The connected components of the idempotents in the Calkin algebra, and their liftings

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Abstract : We study surjective maps between Banach algebras and obtain the lifting of (analytic families of) idempotents for the case when the kernel consists of elements whose spectra are totally disconnected. Our method gives a new proof and a global analytic version of the lifting theorem modulo the radical. The global analytic lifting is not possible in general, which leads to the interesting question : to how large domain does an analytic lifting exist. We obtain a local version. In particular, our results render it possible to describe

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the connected components of idempotents in the Calkin algebra of a Hilbert space.

1 Notations

For a Banach algebra A, let

$$E(A) := \{p \in A : p^2 = p\}$$

be the set of idempotents. If there is an involution $*$ on A, let

$$S(A) := \{p \in A : p^2 = p = p^*\}$$

be the set of self-adjoint idempotents.

For a Banach space X, let $B(X)$ or $K(X)$ denote the Banach algebras of all bounded or compact linear operators on X, respectively. The quotient $C(X) = B(X)/K(X)$ is called the Calkin algebra (as it was first studied in [C] for the Hilbert space case). Let

$$\pi : B(X) \to C(X)$$

be the canonical map. More generally, we shall be considering situations where π denotes a surjective map between Banach algebras.

For a linear operator T on X, let $R(T) := TX$ and $N(T) := T^{-1}(0)$ denote the range and the kernel of T, respectively.

In this paper, the linear structures are considered over the complex scalars, and the algebras are assumed to be unital. We denote by $\sigma(x)$ the spectrum of an element x.

2 Historical background

The set of idempotents splits naturally into connected components. The structure of these components has been studied in a number of papers and is to be briefly described below. The origins seem to start with the observation (arising from perturbation theory) that close projections are similar [SzN1, p. 58], [SzN2, p. 350], [RieSzN, p. 266]. The proof given in [Kat] made it possible to see that the similarity can be accomplished by exponentials [Ze1], which yields analytic idempotent-valued connections between any two idempotents in the same component.

See also [Ka, Problem I.4.13] and [Ko, Theorem 1]. Analogous results also hold for real Banach algebras [Au1]. Surprisingly, even polynomial connections can be constructed within a given component [Es], whose degrees are actually rather low in a number of cases [T1], [MaZe], [D]. This leads to the interesting question of whether there is a universal bound on the minimum degree of these polynomial connections.

In particular, the question seems to be open even for two Hilbert space projections, if the direct sum of the kernel of one of them and the image of the other is only dense in the space, cf. [D, Théorème 3.3]. More precisely, can such projections be approximated by projections for which the corresponding sum (without closure) is equal to the whole space ? (A positive answer has now been given by Vladimir Kadets.)

For C^*-algebras, the problem has been studied in [T2], [T3]. On the other hand, there are piecewise linear idempotent-valued connections in general [Ko, Theorem 1].

The local arcwise connectedness of the set of idempotents was noticed in [Ze1, Theorem 3.2] and further studied in [Ho1], [Ho2]. See also [Ra].

Two idempotents belong to the same component if and only if they are similar via a finite product of exponentials [Ze1], [Au1]. Spectral characterizations of this property have been studied in [AuLaZe], [AuZe], [MaZe].

Central idempotents were characterized in [Ze1], [HLaZe]. For instance, an idempotent is central if and only if its component is a singleton [Ze1, Corollary 3.5]. Otherwise the components are unbounded, lying at distance at least $1/2$ from the centre [Ze1, Theorems 3.4 and 3.7].

In this paper, we shall study the relationship between $E(A)$ and $E(A/I)$, where I is a closed two-sided ideal in A. Note that

$$\pi E(A) \subset E(A/I),$$

where $\pi : A \to A/I$ is the canonical map. Thus, the lifting problem consists in establishing the converse inclusion.

