THE CROSS-SECTION BODY, PLANE SECTIONS OF CONVEX BODIES AND APPROXIMATION OF CONVEX BODIES, II*

E. Makai Jr., and H. Martini*

Abstract

We compare the volumes of projections of convex bodies and the volumes of the projections of their sections, and, dually, those of sections of convex bodies and of sections of their circumscribed cylinders. For $L \subset \mathbb{R}^d$ a convex body, we take n random segments in L and consider their 'Minkowski average' D. For fixed n, the pth moments of $V(D)$ $(1 \leq p < \infty)$ are minimized, for $V(L)$ fixed, by the ellipsoids. For $k = 2$ and fixed n, the pth moment of $V(D)$ is maximized for example by triangles, and, for L centrally symmetric, for example by parallelograms. Last we discuss some examples for cross-section bodies.

Keywords: projections, sections, random polytopes, approximation, ellipsoids, cross-section body

Mathematics Subject Classifications (1991): 52A20, 52A22, 52A27, 52A40

*Research (partially) supported by Hungarian National Foundation for Scientific Research, grant no. 41

Part I of this paper (cf. [MM]) contains a more detailed summary of Part II as well, and also all the references, to both parts. Part I contains sections 1 – 7, Part II contains sections 8 – 11.

Let $\mathbb{R}^d$, $d \geq 2$, denote the d-dimensional Euclidean vector space, and a convex body $K \subset \mathbb{R}^d$ is a compact, convex set with non-empty interior. We denote by V_k k-dimensional Lebesgue measure, for $k = d$ (the dimension of the space) we write V for V_d. For a convex body $K \subset \mathbb{R}^d$ its *central symmetral* is $\triangle K = (K + (-K))/2$, and *centred* means: symmetric about the origin.

§ 8 Volume estimates for projections of convex bodies and projections of their sections

Analogously to section 3 in [MM], also at Theorem 7.1 we can investigate the more general question concerning k-planes rather than hyperplanes ($1 \leq k \leq d-1$), and volumes of sections by k-planes and of projections to k-planes. For this we will use the geometric reformulation of the problem from the proof of Theorem 7.1, i.e., for a convex body $K \subset \mathbb{R}^d$ we looked for a constant c satisfying the following: $\forall u \in S^{d-1}\ \exists v \in S^{d-1}\ V_{d-1}(K \mid v^\perp) \leq c \cdot \max_{\lambda \in \mathbb{R}} \cdot V_{d-1}((K \cap (u^\perp + \lambda u)) \mid v^\perp)$, where $A \mid L$ means the orthogonal projection of a set A to a subspace L, and $v^\perp = (\mathbb{R} \cdot v)^\perp$, where $^\perp$ denotes orthogonal complement.

We let

$$f(k,d) = \sqrt{k} \cdot \frac{\sqrt{k} + \sqrt{(d-1)(d-k)}}{d}$$

for $d \leq \frac{k(k+1)}{2}$, and

$$f(k,d) = \frac{2 + (k-1)\sqrt{k+2}}{k+1}$$

for $d > \frac{k(k+1)}{2}$.

Theorem 8.1. *Let $K \subset \mathbb{R}^d$ be any convex body and k with $1 \leq k \leq d-1$ be any integer. Then for any linear k-subspace L_k there exists a linear k-subspace L'_k such that*

$$V_k(K \mid L'_k) \leq f(k,d)^k \cdot \left(\frac{k}{2}\frac{\kappa_k}{\kappa_{k-1}}\right)^k \cdot \max_{z \in L_k^\perp} V_k((K \cap (L_k + z)) \mid L'_k).$$

If K is also centrally symmetric, then

$$V_k(K \mid L'_k) \leq f(k,d)^k \cdot \max_{z \in L_k^\perp} V_k((K \cap (L_k + z)) \mid L'_k).$$

Proof. The proof is similar to that of Theorem 7.1. First let K be centrally symmetric. We may suppose that K is centred, and let M^d be the Minkowski space with unit ball K. By [KLL], Theorem 1 (for $d \leq k(k+1)/2$) and [KT-J], Theorem 1.1 (for $d > k(k+1)/2$), there exists a projection $P : M^d \to M^d$ (i.e., a bounded linear operator satisfying $P^2 = P$) with image $P(M^d) = L_k$ satisfying $\| P \| \leq f(k,d)$ (the mentioned papers assert this for any linear k-subspace). We let $L'_k = (P^{-1}(0))^{\perp}$, and then we have from $P(K) \subset f(k,d) \cdot (K \cap L_k)$ that

$$\frac{V_k(K \mid L'_k)}{V_k((K \cap L_k) \mid L'_k)} = \frac{V_k(P(K))}{V_k(P(K \cap L_k))} = \frac{V_k(P(K))}{V_k(K \cap L_k)} \leq f(k,d)^k.$$

