non-parallel, a contradiction. Therefore $a$ or $b$ equals 1.

Since $a$ and $b$ have symmetric roles, we may suppose $a = 1$. By $ab \neq 1$ then $b \neq 1$. Corresponding to the vertices of $P+(-P)$ on its facets $G$ and $-G$, $P$ has four strictly antipodal vertex pairs, each pair consisting of a vertex of $\bar{F}$ and a vertex of $F$. Also the vertex $x$, which lies in the slab $0 < \zeta < 1$, belongs to at least one strictly antipodal vertex pair. Thus it suffices to show that either $x$ belongs to at least two strictly antipodal vertex pairs of $P$, or two vertices of $F$, or of $\bar{F}$ form a strictly antipodal vertex pair of $P$.

Suppose first $x$ lies in the open half-space $\xi < 0$, or in the open half-space $\xi > 1$. Then $x$ belongs to at least three strictly antipodal vertex pairs, namely together with any of the three vertices of $Q$ lying on the plane $\xi = 1$, or $\xi = 0$, resp. If $x$ lies on the plane $\xi = 0$ or $\xi = 1$, then $P$ has parallel facets in these planes, one quadrangular, one triangular, which case has already been settled, under (a). Lastly let $x$ lie in the open slab $0 < \xi < 1$. Recall $b \neq 1$. If $b < 1$, then $(0,0,1), (1,1,1)$ is a strictly antipodal vertex pair, belonging to $F$. If $b > 1$, then $(0,0,0), (1,b,0)$ is a strictly antipodal vertex pair, belonging to $\bar{F}$. Hence, the case $n = 7$ is finished.

C. For $n = 9$, we have only to exclude

$$(*) \qquad |\mathrm{vert}\,P| + 1 = |\mathrm{vert}\,(P + (-P))|.$$

Remind that for a convex polytope $Q$ in $\mathbf{R}^d$, if we translate the outer normal cones of its vertices so that the apices are translated to 0, and then intersect the translated copies with the unit sphere $\mathbf{S}^{d-1}$ about 0, then we obtain a tiling of $\mathbf{S}^{d-1}$ that (together with the intersections of all faces of all dimensions of these translated cones with $\mathbf{S}^{d-1}$) is dually combinatorially isomorphic to the boundary complex of $Q$. Using this for $Q = P$ and for $Q = P + (-P)$, and using the fact that the tiling associated to $P + (-P)$ is a refinement (=subdivision) of the tiling associated to $P$, we see the following. If $P \subset \mathbf{R}^3$ is a convex polyhedron satisfying (*), then all edges containing one vertex of any strictly antipodal vertex pair are pairwise parallel and oppositely directed, with exactly one exception: $P$ has precisely one edge $\mathrm{co}\,\{x, y\}$ such that to all edges meeting $\mathrm{co}\,\{x, y\}$ (except to $\mathrm{co}\,\{x, y\}$ itself) there are respectively parallel and oppositely directed edges forming the edge system around a third vertex $z$ of $P$. Then clearly $\{x, z\}$ and $\{y, z\}$ are strictly antipodal pairs, and $P+(-P)$ has precisely one edge more than $P$, that edge being parallel to co $\{x,y\}$. The number of facets of $P$, and of $P+(-P)$ (and in fact their sets of outer unit facet normals) coincide. Using further $n=9$, let $F_1$, $F_2$ be the facets of $P+(-P)$ adjacent to the edge $E$ of $P+(-P)$, that corresponds to the edge co $\{x,y\}$ of $P$. (We say that facets, edges or vertices of $P+(-P)$ correspond to those of $P$, if they have the same outer normal cone.) Since the facets of $P$ corresponding to $-F_1$, $-F_2$ have one edge less than $-F_1$, $-F_2$ (because of $-E$), the angles of $-F_1$, and similarly those of $-F_2$, at the endpoints of $-E$ have a sum $>\pi$. In particular, $-F_1$, $-F_2$ have $\geq 4$ vertices.