3 Lifting of analytic families of idempotents

Lifting of a single self-adjoint idempotent has been studied for the case $A = B(H)$ and $I = K(H)$, where H is a Hilbert space, with the result that $\pi S(B(H)) = S(C(H))$; see [C, Theorem 2.4] and [Ha, Proposition 7]. For a general Banach algebra A and $I = \text{rad } A$, the lifting $\pi E(A) = E(A/\text{rad } A)$ can be found in [Ri, Theorem 2.3.9] and [K, p. 125]. The formula $\pi E(B(H)) = E(C(H))$, with a local analytic version, has been obtained in [L, Theorem 2] ; we are grateful to Professor Jean-Philippe Labrousse for personally explaining the difficult passages of his paper. In this section, we study lifting of idempotents modulo ideals whose elements have totally disconnected spectra, extending the preceding results to general Banach algebras.

Theorem 3.1 *Let $\pi : A \to B$ be a surjective homomorphism between Banach algebras. Suppose that the spectrum of each element $x \in \pi^{-1}(0)$ is totally disconnected. Let U be an open set in the complex plane, and let $q : U \to E(B)$ be an analytic map.*

Then there exists an open set $V \subset U$ and an analytic map $p : V \to E(A)$ such that $\pi p(\lambda) = q(\lambda)$ for all $\lambda \in V$. Moreover, the set V can contain an arbitrarily prescribed point in U.

Proof : Suppose that 0 is the prescribed point in U. If $q(0)$ is 0 or 1, then the function q is constant on a neighbourhood of 0, and a natural solution p is the corresponding constant. So we may assume that $\sigma\big(q(0)\big) = \{0,1\}$.

By [Le, Theorem 5.1], there exists an analytic map $a : U \to A$ such that

$$\pi a(\lambda) = q(\lambda) \text{ for all } \lambda \in U.$$

Since $\big(a(0)\big)^2 - a(0)$ belongs to $\pi^{-1}(0)$, it follows from our spectral assumption and the spectral mapping theorem that $\sigma\big(a(0)\big)$ is totally disconnected, that is, its connected components cannot consist of more than one point (a concept introduced in [S], see also [En, p. 369 and Theorem 6.1.23]). This, in turn, implies that $\mathbb{C} \setminus \sigma\big(a(0)\big)$ is connected, by [Kur2, §59, II, Theorem 1 and §47, VIII, Theorem 1]).

Thus, we can construct two Jordan curves Γ_0 and Γ_1 (consisting of polygonal lines, see [Kur2, §59, I, Theorem 1]), whose interiors are disjoint and cover $\sigma(a(0))$, with $0 \in \operatorname{int}\Gamma_0$ and $1 \in \operatorname{int}\Gamma_1$ (note that $\sigma(a(0)) \supset \sigma(q(0)) = \{0,1\}$).

By [Au2, pp. 50–51], this decomposition of the spectrum also holds for the elements $a(\lambda)$, with λ belonging to a neighbourhood V of 0.

For $\lambda \in V$, consider the Riesz idempotent

$$p(\lambda) := \frac{1}{2\pi i}\int_{\Gamma_1} (z - a(\lambda))^{-1} dz,$$

and the elements

$$a_0(\lambda) := \frac{1}{2\pi i}\int_{\Gamma_0} (1-z)^{-1}(z - a(\lambda))^{-1} dz$$

and

$$a_1(\lambda) := \frac{1}{2\pi i}\int_{\Gamma_1} z^{-1}(z - a(\lambda))^{-1} dz.$$

These are analytic functions of $\lambda \in V$ and

$$\begin{aligned} p(\lambda) &= a(\lambda)a_1(\lambda), \\ 1 - p(\lambda) &= (1 - a(\lambda))a_0(\lambda). \end{aligned}$$

Hence

$$\begin{aligned} a(\lambda) - p(\lambda) &= a(\lambda)(1 - p(\lambda)) - (1 - a(\lambda))p(\lambda) = \\ &= a(\lambda)(1 - a(\lambda))a_0(\lambda) - (1 - a(\lambda))a(\lambda)a_1(\lambda) = \\ &= ((a(\lambda))^2 - a(\lambda))(a_1(\lambda) - a_0(\lambda)) \end{aligned}$$

belongs to the ideal $\pi^{-1}(0)$.

