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Interior Points of the Convex Hull of Few Points in $\mathbb{E}^d$

By

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Abstract. We show that if $P \subset \mathbb{E}^d$, $|P| = d + k$, $d \geqslant k \geqslant 1$ and $O \in \operatorname{int conv} P$, then there exists a simplex S of dimension $\geqslant \left\lceil \frac{d}{k} \right\rceil$ with vertices in P, satisfying $O \in \operatorname{rel int} S$, the bound being sharp. We give an upper bound for the minimal number of vertices of facets of a $(j-1)$-neighbourly convex polytope in $\mathbb{E}^d$ with v vertices.

1. Introduction

Let P be a subset of d-dimensional Euclidean space $\mathbb{E}^d$ and let conv P denote its convex hull. Steinitz [4] has proved that if $O \in \operatorname{int conv} P$, then there exists a subset P' of P such that $O \in \operatorname{int conv} P'$ and $d + 1 \leqslant |P'| \leqslant 2d$, where these bounds are sharp. (Throughout this paper int, rel int, aff, $| \, |$ denote interior, relative interior, affine hull and cardinality of a set. For real x, $\lfloor x \rfloor$ and $\lceil x \rceil$ denote the lower and upper integer part of x. If Q is a convex polytope, vert Q denotes its vertex set.) So, in investigating the property $O \in \operatorname{int conv} P$, in what follows, we shall assume that $d + 1 \leqslant |P| \leqslant 2d$. Recalling Carathéodory's theorem [3, p. 26], $O \in \operatorname{int conv} P$ implies that O is the relative interior point of some i-simplex, with vertices in P, where $i \geqslant 1$. Thus there arises the question if one can give a lower estimate for i in term of $|P|$. In this paper we prove[3]

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[3] After proving Th. 1 we have learnt that P. McMullen also has shown the same result.

Theorem 1. *Let $d \geqslant k \geqslant 1$, let P be a set in $\mathbb{E}^d$ consisting of $d+k$ points, and let O be a point in the interior of the convex hull of P. Then there exists a simplex of dimension not less than $\lceil \frac{d}{k} \rceil$ whose vertices are points of P and the relative interior of which contains O. The bound $\lceil \frac{d}{k} \rceil$ is sharp.*

Generalizing the question of Theorem 1, we pose the following.

Conjecture 1. Let $d \geqslant 2, j \geqslant 1, 2j-1 \leqslant k \leqslant (2j-1)d$, and let P be a set in $\mathbb{E}^d$ consisting of $d+k$ points such that each open half-space bounded by a hyperplane passing through the point O contains at least j points of P. Then there exists a simplex of dimension not less than $\lceil (2j-1) \cdot \frac{d}{k} \rceil$ whose vertices are points of P and whose relative interior contains O.

The case $j=1$ is our theorem. If Conjecture 1 is true, then its bound is sharp if $(2j-1)|k$. Namely, for any integers $d, j \geqslant 1$ there is a set $P(d,j) \subset \mathbb{E}^d$, such that $|P(d,j)| = d+2j-1$ and each open half-space bounded by a hyperplane passing through the point O contains at least j points of P ([2]). Now let $d = \sum_{i=1}^{k/(2j-1)} d_i$, where $\lfloor d/(k/(2j-1)) \rfloor \leqslant d_i \leqslant \lceil d/(k/(2j-1)) \rceil$ for each i. Representing $\mathbb{E}^d$ as the direct sum of $\mathbb{E}^{d_i}$'s, $1 \leqslant i \leqslant k/(2j-1)$, we choose in each $\mathbb{E}^{d_i}$ a set $P(d_i, j)$. Finally, let P be the union of all these $P(d_i, j)$'s. It is easy to check that if S is a simplex whose vertices belong to P and $O \in \operatorname{rel\,int} S$, then $\dim S \leqslant \lceil d/(k/(2j-1)) \rceil$.

One can ask for generalizations of Steinitz's and Carathéodory's theorems in the same spirit.

Conjecture 2. Let P be a set in $\mathbb{E}^d$ (in $\mathbb{E}^d \setminus \{O\}$, respectively) such that each open (closed, resp.) half-space bounded by a hyperplane passing through the point O contains at least j points of P. Then P possesses a subset P' satisfying the same property as P above and $d+2j-1 \leqslant |P'| \leqslant 2dj$ ($2j \leqslant |P'| \leqslant (d+1)j$, respectively).

Here the lower bounds are trivial. The conjectured upper bounds can be attained if P contains j points on each of the positive and negative coordinate semiaxes — this might be the unique extremal case — (if P contains j points on each of $d+1$ half-lines emanating from O and passing through vertices of a simplex with centroid O, resp.).