$P+(-P)$ lies in the infinite prism $Q$ bounded by the planes spanned by $F_1$, $F_2$, $-F_1$, $-F_2$. Since these four facets have altogether at most 10 vertices, $F_1$ and $-F_2$ have at least one vertex in common. More exactly, there are the following possibilities: 1) $F_1$ is a pentagon, $-F_2$ a quadrangle (or conversely), and they have a common edge; 2) $F_1$, $-F_2$ are quadrangles, and they have a common edge; 3) $F_1$, $-F_2$ are quadrangles, they only have a common vertex, and this is adjacent by edges of $F_1$, $-F_2$ to opposite vertices of $E$, $-E$ (in the polyhedron $P+(-P)$); 4) $F_1$, $-F_2$ are quadrangles, they only have a common vertex, and this is adjacent by edges of $F_1$, $-F_2$ to non-opposite vertices of $E$, $-E$. In cases 1), 3), 4) the facets $F_1$, $F_2$, $-F_1$, $-F_2$ comprise all vertices of $P+(-P)$, while in case 2) $P+(-P)$ has still one opposite vertex pair, and this lies in the interior of $Q$.

In any of these four cases the union of the facets $F_1$, $F_2$, $-F_1$, $-F_2$ splits the boundary of $P+(-P)$ into two regions, both bounded by a spatial 1) pentagon; 2) quadrangle; 3) hexagon; 4) hexagon, in the respective cases. These closed regions $R, -R$ in the boundary of $P+(-P)$ consist either only of triangular facets, or not only of triangular facets. In the first of these cases let us consider the corresponding facets of $P$. These are triangles with sides parallel to and similarly oriented as those of the corresponding facets of $P+(-P)$. Since in both of the corresponding regions in the boundary of $P$ (i.e. consisting of facets corresponding to those in $R$, $-R$), say $R'$, $R''$, these triangles join by entire corresponding edges, therefore $R'$, $R''$ are positive homothets of $R$, $-R$. Hence $R'$, $R''$ are negative homothets of each other. Consider now the edges (cases 1), 2)) resp. the vertices (cases 3), 4)) of $P$ corresponding to those of $P+(-P)$ lying on the edges of $Q$ not containing $E$, $-E$. These edges are parallel, resp. these vertices are corresponding points at the homothety between $R'$ and $R''$. Therefore in any of the four cases the coefficient of homothety between $R'$ and $R''$ is $-1$. However this implies $P$ is centrosymmetric, contradicting $n=9$.

Now we turn to the case that $R$, $-R$ also contain facets with $\geq 4$ sides.

In case 1) (supposing, as we may, that $F_1$ is pentagonal, $F_2$ is quadrangular) both $R$, $-R$ consist of a trapezoidal and a triangular facet, sharing a common edge. Then both $R'$, $R''$ consist of a trapezoidal and a triangular facet sharing a common edge, with parallel and oppositely oriented respective sides. Further for the trapezoidal facets the longer bases – resp. the legs – have their endpoints on the facets corresponding to $F_2$ and $-F_2$ – resp. to $F_1$ and $-F_1$ – thus have respectively equal lengths for both $R'$ and $R''$. This proves the negative homothety of $R'$, $R''$, with coefficient $-1$, leading to a contradiction like above.

In case 2) both $R$, $-R$ consist of one quadrangular and two triangular facets, sharing a common trivalent vertex. Then however $R'$, $R''$ are negative homothets of each other. Namely their triangular and quadrangular facets have parallel and oppositely oriented respective edges, and because of the two triangular facets having a common edge, the sides of the quadrangular facets common to the two triangular facets have the same length ratio for both $R'$ and $R''$. This negative homothety property leads to a contradiction like above.

In cases 3) and 4) $R$, $-R$ are bounded by spatial hexagons. Using the property that the sum of the angles of the facet $-F_1$, resp. $-F_2$, at the endpoints of $-E$ is $> \pi$, we see the following. In both of these two cases some three consecutive vertices of these spatial hexagons are the vertices of triangular facets of $P + (-P)$. (In one of these hexagons in case 3) the middle one among these three vertices is the vertex corresponding to $x$ or $y$, in case 4) the middle one lies in the intersection of the planes spanned by $F_1$, $-F_2$; in the other hexagon it is the opposite vertex, in $P + (-P)$.) Deleting these above described triangles from the hexagonal regions $R$, $-R$, both remaining pentagonal regions consist of a quadrangular and a triangular facet. These two facets are separated by a diagonal of the pentagonal region, adjacent to its vertex which is the opposite vertex, in the hexagonal region, to the above middle vertex. For the facets of the corresponding hexagonal regions $R'$, resp. $R''$ of $P$ the edges are parallel to and similarly oriented as the respective edges for the facets of the regions $R$, resp. $-R$ of $P + (-P)$. Hence we have parallelity and opposite orientation of the respective edges for $R'$, $R''$. Further for $R'$ two neighbouring edges, say $E_1'$, $E_2'$, of its quadrangular facet have their endpoints on the parallel facets corresponding to $F_1$, $-F_1$, resp. to $F_2$, $-F_2$. The same holds for the neighbouring edges $E_1''$, $E_2''$ of the quadrangular facet in $R''$, where the outer normal cones of $E_1'$, $E_1''$, resp. of $E_2'$, $E_2''$ are opposite to each other. Therefore $E_1'$, $E_1''$, resp. $E_2'$, $E_2''$ have equal lengths. This implies the negative homothety of $R'$, $R''$, with coefficient $-1$, yielding a contradiction like above. Thus $c(9) > 5$.