If K is not centrally symmetric, then we have, for the k-plane L'_k associated to $\triangle K$ and L_k, by the Brunn-Minkowski theorem, the above established inequality and Theorem 6.3

$$\begin{aligned} V_k(K \mid L'_k) &\leq V_k(\triangle K \mid L'_k) \leq f(k,d)^k V_k(((\triangle K) \cap L_k) \mid L'_k) \leq \\ &\leq f(k,d)^k \left(\frac{k}{2} \frac{\kappa_k}{\kappa_{k-1}} \right)^k \cdot \max_{z \in L_k^{\perp}} V_k((K \cap (L_k + z)) \mid L'_k), \end{aligned}$$

as asserted. □

There arises the question if in Theorem 8.1 one can interchange the role of intersections and projections. With the notation $f(k,d)$ from Theorem 8.1 we prove

THEOREM 8.2. *Let $K \subset \mathbb{R}^d$ be any convex body and k be any integer with $1 \leq k \leq d-1$. Then for any linear k-subspace L_k there exists a linear k-subspace L'_k such that*

$$V_k(K \mid L_k) \leq f(k,d)^k \left(\frac{k}{2} \frac{\kappa_k}{\kappa_{k-1}} \right)^k \cdot \max_{z \in L_k'^{\perp}} \cdot V_k((K \cap (L'_k + z)) \mid L_k).$$

If K is also centrally symmetric, then

$$V_k(K \mid L_k) \leq f(k,d)^k \cdot \max_{z \in L_k'^{\perp}} V_k((K \cap (L'_k + z)) \mid L_k).$$

For the proof of this theorem we need some lemmas about normed linear spaces. In Lemmas 8.3 – 8.5 subspace will mean a linear subspace. By a map between normed linear spaces we mean a bounded linear map. We recall that a map between normed linear spaces ($T : X \to Y$, say) is called a *quotient map* if $T(X) = Y$ and $\forall y \in Y$ the relation
$\| y \| = \inf\{\| x \| : x \in X, T(x) = y\}$ holds. The value of a bounded linear functional x^* at x will be denoted by $\langle x, x^* \rangle$. The dual space of a normed linear space X (i.e., all bounded linear functionals on X) will be denoted by X^*, and for a map $T : X \to Y$ of normed linear spaces $T^* : Y^* \to X^*$ denotes the adjoint map.

Then we have the following lemma whose proof uses similar ideas to those of [D].

Lemma 8.3. *Let $1 \le k < d$ be integers, let M^d, M^k be real normed linear spaces of dimensions d and k, respectively, and let $Q : M^d \to M^k$ be a quotient map. Then there exists a map $S : M^k \to M^d$ such that QS is the identity map on M^k, and $\| S \| \le f(k,d)$.*

Proof. Let us consider the dual spaces $(M^d)^*, (M^k)^*$ and the adjoint map $Q^* : (M^k)^* \to (M^d)^*$ of Q. Since Q is a quotient map, Q^* is an isometric embedding of $(M^k)^*$ to a k-subspace of $(M^d)^*$, namely to $Q^*((M^k)^*)$. Then by [KLL] and [KT-J] there exists a projection $P : (M^d)^* \to (M^d)^*$ with image $Q^*((M^k)^*)$ and of norm $\| P \| \le f(k,d)$. This, considered as a map $(M^d)^* \to Q^*((M^k)^*)$, composed with the inverse of Q^* (defined on $Q^*((M^k)^*)$) gives a map $R : (M^d)^* \to (M^k)^*$ such that RQ^* is the identity map on $(M^k)^*$ and $\| R \| \le f(k,d)$. Turning to adjoints again, we have $S := R^* : M^k = (M^k)^{**} \to (M^d)^{**} = M^d$, with $QS = (RQ^*)^*$ being the identity map on M^k and $\| S \| = \| R \| \le f(k,d)$. □