Consequently, $\pi p(\lambda) = \pi a(\lambda) = q(\lambda)$ for all $\lambda \in V$. □

Corollary 3.2 *If, in Theorem 3.1, $B = A/\operatorname{rad} A$ and π is the canonical map, then $V = U$ satisfies the conclusion.*

Proof : In this case, we have $\sigma(\pi a(\lambda)) = \sigma(q(\lambda)) \subset \{0, 1\}$ for all $\lambda \in U$, so that the preceding proof works with $V = U$. $\square$

This improves the classical result [Ri, Theorem 2.3.9]. However, in the general situation of Theorem 3.1, it is not always possible to find a global lifting with $V = U$, for instance, when U is an annulus, see [Gr]. This leads to the natural question : how large can be the set of those $\lambda \in U$ for which $\sigma(a(\lambda))$ meets, for instance, the line Re $z = 1/2$? Outside such a set, the above argument applies.

Also, is it always possible to have $V = U$ when U is simply connected ?

More generally, suppose that the spectra of elements in $\pi^{-1}(0)$ are just disconnected or $\{0\}$. If $\pi(a^2 - a) = 0$, is it true that 0 and 1 belong to different components of $\sigma(a)$? If so, then we could proceed as in the proof of Theorem 3.1, replacing the topological considerations by [RRo, Theorem 2.10].

There is another approach to (a particular case of) Theorem 3.1, assuming the existence of a lifting for the given single class [L, Theorem 2]. This we have obtained by the way :

Corollary 3.3 *In the situation of Theorem 3.1 we have* $\pi E(A) = E(B)$.

If, moreover, A and B possess continuous involutions and π is a $*$-homomorphism, then also $\pi S(A) = S(B)$.

Proof : The first part corresponds to the constant functions q.

If $q^* = q = \pi a$, then also $q = \pi(a + a^*)/2$, so we may suppose that $a = a^*$, which implies $\sigma(a) = \overline{\sigma(a)}$. Choosing Γ_1 symmetric with respect to the real axis yields $p = p^*$. This proves the second part. $\square$

The lifting $\pi E(A) = E(B)$ can also be obtained as in the radical case [Ri, p. 58], [K, p. 124], by using a theorem of Harte [Au2, Theorem 3.3.8]. This approach only requires the weaker hypothesis that the spectra of elements in the kernel of the surjection π do not disconnect the complex plane.

Nevertheless, the lifting is not possible modulo a general kernel. For instance, $A = C([0, 1])$, $B = C(\{0, 1\})$, $\pi f = f|\{0, 1\}$ for $f \in C([0, 1])$, does not admit lifting of idempotents.

We intend to elucidate the role of the spectral hypothesis for the simple example where $A = C(X)$ and $B = C(Y)$ are the Banach algebras of continuous functions on compact metric spaces X and Y, respectively. The surjection $\pi : C(X) \to C(Y)$ gives, by the Gelfand representation, an injection $Y \subset X$, such that for $f \in C(X)$ we have $\pi f = f|Y \in C(Y)$. Recall that the spectrum of $f \in C(X)$ equals the range $R(f) := f(X)$ of f.

If the difference $X \setminus Y$ is at most countable, then also $\sigma(f)$ is at most countable for each $f \in \pi^{-1}(0) = \{f \in C(X) : f|Y = 0\}$ and our theorem applies.

If $X \setminus Y$ is uncountable, then there is an n such that the compact set $X_n := \{x \in X : \mathrm{dist}(x, Y) \geq 1/n\}$ is uncountable. Hence X_n contains a homeomorphic copy C of the Cantor set, see [Kur1, §36, V, Corollary 1]. Moreover, there exists a continuous map of C onto $[0, 1]$, see [Kur2, §41, VI, Corollary 3a], which we can extend to a continuous map $C \cup Y \to [0, 1]$, by defining it as 0 on Y.