The sharpness of (1) is shown by examples with $c(n) = 6$ and $n = 5, 7, 9$. The first case holds for a pyramid over a parallelogram, the second one for the convex hull of a regular octahedron $\operatorname{co}(T_1, T_2)$ (with $T_1, T_2$ being opposite facets) and a point $q$ outside the octahedron and close to the centroid of $T_1$. Cutting off that vertex $q$ by a plane close to $q$, parallel to $\operatorname{aff}(T_1)$ and having $\operatorname{co}(T_1, T_2)$ in an open half-space, the resulting truncation has 9 vertices and 6 strictly antipodal vertex pairs. ■

## § 4. Proof of Proposition 1

For proving Proposition 1, we shall need a

LEMMA. *Let in $\mathbf{R}^d$, $d \geq 2$, on a hemisphere of the unit sphere $d+1$ points $p_i$ be given. Then we have (with $i \neq j$)*

$$\frac{\max p_i p_j}{\min p_i p_j} \geq \sqrt{1 + \frac{1}{d-1}}.$$

*Equality stands if and only if* $\mathrm{co}\{p_1,\dots,p_{d+1}\}$ *is a right pyramid with base a regular* $(d-1)$*-simplex, inscribed in a great* $(d-2)$*-sphere of the unit* $(d-1)$*-sphere.*

PROOF. The statement is evident for $d=2$. Supposing it holds for $d-1$, we will prove it for $d$, where $d \geq 3$. Some facet hyperplane of $\mathrm{co}\{p_1,\dots,p_{d+1}\}$, say $\mathrm{aff}\{p_1,\dots,p_d\}$, separates the centre $o$ of the unit sphere from the simplex $\mathrm{co}\{p_1,\dots,p_{d+1}\}$. We may suppose the circumcentre $o'$ of $\mathrm{co}\{p_1,\dots,p_d\}$ lies in $\mathrm{co}\{p_1,\dots,p_d\}$, otherwise induction gives

$$\frac{\max p_ip_j}{\min p_ip_j} \geq \frac{\max\limits_{i,j\leq d} p_ip_j}{\min\limits_{i,j\leq d} p_ip_j} \geq \sqrt{1+\frac{1}{d-2}} > \sqrt{1+\frac{1}{d-1}}.$$

Let the circumradius of $\mathrm{co}\{p_1,\dots,p_d\}$ be $r'$. By [Bá], Lemma 3, $o' \in \mathrm{co}\{p_1,\dots,p_d\}$ implies

$$\max_{i,j\leq d} p_ip_j \geq r'\sqrt{2(1+\frac{1}{d-1})}$$

(=edge length of a regular $(d-1)$-simplex of circumradius $r'$). On the other hand, let $p'_{d+1}$ be the projection of $p_{d+1}$ on $\mathrm{aff}\{p_1,\dots,p_d\}$. For $p'_{d+1}=o'$ we have with $i \leq d$

$$p_ip_{d+1} = \sqrt{{p_ip'_{d+1}}^2 + {p'_{d+1}p_{d+1}}^2} \leq r'\sqrt{2},$$

because of

$$p_ip'_{d+1} = p_io' = r' \text{ and } p'_{d+1}p_{d+1} \leq r'$$

(since the spherical cap cut from our sphere by $\mathrm{aff}\{p_1,\dots,p_d\}$ and containing $p_{d+1}$ is at most a hemisphere).