Lemma 8.4. *Let $1 \le k < d$ be integers, let M^d, M^k be real normed linear spaces of dimensions d and k, respectively, and let $Q : M^d \to M^k$ be a quotient map. Then there exists a subspace N^k of M^d, of dimension k, such that for $x \in N^k$ we have $\| Qx \| \ge \| x \| / f(k,d)$.*

Proof. By Lemma 8.3 we have $S : M^k \to M^d$, such that QS is the identity map on M^k, and $\| S \| \le f(k,d)$. Then $N^k := S(M^k)$ is a k-dimensional subspace of M^d. For $x \in N^k \setminus \{0\}$ we have $x = Sy$ with $y \in M^k \setminus \{0\}$, hence $\| Qx \| / \| x \| = \| QSy \| / \| Sy \| = \| y \| / \| Sy \| \ge 1 / \| S \| \ge 1/f(k,d)$. □

For the sake of completeness we prove the dual statement as well.

Lemma 8.5. *Let $1 \le k < d$ be integers, let M^d, M^k be real normed linear spaces of dimensions d and k, respectively, such that M^k is a subspace of M^d. Then there exists a quotient map Q from M^d to a k-dimensional real normed linear space such that for $x \in M^k$ we have $\| Qx \| \ge \| x \| / f(k,d)$.*

Proof. By [KLL] and [KT-J] there exists a projection $P : M^d \to M^d$, with image M^k, and of norm $\| P \| \le f(k,d)$. Let Q denote the quotient map of M^d to $M^d/P^{-1}(0)$, the second space taken with the usual quotient normed linear space structure. Then we have $P = P_1 Q$, for a unique map $P_1 : M^d/P^{-1}(0) \to M^d$, and we have $\| P_1 \| = \| P \|$. Then for $x \in M^k \setminus \{0\}$ we obtain $\| Qx \| / \| x \| = \| Qx \| / \| Px \| = \| Qx \| / \| P_1 Qx \| \ge 1 / \| P_1 \| = 1 / \| P \| \ge 1/f(k,d)$. □

Proof of Theorem 8.2. We proceed as in Theorems 7.1 and 8.1. First let K be centrally symmetric. We may suppose that K is centred, and let M^d be the Minkowski space with unit ball K. Then the orthogonal projection of M^d to L_k is a quotient map Q from M^d to the Minkowski space on L_k with unit ball $K \mid L_k$. By Lemma 8.4 there exists a k-subspace $L'_k = N^k$ of M^d such that for $x \in L'_k$ we have $\| Qx \| \ge \| x \| / f(k,d)$, where $\| Qx \|$ is considered with respect to the above quotient norm on L_k. Hence the restriction of Q to L'_k is an isomorphism of L'_k to L_k, and the image by Q of the unit ball of the subspace L'_k of M^d contains $f(k,d)^{-1}$ times the unit ball of L_k, with the quotient norm, i.e., $(K \cap L'_k) \mid L_k \supset f(k,d)^{-1}(K \mid L_k)$. This implies $V_k((K \cap L'_k) \mid L_k) \ge f(k,d)^{-k} V_k(K \mid L_k)$, as we had to prove.

The statement for the general case follows from that in the centrally symmetric case as in Theorems 7.1 and 8.1. $\square$

Now we give a reformulation of Theorems 8.1 and 8.2 which can also be considered as a statement dual to them, comparing the volumes of sections of convex bodies and the volumes of sections of their circumscribed cylinders.

COROLLARY 8.6. *Let $K \subset \mathbb{R}^d$ be any convex body and k be any integer with $1 \le k \le d-1$. Then for any linear k-subspace L_k there exists a linear k-subspace L'_k such that $L_k \cap L'^{\perp}_k = \{0\}$ and*

$$V_k((K + L'^{\perp}_k) \cap L_k) \le f(k,d)^k \left(\frac{k}{2}\frac{\kappa_k}{\kappa_{k-1}}\right)^k \cdot \max_{z \in L_k^{\perp}} V_k(K \cap (L_k + z)).$$

If K is also centrally symmetric, then

$$V_k((K + L'^{\perp}_k) \cap L_k) \le f(k,d)^k \cdot \max_{z \in L_k^{\perp}} V_k(K \cap (L_k + z)).$$

Moreover, these statements remain valid if in these equalities and inequalities L_k and L'_k are interchanged.