By the Tietze extension theorem [Kur1, §14, IV, p. 127], we can further extend the latter map to a function $g \in C(X)$, with values in $[0,1]$. Then $g \in \pi^{-1}(0)$ and $\sigma(g) = R(g) = [0,1]$ is not totally disconnected. The function $e^{2\pi i g} - 1$ also belongs to $\pi^{-1}(0)$, and its spectrum $\{z \in \mathbb{C} : |z| = 1\} - 1$ even disconnects $\mathbb{C}$. Summing up : if $X \setminus Y$ is uncountable, then the hypothesis of our theorem is not satisfied, and even the weaker hypothesis, about not disconnecting $\mathbb{C}$, is not satisfied either.

Nevertheless, the conclusion of our theorem is valid, in the example considered, if $X \setminus Y$ is 0-dimensional (i.e., each point has a neighbourhood base consisting of open-and-closed sets). Namely, here lifting of idempotents is equivalent to the following : for every partition of Y consisting of two open-and-closed sets Y_1, Y_2 there is a partition of X formed by open-and-closed sets X_1, X_2, such that $Y_1 \subset X_1$ and $Y_2 \subset X_2$. We shall show that here this property holds. We have $\mathrm{dist}(Y_1, Y_2) > 0$; let $0 < 5\varepsilon < \mathrm{dist}(Y_1, Y_2)$. Consider the open ε-neighbourhoods $U_\varepsilon(Y_i)$ of Y_i in X $(i = 1, 2)$, and their closures $\mathrm{cl}\, U_\varepsilon(Y_i)$. For each boundary point $x_i \in \mathrm{bd}\, U_\varepsilon(Y_i)$, let $V(x_i)$ be an open-and-closed neighbourhood of x_i, contained in $U_\varepsilon(x_i)$. The compact set $\mathrm{cl}\, U_\varepsilon(Y_i)$ is covered by the open sets $U_\varepsilon(Y_i)$ and $V(x_i)$, $x_i \in \mathrm{bd}\, U_\varepsilon(Y_i)$. Their finite subcover has a union A_i, which is an open-and-closed set, with $\mathrm{dist}(A_1, A_2) > \varepsilon$, so $A_1 \cap A_2 = \emptyset$. Hence $\{A_1, X \setminus A_1\}$ is a desired partition of X.

Thus, the spectral hypothesis in our theorem is not necessary for the validity of its conclusion, even in the case of commutative algebras.

On the other hand, Banach algebras where every element has totally disconnected spectrum admit elegant characterizations and natural examples, see [Z1], [Z2].

In the next sections we shall study the details of the following

Corollary 3.4 *Let X be a Banach space. Then $\pi E(B(X)) = E(C(X))$.*
If H is Hilbert space, then also $\pi S(B(H)) = S(C(H))$.

4 The connected components of the idempotents in the Calkin algebra of a Hilbert space

A Hilbert space is determined, up to a bounded linear isomorphism, by its dimension [AG, p. 39]. Thus, two idempotents in $B(H)$ are similar if and only if their ranges and kernels have the same dimensions, respectively. Since the group of invertible operators in $B(H)$ is connected [Ri, p. 280], the components of $E(B(H))$ just correspond to the pairs of cardinals, call them *coordinates*, indicating the corresponding dimensions. The same is also true for $S(B(H))$, by the connectedness of the unitary group (see [Ri, p. 281], [Ku]), or by [BFMiL, p. 61], [M], [T2, Lemma 1]. See also [MaZe, p. 94].

Of course, the sum of coordinates is equal to the dimension of the Hilbert space H.