For $p'_{d+1} \neq o'$ consider the closed half $(d-1)$-plane $H$ of $\mathrm{aff}\{p_1,\dots,p_d\}$ containing $p'_{d+1}$, whose boundary contains $o'$ and is perpendicular to $o'p'_{d+1}$. Since $o' \in \mathrm{co}\{p_1,\dots,p_d\}$, some $p_i \in H$ $(i \leq d)$. Then

$$p_ip_{d+1} = \sqrt{{p_ip'_{d+1}}^2 + {p'_{d+1}p_{d+1}}^2} \leq \sqrt{\max\{{pp'_{d+1}}^2 | o'p \leq r', p \in H\} + {p'_{d+1}p_{d+1}}^2} =$$

$$= \sqrt{r'^2 + {o'p'_{d+1}}^2 + {p'_{d+1}p_{d+1}}^2} \leq r'\sqrt{2}$$

(since $p'_{d+1}p_{d+1}$ attains its maximum among the above circumstances, if the described spherical cap is just a hemisphere, when ${o'p'_{d+1}}^2 + {p'_{d+1}p_{d+1}}^2 = r'^2$). Hence in both cases

$$\min_{i\leq d} p_ip_{d+1} \leq r'\sqrt{2}.$$

From the two estimates we have

$$\frac{\max p_ip_j}{\min p_ip_j} \geq \frac{\max\limits_{i,j\leq d} p_ip_j}{\min\limits_{i\leq d} p_ip_{d+1}} \geq \sqrt{1+\frac{1}{d-1}}.$$

Obviously, equality holds e.g. for the case of the mentioned right pyramid.

Now we show that this is the only case of equality. If $o' \notin \operatorname{co}\{p_1, \dots, p_d\}$, then we have shown that there is strict inequality. If $o' \in \operatorname{co}\{p_1, \dots, p_d\}$ and we have equality, then

$$\max_{i,j \le d} p_i p_j = r' \sqrt{2(1 + \frac{1}{d-1})},$$

and by the proof in [Bá] this is only possible if $o'$ does not lie on any proper face of $\operatorname{co}\{p_1, \dots, p_d\}$, and then $\operatorname{co}\{p_1, \dots, p_d\}$ is a regular $(d-1)$-simplex with circumcentre $o'$. Then in case of $p'_{d+1} = o'$ we must have $p'_{d+1} p_{d+1} = r'$, which is the asserted case of equality. In case of $p'_{d+1} \ne o'$ for each $p_i \in H$ we have

$$p_i p'_{d+1} = \max\{pp'_{d+1} | o'p \le r', p \in H\} = \sqrt{r'^2 + o'p'_{d+1}{}^2},$$

thus each $p_i \in H$ lies on $\operatorname{bd} H$ (taken in $\operatorname{aff}\{p_1, \dots, p_d\}$), contradicting the fact that $o'$ is the circumcentre of the regular $(d-1)$-simplex $\operatorname{co}\{p_1, \dots, p_d\}$. ∎

Proof of the Proposition. The statement is evident for $d = 2$. Again we suppose that it holds for $d-1$ and will prove it for $d, d \ge 3$. Let $o'$ denote the circumcentre of the $(d-2)$-face $a_1 \dots a_{d-1}$. If $o'$ does not belong to the face $a_1 \dots a_{d-1}$, then $d > 3$ and by the Lemma

$$\frac{\text{maximal edge length of } \operatorname{co}\{a_1, \dots, a_{d-1}, b_1, b_2\}}{\text{minimal edge length of } \operatorname{co}\{a_1, \dots, a_{d-1}, b_1, b_2\}} \ge \frac{\max a_{i_1} a_{i_2}}{\min a_{i_1} a_{i_2}} \ge$$

$$\ge \sqrt{1 + \frac{1}{d-3}}.$$

Further let $b'_j$ denote the projection of $b_j$ on $\operatorname{aff}\{a_1, \dots, a_{d-1}\}$. If $b'_j$ does not belong to the face $a_1 \dots a_{d-1}$, then $\operatorname{co}\{a_1, \dots, a_{d-1}, b_j\}$ is a $(d-1)$-simplex, having two facets (one being $\operatorname{co}\{a_1, \dots, a_{d-1}\}$) enclosing an angle $\ge \beta > \frac{\pi}{2}$, where for $d \ge 4$

$$\cos \beta \ge -\frac{1}{(d-1)^2 - 3(d-1) + 1}.$$

Then by the induction hypothesis we have for this fixed $j$

$$\frac{\max\{a_{i_1} a_{i_2}, a_i b_j\}}{\min\{a_{i_1} a_{i_2}, a_i b_j\}} \ge \sqrt{\frac{2 - 2\cos\beta}{1 + \frac{d-3}{d-2}(1 - \cos\beta)}} > \sqrt{\frac{2}{1 + \frac{d-3}{d-2}}} = \sqrt{1 + \frac{1}{2d-5}}.$$