PROOF. In Theorem 8.1 we have $L_k \cap L'^{\perp}_k = \{0\}$. Using

$$K \mid L'_k = (K + L'^{\perp}_k) \mid L'_k = ((K + L'^{\perp}_k) \cap L_k) \mid L'_k,$$

we obtain

$$\frac{V_k(K \cap (L_k + z))}{V_k((K + L'^{\perp}_k) \cap L_k)} = \frac{V_k((K \cap (L_k + z)) \mid L'_k)}{V_k[((K + L'^{\perp}_k) \cap L_k) \mid L'_k]} = \frac{V_k((K \cap (L_k + z)) \mid L'_k)}{V_k(K \mid L'_k)},$$

and then 8.1 implies the first two inequalities.

If we interchange in our equalities and inequalities L_k and L'_k, then we proceed similarly, applying 8.2. $\square$

REMARK 8.7. Theorems 8.1 and 8.2, for $k = 1$, are sharp, but then the statements are evident. For $k = d-1$ the statement of Theorem 8.1 reduces to that of Theorem 7.1. Also for Theorem 8.2, for the case $k = d-1$, we can conjecture that the sharp constants in the inequalities for general, or centrally symmetric convex bodies are those given in Conjecture 7.2, and are attained for K a simplex, or a cross-polytope (with the choice of the normals u and v of L_k and L'_k as described before Conjecture 7.2), respectively. We mention some estimates of $f(k,d)$ from [KLL], p. 341, and [KT-J], p. 255: For $d \le k(k+1)/2$ one has

$$f(k,d) \le \min\left(\sqrt{k}\left(1 - \frac{(\sqrt{k}-1)^2}{2d}\right), \quad \frac{k}{d} + \sqrt{1 - \frac{k}{d}} \cdot \sqrt{k}, \ \sqrt{k} - \frac{1}{\sqrt{k}} + O\left(\frac{1}{k}\right)\right),$$

while for $d > \frac{k(k+1)}{2}$ one has

$$\sqrt{k}\left(1 - \frac{(\sqrt{k}-1)^2}{2d}\right) > f(k,d) = \sqrt{k} - \frac{1}{\sqrt{k}} + O\left(\frac{1}{k}\right).$$

Let us assume that $k > 1$. The above estimates of $f(k,d)$ imply that our estimates from Theorems 8.1 and 8.2 are in most cases better than those by John's ellipsoids, see [J], pp. 202-203. For the centrally symmetric case John's ellipsoids give a factor

$$d^{k/2} \geq k^{k/2} > f(k,d)^k$$

on the right hand side of the inequality. For the general case ([J], p. 202, Theorem III) they give a factor d^k, and, using the upper estimate $\sqrt{k}\left(1-(\sqrt{k}-1)^2/(2d)\right)$ for $f(k,d)$, one can establish the inequality that d^k is larger than our factor, except possibly finitely many pairs (k,d). (This inequality is quadratic for d, with discriminant negative for k large enough.) Numerical checking suggests that actually there are no exceptional pairs. ■

§9 Approximation of convex bodies by random polytopes; lower estimates

Now we prove analogues of Lemmas 5.6, 5.7 and Theorems 5.8, 5.9. Analogously to [Bu 2], [Gro 2], [Gro 3], [Gro 4], [BMMP], we can investigate not only the supremum of $V(D(x_1,y_1,\cdots,x_n,y_n))$, where $x_1,y_1 \in L_1,\cdots,x_n,y_n \in L_n$, $L_1,\cdots,L_n \subset \mathbb{R}^d$ are convex bodies, $n \geq d$, and $D(x_1,y_1,\cdots,x_n,y_n) = ([x_1,y_1]+\cdots+[x_n,y_n])/n$ ('Minkowski average' of the segments $[x_i,y_i]$, $1 \leq i \leq n$, also written as $D(x_i,y_i;\ 1 \leq i \leq n)$), but also its mean value, or p-th moment, or other quantities related to its "average behaviour". In the special case $L_1 = \cdots = L_n = L$ we can ask, which are the convex bodies L of given volume that have these quantities minimal, that is, which are worst approximated by the bodies $D(x_1,y_1,\cdots,x_n,y_n)$ "in the average".