Now we point out an equivalent formulation of Conjecture 1. In [1] the following quantity $m(v, d)$ has been investigated. For $d \geqslant 2$ and

$v > d+1$ let $m(v, d)$ be the smallest number m, such that every convex d-polytope with v vertices has a facet with at most m vertices. In [1] bounds for $m(v, d)$ are found; in particular, for $v > d \geqslant 5$ we have

$$\lfloor ((r-1)\cdot v + s)/r \rfloor + 2 \leqslant m(v, d) \leqslant$$

$$\leqslant \min\{ \lfloor ((d-2)\cdot v + 2)/(d-1) \rfloor , \lfloor ((d-3)\cdot v + 5)/(d-2) \rfloor \},$$

where $r = \lfloor (d+1)/3 \rfloor$ and $s = d + 1 - 3r$. Furthermore, we have

$$m(v, 2) = 2, 3 \leqslant m(v, 3) \leqslant 5$$

and

$$\liminf_{v \to \infty} m(v, 4)/\sqrt{v} \geqslant 2\sqrt{2}$$

(for more details see [1]). Let $v > d+1$ and let $m_j(v, d)$, where $2 \leqslant j \leqslant \frac{d}{2} + 1$, be the smallest number m such that every convex d-polytope with v vertices also satisfying the property that the convex hull of any $j-1$ vertices is a $(j-2)$-face (i.e. being $(j-1)$-neighbourly, which is for $v > d+1$ only possible if $j \leqslant \frac{d}{2} + 1$ [3, p. 92]) has a facet with at most m vertices. Here we prove

Theorem 2. (i) *Let* $v > d+1$, $2 \leqslant j \leqslant \frac{d}{2} + 1$ *and let* $f_j(v, d)$ *be the sharp lower bound in Conjecture 1 for* $P \subset \mathbb{E}^d$, $|P| = v$, $d + 2j - 1 \leqslant$ $\leqslant v < \infty$. *Then*

$$m_j(v, d) = v - f_j(v, v-d-1) - 1.$$

(ii) *Let* $v > d+1$ *and* $3 \leqslant j \leqslant \frac{d}{2} + 1$. *Then*

$$m_j(v, d) \leqslant \lfloor (v(d-2j+2) + 2j - 2)/(d-2j+3) \rfloor ,$$

or equivalently,

$$f_j(d+k, d) \geqslant \left\lceil \frac{d}{k-2j+2} \right\rceil \quad \textit{for} \quad 3 \leqslant j \leqslant \frac{k+1}{2}.$$

Since $m_2(v, d) = m(v, d)$, [1] and (i) imply the following.

Corollary. *For* $d \geqslant 2$, $k \geqslant 4$ *we have*

$$f_2(d+k, d) \geqslant \max\{ \lceil d/(k-2) \rceil , \lceil (d-2)/(k-3) \rceil \}.$$

2. Proof of Theorem 1

If $d = 1$ or $k = 1$, then the existence of the corresponding simplex is obvious. Hence suppose that the existence part of the theorem is proved for any set in $\mathbb{E}^{d'}$ consisting of $d' + k'$ points ($d' \geqslant k' \geqslant 1$) with $d > d' \geqslant 1$, and consider a set P in $\mathbb{E}^d$ consisting of $d + k$ points ($d \geqslant k \geqslant 2$) satisfying $O \in \operatorname{int}\operatorname{conv} P$. Because of Carathéodory's theorem [3, p. 26] there exists a simplex S' of dimension $l \geqslant 1$ whose vertices are points of P and whose relative interior contains O. If $l \geqslant d/k$, then we are done. Therefore suppose that $d/k > l \geqslant 1$. Let $\mathbb{E}^{d-l}$ be a $(d-l)$-dimensional affine subspace of the d-dimensional Euclidean space which is orthogonal to the affine hull $\operatorname{aff} S'$ of S' and let π denote the orthogonal projection of the d-dimensional Euclidean space onto $\mathbb{E}^{d-l}$ parallel to $\operatorname{aff} S'$. Since $O \in \operatorname{int}\operatorname{conv} P$ therefore $\pi(S') = \pi(O) = O'' \in \operatorname{rel\,int}\operatorname{conv} \pi(P)$. On the other hand $\pi(P)$ is a set in $\mathbb{E}^{d-l}$ with at most $(d+k) - (l+1) = (d-l) + (k-1)$ points different from O''. Consequently, by induction ($d' = d - l$, $k' \leqslant k - 1$) we see that there exists a simplex S'' of dimension at least $(d-l)/(k-1)$ whose vertices are points of $\pi(P) \setminus \{O''\}$ and the relative interior of which contains O''. Now it is easy to see that the existence of S'' implies the existence of a simplex S whose vertices are points of P and for which $\pi(S) = S''$ and $\dim S = \dim S'' \geqslant (d-l)/(k-1)$. Hence $\operatorname{aff} S \cap \operatorname{aff} S' = \operatorname{rel\,int} S \cap \operatorname{aff} S' = O'$ is a point of the d-dimensional Euclidean space. If $O = O'$, then the simplex S is the simplex we are looking for since

$$\dim S \geqslant \frac{d-l}{k-1} > \frac{d - \frac{d}{k}}{k-1} = \frac{d}{k}.$$

If $O \neq O'$, then the half-line $\overrightarrow{O'O}$ will intersect the relative interior of a face F^* of S' in a point O^* with O in the relative interior of the segment $O'O^*$. Thus $O \in \operatorname{rel\,int}\operatorname{conv}(S \cup F^*)$ where $\operatorname{conv}(S \cup F^*)$ is a simplex of dimension $> \dim S \geqslant (d-l)/(k-1) > d/k$ finishing the proof of the first part of the theorem.