Since for $d > 3$ we have $1/(d-3) > 1/(2d-5)$, it is clear that if either $o'$, or $b'_1$, or $b'_2$ does not belong to $\operatorname{co}\{a_1, \dots, a_{d-1}\}$, then the quotient of the lengths of the longest and shortest edges is

$$> \sqrt{1 + \frac{1}{2d-5}}, \text{ which is } \ge \sqrt{\frac{2 - 2\cos\phi}{1 + \frac{d-2}{d-1}(1 - \cos\phi)}}, \text{ by } \cos\phi \ge -\frac{1}{d^2 - 3d + 1}.$$

This shows the statement of the Proposition in any of these cases.

Henceforward we may and will suppose $o', b'_1, b'_2 \in \operatorname{co}\{a_1, \dots, a_{d-1}\}$. In this case we will show

$$\frac{\max\{\max a_{i_1} a_{i_2}, b_1 b_2\}}{\min a_i b_j} \ge \sqrt{\frac{2 - 2\cos\phi}{1 + \frac{d-2}{d-1}(1 - \cos\phi)}}.$$

The inclusion $o' \in \mathrm{co}\{a_1, \ldots, a_{d-1}\}$ implies by [Bá], Lemma 3, that $\max a_{i_1}a_{i_2} \geq r'\sqrt{2\left(1+\frac{1}{d-2}\right)}$, where $r'$ is the circumradius of $\mathrm{co}\{a_1, \ldots, a_{d-1}\}$. Further $b_1b_2 \geq b_1''b_2''$, where $b_1''$, $b_2''$ are the orthogonal projections of $b_1$, $b_2$ on a 2-plane orthogonal to $\mathrm{aff}\{a_1, \ldots, a_{d-1}\}$. We have

$$b_1''b_2'' = \sqrt{b_1'b_1^2 + b_2'b_2^2 - 2b_1'b_1 \cdot b_2'b_2 \cdot \cos\alpha} \geq$$
$$\geq \sqrt{b_1'b_1^2 + b_2'b_2^2 - 2b_1'b_1 \cdot b_2'b_2 \cdot \cos\phi},$$

where $\alpha(\geq \phi)$ is the angle of the facets $\mathrm{co}\{a_1, \ldots, a_{d-1}, b_1\}$, $\mathrm{co}\{a_1, \ldots, a_{d-1}, b_2\}$.

On the other hand $a_ib_j = \sqrt{a_ib_j'^2 + b_j'b_j^2}$, and we assert that, for $j$ fixed, $\min_i a_ib_j = \min_i \sqrt{a_ib_j'^2 + b_j'b_j^2} \leq \sqrt{r'^2 + b_j'b_j^2}$, i.e. $\min_i a_ib_j' \leq r'$. We may suppose $b_j' \neq o'$. We have $b_j' \in \mathrm{co}\{a_1, \ldots, a_{d-1}\}$, thus also $b_j'$ lies in the circumsphere of $\mathrm{co}\{a_1, \ldots, a_{d-1}\}$ in $\mathrm{aff}\{a_1, \ldots, a_{d-1}\}$. Let $H$ be a $(d-3)$-plane in $\mathrm{aff}\{a_1, \ldots, a_{d-1}\}$ (where $d-3 \geq 0$), passing through $b_j'$ and (for $d > 3$) orthogonal to $o'b_j'$. $H$ cuts off from the circumsphere of $\mathrm{aff}\{a_1, \ldots, a_{d-1}\}$ a closed smaller spherical segment, each point of which has a distance $< r'$ to $b_j'$. Hence if $\min_i a_ib_j' > r'$ (actually if $\min_i a_ib_j' \geq r'$), then each $a_i$ lies in the open larger spherical segment. Thus $b_j' \notin \mathrm{co}\{a_1, \ldots, a_{d-1}\}$, a contradiction.

