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Facets with Fewest Vertices

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Abstract. For $v > d \geqslant 3$, let $m(v, d)$ be the smallest number m, such that every convex d-polytope with v vertices has a facet with at most m vertices. In this paper, bounds for $m(v, d)$ are found; in particular, for fixed $d \geqslant 3$,

$$\frac{r-1}{r} \leqslant \liminf_{v\to\infty} \frac{m(v,d)}{v} \leqslant \limsup_{v\to\infty} \frac{m(v,d)}{v} \leqslant \frac{d-3}{d-2},$$

where $r = \lfloor \frac{1}{3}(d+1) \rfloor$.

1. Introduction

It is a well-known fact, which is easy to deduce from the Euler relation, that an ordinary (three-dimensional) convex polyhedron has a face with at most five vertices. However, the examples of the duals of the cyclic polytopes show that, in higher dimensions, there can exist no such fixed bound for the minimal number of vertices of a facet of a convex polytope. We are thus led to consider the following problem: determine $m(v, d)$, defined to be the smallest number m, such that every convex d-polytope with v vertices has a facet with at most m vertices.

In this paper, we shall establish both upper and lower bounds for $m(v, d)$. These bounds (except the lower bound for $d = 4$) are linear, but there is a considerable gap between the constants involved. As we shall point out, our lower bound can be somewhat improved, but we shall not specificly write down the better bound, because the asymptotic behaviour is not affected.

Some remarks should be made on the origins of this paper. Originally, the first two authors found a (somewhat weak) upper bound for $m(v, d)$, using the technique of Gale diagrams. Subsequently, they and the third author improved the upper bound and found a lower bound, still using Gale diagrams. Meanwhile, the last author had found, by the direct method described here, a further improved upper bound. Finally, the use of Gale diagrams is avoided in the present exposition.

2. The Upper Bound

In this section, we obtain an upper bound for $m(v, d)$.

Theorem 1. *For* $v > d \geqslant 3$,

$$m(v, d) \leqslant \min\left\{\left\lfloor \frac{(d-2)v+2}{d-1} \right\rfloor, \left\lfloor \frac{(d-3)v+5}{d-2} \right\rfloor\right\}.$$

Here, $\lfloor \alpha \rfloor$ is, as usual, the integer part of the real number α. We write $v(Q)$ for the number of vertices of the polytope Q. Let P be a d-polytope with $v = v(P)$ vertices. Write $P = F_d$, and for $j = d-1, \ldots, 0$, choose F_j to be a j-face of F_{j+1} with $v(F_j)$ minimal. Let $n_j = v(F_j) - v(F_{j-1})$ $(j = 0, \ldots, d; F_{-1} = \emptyset)$. Then

$$n_d \geqslant n_{d-1} \geqslant \ldots \geqslant n_1 \geqslant n_0,$$

because if, say, $n_{j+1} < n_j$, then the j-face F of P different from F_j with $F_{j-1} \subset F \subset F_{j+1}$ satisfies

$$v(F) \leqslant v(F_{j-1}) + n_{j+1} < v(F_{j-1}) + n_j = v(F_j),$$

contrary to the choice of F_j.

Now F_0 is a vertex and F_1 an edge of P, so that $n_0 = 1 = n_1$. It thus follows that

$$n_d \geqslant \frac{v-2}{d-1}.$$

However, as remarked in § 1, we must also have $v(F_2) \leqslant 5$, so that $n_2 + n_1 + n_0 \leqslant 5$, and hence

$$n_d \geqslant \frac{v-5}{d-2}.$$

Since $v(F_{d-1}) = v - n_d$, we conclude that

$$v(F_{d-1}) \leqslant \min\left\{v - \frac{v-2}{d-1}, v - \frac{v-5}{d-2}\right\},$$

which (bearing in mind that $v(F_{d-1})$ is an integer) yields the bound of the theorem.

We may observe that the bound $\left\lfloor \frac{(d-2)v+2}{d-1} \right\rfloor$ is only better for certain small values of v (actually for $v \leqslant d+3$ or $v = 2d+1$).

The technique employed here was first used by the last author to provide a short proof of a result by Reay on positive bases (it appears in [4]).

3. The Lower Bound for $d \geqslant 5$

We shall now obtain a lower bound for $m(v,d)$, in case $d \geqslant 5$.

Theorem 2. *Let* $v > d \geqslant 5$, *let* $r = \lfloor \frac{1}{3}(d+1) \rfloor$ *and let* $s = d + 1 - 3r$. *Then*

$$m(v,d) \geqslant \left\lfloor \frac{(r-1)v+s}{r} \right\rfloor + 2.$$

To prove the theorem, first recall that the (free) *join* $P \circledv Q$ of two polytopes P and Q is obtained by placing them in independent affine subspaces and taking their convex hull (for the concept, if not the terminology and notation, see [2], Ex. 4.8.1). The construction of $P \circledv Q$ is well-defined up to combinatorial isomorphism (which is all that concerns us here); we have $\dim(P \circledv Q) = \dim P + \dim Q + 1$, and the faces of $P \circledv Q$ are all polytopes of the form $F \circledv G$, with F a face of P and G a face of Q (including the improper faces $\emptyset$ and P or Q itself).