Let therefore $\varphi : [0,\infty) \to \mathbb{R}$ be a non-decreasing convex function (which is therefore continuous), for example $\varphi(t) = t^p$, $1 \leq p < \infty$. For $n \geq d$ and convex bodies $L_1,\cdots,L_n \subset \mathbb{R}^d$, define

$$\beta_{n,\varphi}(L_1,\cdots,L_n) = \int_{L_n}\int_{L_n}\cdots\int_{L_1}\int_{L_1} \varphi(V(D(x_1,y_1,\cdots,x_n,y_n)))dx_1dy_1\cdots dx_ndy_n,$$

the integrals taken with respect to Lebesgue measure. For $\varphi(t) = t^p$, $1 \leq p < \infty$, we write

$$\beta_{n,\varphi}(L_1,\cdots,L_n) = \beta_n^p(L_1,\cdots,L_n),$$

that is the pth moment of $V(D(x_1, y_1, \cdots, x_n, y_n))$. In the special case $L_1 = \cdots = L_n = L$ we write

$$\beta_{n,\varphi}(L, \cdots, L) = \beta_{n,\varphi}(L)$$

and

$$\beta_n^p(L, \cdots, L) = \beta_n^p(L).$$

Now, following [Bu 2], [Gro 2], [Gro 3], [Gro 4], [BMMP], we prove an analogue of Lemma 5.7.

LEMMA 9.1. *Let $n \geq d$ be an integer, $L_1, \cdots, L_n \subset \mathbb{R}^d$ be convex bodies, and $S(L_1), \cdots, S(L_n)$ their Steiner symmetrals with respect to some hyperplane H. Furthermore, let $\varphi : [0, \infty) \to \mathbb{R}$ be a non-decreasing convex function. Then*

$$\beta_{n,\varphi}(L_1, \cdots, L_n) \geq \beta_{n,\varphi}(S(L_1), \cdots, S(L_n)).$$

Here we have equality if for each $i \in \{1, \cdots, n\}$ the midpoints of all segments, that are the intersections of L_i and some line orthogonal to H, lie in some hyperplane H_i and these hyperplanes H_i are parallel, and, for φ strictly increasing, only in this case.

PROOF. First we establish that in the asserted case of equality in fact equality holds. This follows from the fact that in this case the Steiner symmetrization of L_i (i.e., the map $S : L_i \to S(L_i)$, mapping each nonempty intersection $L_i \cap l$, where l is a line orthogonal to H, to $S(L_i) \cap l$ by translation) is a volume preserving affinity.

Further on, we use the notations of the proof of Lemma 5.7. In particular, we suppose that H is spanned by 0 and the first $d-1$ basic vectors. Points x_i, y_i (etc.) will be written as $(x_i', \xi_i), (y_i', \eta_i)$ (etc.), where x_i', y_i' are the projections of x_i, y_i to H, and ξ_i, η_i are the dth coordinates of x_i, y_i. Let (x_i', ξ_i), $(y_i', \eta_i) \in S(L)$, $1 \leq i \leq n$. We write $D^+ = D((x_i', \xi_i), (y_i', \eta_i); 1 \leq i \leq n)$ and $D^- = D((x_i', -\xi_i), (y_i', -\eta_i); 1 \leq i \leq n)$. We denote the projection of L_i to H by L_i'. For $x_i' \in L_i'$ we denote by $(x_i', h_i(x_i'))$ the midpoint of the segment which is the intersection of L_i and the line orthogonal to H and passing through x_i'. By the proof of 5.7, h_i is continuous on rel int L_i' and is Borel on L_i'. We have $(x_i', \pm\xi_i + h_i(x_i'))$, $(y_i', \pm\eta_i + h_i(y_i')) \in L_i$, and we write $D_a^\pm = D((x_i', \pm\xi_i + h_i(x_i')), (y_i', \pm\eta_i + h_i(y_i')); 1 \leq i \leq n)$.

From formulas (2) and (3) in the proof of 5.7, we have

$$V(D_a^+) + V(D_a^-) \geq 2V(D^+) = V(D^+) + V(D^-).$$

This implies

$$(4) \begin{cases} [\varphi(V(D_a^+)) + \varphi(V(D_a^-))]/2 \geq \varphi([V(D_a^+) + V(D_a^-)]/2) \geq \\ \geq [\varphi(V(D^+)) + \varphi(V(D^-))]/2. \end{cases}$$

Observe further that if we have random pairs of points x_i, y_i in L_i, for $i = 1, \cdots, n$, then the Steiner symmetrization of L_i (i.e., the above map $S : L_i \to S(L_i)$) maps these random pairs of points to random pairs Sx_i, Sy_i in $S(L_i)$, for $i = 1, \cdots, n$. That is, these new pairs of points also are independent and uniformly distributed in the respective $S(L_i)$'s.