By the results of the last two paragraphs we have

$$\frac{\max\{\max a_{i_1}a_{i_2}, b_1b_2\}}{\min a_ib_j} \geq \frac{\max\left\{r'\sqrt{2\left(1+\frac{1}{d-2}\right)}, \sqrt{h_1^2+h_2^2-2h_1h_2\cos\phi}\right\}}{\min\sqrt{r'^2+h_j^2}},$$

where $h_j = b_j'b_j$. Letting $r'^2 \cdot 2\left(1+\frac{1}{d-2}\right) = A$, $h_1^2+h_2^2-2h_1h_2\cos\phi = B$, $r'^2+h_j^2 = C_j$, the square of the last quotient is $\max\{A, B\}/\min C_j$. Thus it suffices to prove

$$(*) \qquad \frac{\max\{A, B\}}{\min C_j} \geq \frac{2-2\cos\phi}{1+\frac{d-2}{d-1}(1-\cos\phi)}.$$

The quantities $A$, $B$, $C_j$ satisfy

$$C_1 + C_2 - \frac{d-2}{d-1}A - 2\cos\phi\sqrt{\left(C_1 - \frac{d-2}{2(d-1)}A\right)\left(C_2 - \frac{d-2}{2(d-1)}A\right)} = B.$$

For $0 \geq \cos\phi$ we have

$$\left(\frac{d-2}{d-1}+1\right)\max\{A,B\} \geq \frac{d-2}{d-1}A+B =$$
$$= C_1+C_2-2\cos\phi\sqrt{\left(C_1-\frac{d-2}{2(d-1)}A\right)\left(C_2-\frac{d-2}{2(d-1)}A\right)}$$
$$\geq 2\min C_j - 2\cos\phi\cdot\left(\min C_j-\frac{d-2}{2(d-1)}A\right) \geq$$
$$\geq 2\min C_j - 2\cos\phi\left(\min C_j-\frac{d-2}{2(d-1)}\max\{A,B\}\right), \text{ which implies}$$
$$\left(\frac{d-2}{d-1}+1-\cos\phi\cdot\frac{d-2}{d-1}\right)\max\{A,B\} \geq (2-2\cos\phi)\min C_j.$$

This is equivalent to (*).

For $0 \leq \cos\phi$ we have

$$C_1+C_2=\frac{d-2}{d-1}A+B+2\cos\phi\sqrt{\left(C_1-\frac{d-2}{2(d-1)}A\right)\left(C_2-\frac{d-2}{2(d-1)}A\right)} \leq$$
$$\leq \frac{d-2}{d-1}A+B+2\cos\phi\left(\frac{C_1+C_2}{2}-\frac{d-2}{2(d-1)}A\right),$$

thus

$$(2-2\cos\phi)\min C_j \leq (C_1+C_2)(1-\cos\phi) \leq \frac{d-2}{d-1}(1-\cos\phi)A+B \leq$$
$$\left(\frac{d-2}{d-1}+1-\cos\phi\cdot\frac{d-2}{d-1}\right)\max\{A,B\},$$

which implies (*).

It is obvious that the inequality in the Proposition becomes an equality for the case asserted in the Proposition. We will show that this is the only case of equality.

If either $o'$, or $b_1'$, or $b_2'$ does not belong to $\mathrm{co}\,\{a_1,\ldots,a_{d-1}\}$, then, as shown above, we have strict inequality in the inequality of the Proposition. For $o', b_1', b_2' \in \mathrm{co}\,\{a_1,\ldots,a_{d-1}\}$, in case if equality holds in the Proposition, then both for $0 \geq \cos\phi$ and $0 \leq \cos\phi$ from the chain of inequalities for $A$, $B$, $C_1$, $C_2$ we see that $A = B = \max\{A,B\}$, $C_1 = C_2 = \min C_j$. Then we must have

$$\max a_{i_1}a_{i_2} = b_1b_2 = r'\sqrt{2\left(1+\frac{1}{d-2}\right)} = \sqrt{h_1^2+h_2^2-2h_1h_2\cos\phi}.$$