Clearly, the components of $E(B(H))$ having at least one finite coordinate are mapped by π to the components $\{0\}$ or $\{1\}$ of $E(C(H))$. We shall show that π induces a one-to-one correspondence between the other components of $E(B(H))$ and $E(C(H))$. In view of Corollary 3.4, it is enough to prove that

$$\|\pi(P-Q)\| \geq 1$$

whenever P and Q belong to different components of $E(B(H))$, with infinite coordinates. Since the finite rank operators F are dense in $K(H)$, it is enough to show that

$$\|P-Q-F\| \geq 1.$$

But this is obvious : since either $R(P) \cap N(Q)$ or $N(P) \cap R(Q)$ has infinite dimension, and $N(F)$ has finite codimension, there are unit vectors x in the suitable intersection such that $\|(P-Q-F)x\| = 1$.

Thus we have

Proposition 4.1 *For a Hilbert space, both correspondences in Corollary 3.4 are one-to-one for the components with both coordinates infinite.* □

5 The connected components of the idempotents in the Calkin algebra of a Banach space

In a Banach space, subspaces of the same infinite dimension may not be isomorphic (see below), the group of invertible operators may not be connected [Do], and the finite rank operators may not be dense in the compact operators [E]. Thus, each of the three main ingredients of the argument in the preceding section may fail in general (perhaps all simultaneously ?), and so is with the conclusion.

Proposition 5.1 *There exists a Banach space X for which the canonical surjection sends two different components of $E(B(X))$, with infinite coordinates, to the same component of $E(C(X))$.*

Proof : There exists an infinite-dimensional Banach space Y that is not isomorphic to $Y \oplus \mathbb{C}$, see [Go, pp. 935–938], [GoMau, Corollary 19]. Let

$$X := Y \oplus \mathbb{C} \oplus \ell_2,$$

and consider the natural projections

$$P : X \to Y \text{ and } Q : X \to Y \oplus \mathbb{C}.$$

Then the difference $P - Q$ has rank one, so it is compact. However, if P and Q were in the same connected component of $E(B(X))$, then they would be similar, hence their ranges isomorphic, a contradiction. Since the ranges and kernels of both projections are infinite-dimensional, the component of $\pi P = \pi Q$ in $E(C(X))$ is different from $\{0\}$ and $\{1\}$. □

Nevertheless, it is possible to prove a weaker conclusion, namely that if πP and πQ belong to the same component of $E(C(X))$, different from $\{0\}$ and

$\{1\}$, then the ranges and kernels of the projections P and Q have the same cardinalities, respectively, even if P and Q may belong to different components of $E(B(X))$. (We observe that the cardinality $|.|$ of the subspace is (in case of infinite dimension) the same as the algebraic dimension, cf. [PBe, p.248, Theorem 7.9]). Indeed, by using the Gelfand measure of non-compactness [Ze2, pp. 222–224], one can follow the argument in the preceding section to show that $\|\pi(P-Q)\| \geq 1$ unless the corresponding cardinalities coincide.

In fact, for $R(P) \cap N(Q)$ infinite-dimensional, we have $\|P - Q - K\| \geq 1$ for any compact operator K. For $R(P) \cap N(Q)$ finite-dimensional, we have $R(P) = (R(P) \cap N(Q)) \oplus Y$ for a closed finite-codimensional subspace $Y \subset R(P)$. Then we have $|R(P)| = |Y| \leq |R(Q)|$ since Q is by $Y \cap N(Q) = \{0\}$ injective on Y. Similarly we gain that unless $\|\pi(P-Q)\| \geq 1$ we have $|R(Q)| \leq |R(P)|$, and also $|N(P)| = |N(Q)|$. Then Corollary 3.4 yields the desired conclusion.

The result can also be formulated in terms of densities.

Let us conclude with some qualitative questions. Is it possible that, for some infinite-dimensional Banach space X, the set $E(C(X))$ can have more than two isolated points, or only isolated points, or even that $C(X)$ can be commutative?

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