Let the segment, that is the intersection of $S(L_i)$ and of the line orthogonal to H and passing through $x'_i \in L'_i$, be given by $\{(x'_i, \xi_i) \mid -f_i(x'_i) \leq \xi_i \leq f_i(x'_i)\}$. Similarly to the case of h_i in Lemma 5.7, here f_i is continuous on rel int L'_i and Borel on L'_i. Then the segment, which is the intersection of L_i and of the line orthogonal to H and passing through x'_i, is given by $\{(x'_i, \xi_i) \mid h_i(x'_i) - f_i(x'_i) \leq \xi_i \leq h_i(x'_i) + f_i(x'_i)\}$. Recall that $x_i = (x'_i, \xi_i)$, $y_i = (y'_i, \eta_i)$. Integrating (4) with respect to $\xi_1, \eta_1, \cdots, \xi_n, \eta_n$, we obtain

$$(5) \begin{cases} \int_{-f_n(y'_n)}^{f_n(y'_n)} \cdots \int_{-f_1(x'_1)}^{f_1(x'_1)} \varphi(V(D_a^+)) d\xi_1 \cdots d\eta_n = \int_{-f_n(y'_n)}^{f_n(y'_n)} \cdots \int_{-f_1(x'_1)}^{f_1(x'_1)} ([\varphi(V(D_a^+)) \\ +\varphi(V(D_a^-))]/2)\, d\xi_1 \cdots d\eta_n \geq \int_{-f_n(y'_n)}^{f_n(y'_n)} \cdots \int_{-f_1(x'_1)}^{f_1(x'_1)} ([\varphi(V(D^+)) + \\ \varphi(V(D^-))\,]/2)\, d\xi_1 \cdots d\eta_n = \int_{-f_n(y'_n)}^{f_n(y'_n)} \cdots \int_{-f_1(x'_1)}^{f_1(x'_1)} \varphi(V(D^+)) d\xi_1 \cdots d\eta_n. \end{cases}$$

Leaving out the middle two terms from (5), and then integrating it with respect to $x'_1, y'_1, \cdots,$
x'_n, y'_n, we obtain the inequality of the lemma.

Now we proceed to prove that, for φ strictly increasing, we have here strict inequality, except in the case mentioned in the lemma. Evidently for $x'_i, y'_i \in$ rel int L'_i, and also for $x'_i \in L'_i \cap l_i$, $y'_i \in L'_i \cap \tilde{l}_i$, where $l_i, \tilde{l}_i$ are lines in H, intersecting rel int L'_i, the first and last quantities in (4) are continuous functions of $x_1, y_1, \cdots, x_n, y_n$. By compactness considerations both sides of (5) are continuous functions of $x'_1, y'_1, \cdots, x'_n, y'_n$, for $x'_i, y'_i \in$ rel int L'_i, and also for $x'_i \in L'_i \cap l_i$, $y'_i \in L'_i \cap \tilde{l}_i$, with $l_i, \tilde{l}_i$ like above. Hence it suffices to establish that, except in the case mentioned in the lemma, for some $x_1, y_1, \cdots, x_n, y_n$ in (4) strict inequality holds between the first and last quantities. Namely, then by the above continuity properties in a non-empty open subset of a neighbourhood of $(x_1, y_1, \cdots, x_n, y_n)$ we will have strict inequality in (4) between the first and last quantities, hence by integrating we will have strict inequality in (5), for a non-empty open subset of $(x'_1, y'_1, \cdots, x'_n, y'_n)$'s. Repeating this consideration, we will have strict inequality in the inequality of the lemma.

To establish, for some $x_1, y_1, \cdots, x_n, y_n$, strict inequality in (4) between the first and last quantities, it is sufficient to prove

$$(6) \quad V(D_a^+) + V(D_a^-) > 2V(D^+).$$

Namely, in this case the second inequality in (4) will be strict since φ is strictly increasing. In other words, we have to establish that in formula (3) in the proof of Lemma 5.7 we have strict inequality, for some $x_1, y_1, \cdots, x_n, y_n$. Turning to the mentioned formula, we have one inequality in that chain of equalities and inequalities. Choosing $\xi_i = 0 \in [-f_i(x_i'), f_i(x_i')]$, $\eta_i = 0 \in [-f_i(y_i'), f_i(y_i')]$, this inequality becomes

$$(7) \begin{cases} V\left(\sum\limits_{i=1}^{n}[(0,0),(y_i'-x_i',h_i(y_i')-h_i(x_i'))]\right)+ \\ +V\left(\sum\limits_{i=1}^{n}[(0,0),(y_i'-x_i',-h_i(y_i')+h_i(x_i'))]\right) \\ \geq 2V\left(\sum\limits_{i=1}^{n}[(0,0),(y_i'-x_i',0)]\right)=0, \end{cases}$$

since all vectors $(y_i' - x_i', 0)$ lie in a linear $(d-1)$-subspace.