Thus by [Bá] $\mathrm{co}\,\{a_1,\ldots,a_{d-1}\}$ is a regular simplex with circumcentre $o'$, and by

$$b_1b_2 \geq b_1''b_2'' = \sqrt{h_1^2+h_2^2-2h_1h_2\cos\alpha} \geq \sqrt{h_1^2+h_2^2-2h_1h_2\cos\phi}$$

in this chain of inequalities we have equalities, thus $\alpha = \phi$. Similarly we must have

$$\min_i a_ib_1 = \min_i a_ib_2 = \sqrt{r'^2+h_1^2} = \sqrt{r'^2+h_2^2},$$

i.e. both for $j = 1, 2$ we must have

$$\min_i a_i b'_j = r'.$$

For $b'_1 = b'_2 = o'$ we have the asserted case of equality. If some $b'_j \neq o'$, then, as shown above,

$$\min_i a_i b'_j \geq r'$$

is impossible. Thus we cannot have $\min_i a_i b'_j = r'$, a contradiction. ■

## § 5. Proof of Proposition 2.

Since the detailed proof is more lengthy, we only have included a sketchy proof, which will be, however, sufficient for reconstructing the details of the discussion.

The sharpness of the estimation can be seen by the example consisting of the middle third segments belonging to three pairwise skew edges of the cube. Now we show that $n \leq 3$ holds under the hypotheses of Proposition 2. Changing the notations, let us suppose there are four such segments $x_1x_2$, $y_1y_2$, $z_1z_2$, $u_1u_2$. We may suppose their endpoints are in as general position, as we want. Thus we may suppose no three of the directions of the lines $x_1x_2, y_1y_2, z_1z_2, u_1u_2$ are coplanar. Then the lines $x_1x_2, y_1y_2, z_1z_2$ are three mutually skew edge-lines of a parallelepiped. Thus in a suitable coordinate system $x_i = (\xi_i, 1, -1)$, $y_i = (-1, \eta_i, 1)$, $z_i = (1, -1, \zeta_i)$. By hypothesis, $u_1u_2$ lies in the intersection of three open parallel slabs, each containing on its two boundary planes two of the edges $x_1x_2, y_1y_2, z_1z_2$, i.e. in the open cube with vertices $(\pm 1, \pm 1, \pm 1)$. Similarly $z_1z_2$ lies in the open slab, whose boundary planes contain $x_1x_2$, resp. $y_1y_2$, i.e. on the open edge $(1, -1, -1)$, $(1, -1, 1)$ of this cube, and similarly for $x_1x_2, y_1y_2$, i.e. $-1 < \xi_i, \eta_i, \zeta_i < 1$. Project now this cube along the line $u_1u_2$. Then, by the general position of the points, the projection is a hexagon, for which the projections of the edges of the cube, containing $x_1x_2, y_1y_2$, resp. $z_1z_2$, are either edges or certain segments in the hexagon. Thus we have to distinguish several cases, and in each of these we proceed as follows.

The projection of $u_1u_2$ is an inner point of the hexagon. The slab bounded by the parallel planes passing through $u_1u_2$, resp. $z_1z_2$, projects to a strip, which we say is between the projections of $z_1z_2$ and $u_i$. However, this strip must intersect the projections of the edges of the cube containing $x_1x_2$, resp. $y_1y_2$. This means the projection of $u_i$ has to lie in some subregion of the hexagon. By considering also $x_1x_2, y_1y_2$ rather than $z_1z_2$, we will have three such subregions, which therefore must have a non-empty intersection. This rules out several of the cases, distinguished above. In the remaining cases the projection of $u_i$ lies in the non-empty intersection of these three subregions. Then the projection of the segment $z_1z_2$ must lie in the strip between the projections of $x_1x_2$ and $u_i$, as well as in the strip between the projections of $y_1y_2$ and $u_i$. Considering also $x_1x_2, y_1y_2$ rather than $z_1z_2$, we will have altogether three such situations, however, in any of the remaining cases, one of them is impossible. This shows that the number $n$ of our 1-simplices $s^1_1, \ldots, s^1_n$ is at most 3. ■

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MATHEMATICAL INSTITUTE
HUNGARIAN ACADEMY OF SCIENCES
PF. 127, H–1364 BUDAPEST
HUNGARY

MATHEMATISCHES INSTITUT, PH DRESDEN
WIGARDSTR. 17
D–O/8060 DRESDEN
Present address:
TECHNISCHE UNIVERSITÄT, CHEMNITZ-ZWICKAU
FB MATHEMATIK
PF. 964
D–O/9009 CHEMNITZ
BRD