The sharpness of the bound $\lceil \frac{d}{k} \rceil$ follows from the construction given after Conjecture 1.

3. Proof of Theorem 2

(i) We recall the definition and some elementary properties of (algebraic) Gale transform [3, Ch. 3]. Let $Q \subset \mathbb{E}^d$ be a convex polytope with $v > d+1$ vertices $x_1, x_2, \ldots, x_v$. An affine dependence is a relation $\sum_{i=1}^{v} \lambda_i x_i = 0$, where $\sum_{i=1}^{v} \lambda_i = 0$. Considering such $(\lambda_1, \ldots, \lambda_v)$'s as points of $\mathbb{E}^v$, they constitute a linear subspace of dimension $v-d-1$. Let $(\lambda_{1j}, \ldots, \lambda_{vj})$, $1 \leqslant j \leqslant v-d-1$ be some basis of this subspace. Then $\bar{x}_i = (\lambda_{i1}, \ldots, \lambda_{i(v-d-1)}) \in \mathbb{E}^{v-d-1}$, $1 \leqslant i \leqslant v$ is called a Gale transform of the vertex x_i of Q and $\overline{Q} = \{\bar{x}_1, \ldots, \bar{x}_v\} \subset \mathbb{E}^{v-d-1}$ is called a Gale transform of vert Q. Several points of Q may coincide. There is considerable arbitrariness in the construction of a Gale transform. Namely, we may choose our basis in many ways, and these lead to different (but linearly equivalent) Gale transforms. For $Z \subsetneqq \operatorname{vert} Q$ we have that $\operatorname{conv} Z$ is a face of Q if and only if, in a Gale transform $\overline{Q}$ of vert Q, there holds $O \in \operatorname{rel\,int} \operatorname{conv} (\overline{\operatorname{vert} Q \setminus Z})$. Furthermore, $\operatorname{conv} Z$ is a facet of Q if and only if $\overline{\operatorname{vert} Q \setminus Z}$ is the set of vertices of a non-degenerate simplex with O in its relative interior. Thus the minimum vertex number of facets of Q equals $v-f-1$, where f is the maximal dimension of a simplex with vertices in $\overline{Q}$ and containing O in its relative interior. That is, $m(v, d) = m_2(v, d) = v - f^* - 1$, where f^* is the minimal value of f for all point systems $\overline{Q}$ in $\mathbb{E}^{v-d-1}$ which are the Gale transforms of the vertex sets of some convex d-polytopes, i.e., which have the properties that the centroid of $\overline{Q}$ is O and each open half-space bounded by a hyperplane passing through O contains at least two points of $\overline{Q}$. However here we can omit the centroid condition and thus obtain $f^* = f_2(v, v-d-1)$. The case $j > 2$ can be proved analogously, noting that the property that the convex hull of any $j-1$ vertices of Q is a $(j-2)$-face transforms to the following property of $\overline{Q}$: for $Z \subsetneqq \operatorname{vert} Q$, $|Z| = j-1$ we have $O \in \operatorname{rel\,int} \operatorname{conv} (\overline{\operatorname{vert} Q \setminus Z})$. However, one easily sees that this property of $\overline{Q}$ is equivalent to the property that each open half-space bounded by a hyperplane passing through O contains at least j points of $\overline{Q}$.

(ii) Observe that the property that the convex hull of any $j-1$ vertices of a convex polytope is a $(j-2)$-face, is inherited by the faces of the polytope. Also, a $(2j-3)$-polytope with the above property is a simplex and thus has $2j-2$ vertices [3, p. 92]. It is shown in [1, p. 90] that if for a convex d-polytope Q we write $Q = F_d$ and for $i = d-1, d-2, \ldots, 0$ we define F_i as an i-face of F_{i+1} with minimum number of vertices, then for $0 \leqslant i \leqslant d$ the number $|\operatorname{vert} F_i|$ is a convex function of i. If Q has the property that the convex hull of any $j-1$ vertices is a $(j-2)$-face, then following from what has been said above we will have $|\text{vert}\, F_{2j-3}| = 2j - 2$ and $|\text{vert}\, F_d| = v = |\text{vert}\, Q|$; hence by $2j - 3 \leqslant d - 1 < d$ and the convexity of $|\text{vert}\, F_i|$ we will get inequality (ii) in Theorem 2.

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