Now we prove that, except in the case mentioned in the lemma, for some x_i', y_i' in (7) strict inequality holds. Assuming the contrary, we look for a contradiction. Fixing $x_1', \cdots, x_n'$, and letting $y_1', \cdots, y_n'$ vary, we see that all vectors $(y_i' - x_i', h_i(y_i') - h_i(x_i'))$ lie in some hyperplane not parallel to the dth basic vector. Thus we have the case of equality described in the lemma, which we have excluded. This contradiction proves our claim. □

Again following [Bu 2], [Gro 2], [Gro 3], [Gro 4], [BMMP], we prove analogues of Theorems 5.8 and 5.9.

Theorem 9.2. *Let $L \subset \mathbb{R}^d$ be any convex body and $B \subset \mathbb{R}^d$ be a ball, with $V(B) = V(L)$. Furthermore, let $n \geq d$ be any integer and $\varphi : [0, \infty) \to \mathbb{R}$ be a non-decreasing convex function. Then*

$$\beta_{n,\varphi}(L) \geq \beta_{n,\varphi}(B).$$

Here equality holds if L is an ellipsoid and, for φ strictly increasing, only in this case. In particular, for $1 \leq p < \infty$ we have

$$\beta_n^p(L) \geq \beta_n^p(B),$$

with equality if and only if L is an ellipsoid.

This theorem follows from

Theorem 9.3. *Let $n \geq d$ be an integer, $L_1, \cdots, L_n \subset \mathbb{R}^d$ be convex bodies, and $B_1, \cdots, B_n \subset \mathbb{R}^d$ balls with $V(B_i) = V(L_i)$, $1 \leq i \leq n$. Furthermore, let $\varphi : [0, \infty) \to \mathbb{R}$ be a non-decreasing convex function. Then*

$$\beta_{n,\varphi}(L_1, \cdots, L_n) \geq \beta_{n,\varphi}(B_1, \cdots, B_n).$$

Here equality holds if the L_i's are homothetic ellipsoids and, for φ strictly increasing, only in this case. In particular, for $1 \leq p < \infty$ we have

$$\beta_n^p(L_1, \cdots, L_n) \geq \beta_n^p(B_1, \cdots, B_n),$$

with equality if and only if the L_i's are homothetic ellipsoids.

PROOF. First we establish that $\beta_{n,\varphi}(L_1, \cdots, L_n)$ is a continuous function of $L_1, \cdots, L_n$. We may suppose $\varphi(0) \geq 0$, otherwise replace φ by $\varphi - \varphi(0)$. Also we may suppose $0 \in \operatorname{int} L_i$, and then consider the bodies $\left(1 - \frac{1}{m}\right) L_i$, $\left(1 + \frac{1}{m}\right) L_i$, where $m \geq 2$ is any integer. Since now $\beta_{n,\varphi}(L_1, \cdots, L_n)$ is monotonic in its arguments, it suffices to prove

$$\beta_{n,\varphi}\left((1-1/m)L_1, \cdots, (1-1/m)L_n\right), \beta_{n,\varphi}\left((1+1/m)L_1, \cdots, (1+1/m)L_n\right) \longrightarrow \beta_{n,\varphi}(L_1, \cdots, L_n), \quad \text{for } m \to \infty. \tag{8}$$

However, $\beta_{n,\varphi}(L_1, \cdots, L_n)$ is defined by a $2n$-fold integral, with domain of integration $L_n \times L_n \times \cdots \times L_1 \times L_1$, which equals $\bigcap_{m=2}^{\infty} [(1+1/m)L_n \times \cdots \times (1+1/m)L_1]$. Similarly, $\bigcup_{m=2}^{\infty} [(1-1/m)L_n \times \cdots \times (1-1/m)L_1] = \operatorname{int} L_n \times \cdots \times \operatorname{int} L_1$, and the integral over $(L_n \times \cdots \times L_1) \setminus (\operatorname{int} L_n \times \cdots \times \operatorname{int} L_1)$ is evidently 0. Hence (8) follows from elements of integration theory.