The polytope P which attains the bound of the theorem is a join of r polygons and s points. The polygons are chosen to have as nearly equal numbers of vertices as possible, namely $\left\lceil \frac{v-s}{r} \right\rceil$ or $\left\lceil \frac{v-s}{r} \right\rceil - 1$, where $\lceil \alpha \rceil = -\lfloor -\alpha \rfloor$ is the smallest integer which is no less than α. (The number of vertices of the polygons totals $v - s$.)

From the description of the join given above, and the choice of the components, a minimal facet F of P excludes $\left\lceil \frac{v-s}{r} \right\rceil - 2$ vertices (that is, all but two vertices, which are those of an edge of a maximal polygon), so that

$$v(F) = v - \left\lceil \frac{v-s}{r} \right\rceil + 2 = \left\lfloor \frac{(r-1)v+s}{r} \right\rfloor + 2,$$

which yields the required bound.

4. The Lower Bound for $d = 4$

We now consider $m(v, 4)$. If P_s is the dual of the cyclic 4-polytope with s vertices (see [2]), then P_s itself has $\frac{1}{2}s(s-3)$ vertices, while each of its facets is a wedge over an $(s-2)$-gon, and so has $2s-6$ vertices (see Figure 1, which illustrates the case $s = 8$).

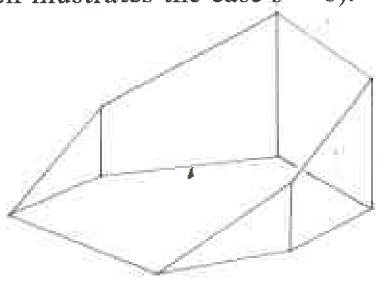

Fig. 1

We proceed to split one or two of these facets as follows. We suppose that $s \geqslant 6$, and we write $m = 2s - 6$. Each splitting (which tilts half the facet about a cutting plane) will produce a new polytope, each of whose facets again has at least m vertices. The splitting plane is depicted by dashed lines in Figures 2a, 2b and 2c. The number

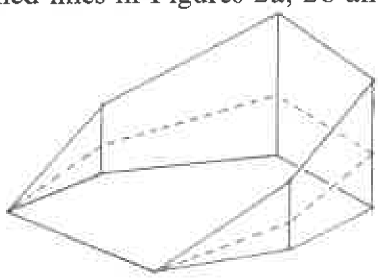

Fig. 2a

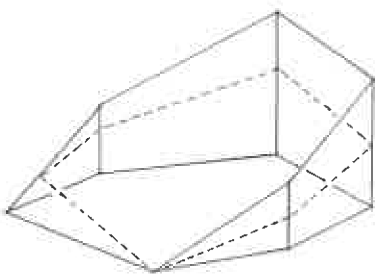

Fig. 2b

of vertices added is $s-4$, $s-3$ and $s-2$, respectively. Note that one (at least) of the new facets is of the same kind as the original, and so the process can be repeated, with the next splitting plane passing through the non-simple vertex or vertices (at which more than four facets meet). If such splittings are performed k times, any number of vertices between $k(s-4)$ and $k(s-2)$ can be added. It is clear that every possible number of vertices, from some point depending on s (in fact, rather before s^2), can be obtained in this way.

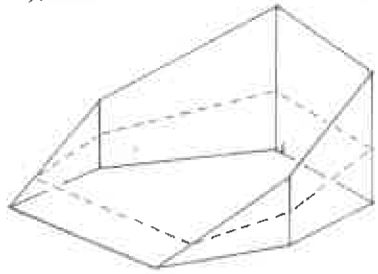

Fig. 2c

However, suppose that, instead of performing all k splittings on one facet, we perform $k-l$ of them on the first, and l of them on a second facet which meets the first in a quadrilateral face. Then (see Figure 3, which illustrates the case $k=5$ and $l=2$), $l(k-l)$ further vertices are added, in addition to those arising from the individual splittings. Bearing in mind the previously obtained range (of length $2k+1$), it is not too hard to see that we can now add on any number of vertices between $k(s-4)$ and $k(s-2)+\lfloor\frac{1}{4}k^2\rfloor$, inclusive.

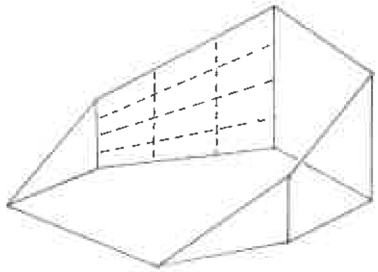

Fig. 3

It follows that we can obtain by these means 4-polytopes whose facets all have at least $m=2s-6$ vertices, with every possible number of vertices from $\frac{1}{2}s(s-3)+k(s-4)$ on, so long as

$$k(s-2)+\lfloor\tfrac{1}{4}k^2\rfloor \geqslant (k+1)(s-4)-1,$$

or (replacing $\lfloor\frac{1}{4}k^2\rfloor$ by the no larger $\frac{1}{4}(k^2-1)$)

$$k \geqslant \sqrt{(4s-3)} - 4.$$

After a little more manipulation, we conclude:

Lemma. $m(v,4) \geqslant m(s) = 2s - 6$ *whenever*

$$v \geqslant v(s) = \tfrac{1}{2}(s^2 - 11s + 16) + \lfloor \sqrt{(4s-3)} \rfloor (s-4).$$

This complicated relationship is not at all easy to analyse, but it is clear that we can deduce from it

Theorem 3. $\liminf_{v\to\infty} \frac{m(v,4)}{\sqrt{v}} \geqslant 2\sqrt{2}.$

5. Asymptotic Bounds

The asymptotic bounds given in the abstract of the paper are immediate consequences of Theorems 1, 2 and 3, and the remarks in the Introduction.

Theorem 4. *For* $d \geqslant 3$,

$$\frac{r-1}{r} \leqslant \liminf_{v\to\infty} \frac{m(v,d)}{v} \leqslant \limsup_{b\to\infty} \frac{m(v,d)}{v} \leqslant \frac{d-3}{d-2},$$

where $r = \lfloor \frac{1}{3}(d+1) \rfloor$.

6. Further Remarks

For $d \geqslant 5$, both the upper and lower bounds for $m(v,d)$ which we have obtained above are linear in v, although there is an increasingly large gap between the respective bounds. It is therefore natural to speculate about the exact asymptotic value of $m(v,d)/v$, if it exists.

There is good reason to suspect that the upper bound for $m(v,d)$ which we have found is too large. For one thing, it is extremely unlikely that there is any d-polytope except the simplex for which (in the notation of Theorem 1) $n_d = n_{d-1} = \ldots = n_3$. For another, it is conjectured that d-polytopes with no triangular faces have at least 2^d vertices (see [1], where this conjecture is proved in the special case of simple polytopes), while it has recently been shown (see [3]) that a d-polytope for $d \geqslant 5$ must have a triangular or quadrilateral face. While this does not necessarily strengthen the estimate $n_2 + n_1 + n_0 \leqslant 5$, it does suggest that there will usually be large gaps between some of the other n_j when v is large compared with d.

Let us also observe that any improvement in the upper bound for $m(v, d-1)$ leads to a corresponding lowering of the upper bound for $m(v, d)$. To see this, let P be a d-polytope with v vertices, whose facets all have at least $m(v, d) = w$ vertices. Again using the notation of Theorem 1, we have $n_d = v - w$ and $n_{d-1} \geqslant w - m(w, d-1)$, and so from $n_d \geqslant n_{d-1}$ follows $v - w \geqslant w - m(w, d-1)$, or

$$w \leqslant \tfrac{1}{2}(v + m(w, d-1)).$$

Consequently, of course, any improvement for $m(v, d)$ for a given dimension d works its way up into all higher dimensions.

In fact, we might generally conjecture that the true value of $m(v, d)$ is much nearer the lower bound of Theorems 2 or 3 than the upper bound of Theorem 1. More particularly, $2\sqrt{2v}$ is probably the correct order of $m(v, 4)$, with the duals of the cyclic polytopes providing the extreme examples; that is

Conjecture 1. $m(v, 4) \leqslant \sqrt{8v + 9} - 3$.

Were this true, it would lead to a corresponding improvement for $m(v, 5)$; with $d = 5$ in the remarks above, the estimates for $w = m(v, 5)$ given by $w \leqslant \frac{1}{2}(v + m(w, 4))$ and $m(w, 4) \leqslant c\sqrt{w} \leqslant c\sqrt{v}$ (for some constant c) yield

Conjecture 2. $\lim\limits_{v \to \infty} \dfrac{m(v, 5)}{v} = \frac{1}{2}$.

Since (asymptotically) the upper and lower bounds for $m(v, 5)$ would then coincide, it is an obvious supplement to Conjecture 2 that the joins of polygons provide the optimal upper bounds for $m(v, 5)$ as well.

Further minor improvements to the lower bound for $m(v, d)$ are also available. First, let us observe that, on the one hand, Euler's theorem and the discussion of some particular cases, and, on the other, direct (and rather uninteresting) geometric constructions show:

Theorem 5.

$$m(v, 3) = \begin{cases} 3, & \text{for } v \leqslant 7,\ v = 9,\ v = 11; \\ 4, & \text{for } v = 8,\ v = 10,\ 12 \leqslant v \leqslant 19,\ v = 21,\ v = 23; \\ 5, & \text{for } v = 20,\ v = 22,\ v \geqslant 24. \end{cases}$$

If $s = 1$ or 2 in Theorem 2 (that is, if d is not of the form $3r - 1$), and if the number v of vertices is sufficiently large, then a judicious replacement of a pair consisting of a polygon and a point by a suitable 3-polytope, or a triple consisting of a polygon and two points by a suitable 4-polytope, will result in an increase of the lower bound estimate for $m(v, d)$. However, the improvement is of order $o(v)$, and so does not affect the asymptotic constant $\frac{r-1}{r}$ of Theorem 4; we shall therefore not give the somewhat intricate details.

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