Now we copy the proof of Theorem 5.9. For L_i being homothetic ellipsoids we have equality in the inequality of the theorem. Like in the proof of Theorem 5.9, we choose for each $i \in \{1, \cdots, n\}$ a sequence $L_i = L_{i,1}, L_{i,2}, \cdots$ of convex bodies, $L_{i,j+1}$ obtained from $L_{i,j}$ by Steiner symmetrization with respect to some hyperplane H_j (independent of i) and $L_{i,j}$ converging to a ball B_i of volume $V(L_i)$. Then from Lemma 9.1 we have

$$\beta_{n,\varphi}(L_1, \cdots, L_n) = \beta_{n,\varphi}(L_{1,1}, \cdots, L_{n,1}) \geq \beta_{n,\varphi}(L_{1,2}, \cdots, L_{n,2}) \geq \cdots,$$

$$\lim_{j\to\infty} \beta_{n,\varphi}(L_{1,j}, \cdots, L_{n,j}) = \beta_{n,\varphi}(B_1, \cdots, B_n),$$

hence

$$\beta_{n,\varphi}(L_1, \cdots, L_n) \geq \beta_{n,\varphi}(B_1, \cdots, B_n).$$

It remains to be verified that, for φ strictly increasing and L_i not being homothetic ellipsoids, we have here strict inequality. By [Gro 2], Lemma 2, or [Gro 3], Lemma 2, for, say, L_1 not an ellipsoid there exists a direction such that the midpoints of all chords of L_1 parallel to that direction do not lie in a hyperplane. In this case choose the first symmetrizing hyperplane (yielding $L_{i,2}$ from $L_i = L_{i,1}$) orthogonal to this direction. Then, beginning with these $L_{i,2}$'s, we can define the sequences $L_{i,2}, L_{i,3}, \cdots$ $(i = 1, \cdots, n)$, like above. Now we have by Lemma 9.1

$$\beta_{n,\varphi}(L_1, \cdots, L_n) > \beta_{n,\varphi}(L_{1,2}, \cdots, L_{n,2}) \geq \beta_{n,\varphi}(L_{1,3}, \cdots, L_{n,3}) \geq \cdots,$$

implying

$$\beta_{n,\varphi}(L_1, \cdots, L_n) > \beta_{n,\varphi}(B_1, \cdots, B_n).$$

Therefore we may suppose that L_1, and each L_i, is an ellipsoid. By equiaffine invariance of our problem we may suppose that L_1 is a ball, and also we may suppose that each L_i has centre 0. We have supposed that the L_i's are not homothetic ellipsoids; suppose, for example, that L_2 is not a ball. Then there is some hyperplane H through 0 that is not a hyperplane of symmetry of L_2. Choosing H as the first symmetrizing hyperplane, by Lemma 9.1

$$\beta_{n,\varphi}(L_1, \cdots, L_n) > \beta_{n,\varphi}(L_{1,2}, \cdots, L_{n,2}) \geq \cdots,$$

hence

$$\beta_{n,\varphi}(L_1, \cdots, L_n) > \beta_{n,\varphi}(B_1, \cdots, B_n). \quad \square$$

Remark 9.4. Evidently Lemma 5.7 and the first inequality in Theorem 5.9 follow (in a longer way) from Lemma 9.1 and Theorem 9.3, respectively, by applying them to $\varphi(t) = t^p$ $(1 \leq p)$ and letting $p \to \infty$, noting $\beta_n(L_1, \cdots, L_n) = \lim_{p\to\infty} \beta_n^p(L_1, \cdots, L_n)^{1/p}$. ■

§ 10 Approximation of convex bodies by random polytopes; upper estimates

We may ask for lower bounds of $b_n(L)$, or upper bounds of $\beta_{n,\varphi}(L)$, with $L \subset \mathbb{R}^d$ a convex body. Note that for $d = 2$ the question of $b(L)$ is answered by Theorem 5.2, both for the centrally symmetric and for the general case. Now we will settle the above posed two questions for $d = 2$, both for the centrally symmetric and for the general case.

Following [DL], we will apply a certain converse of Steiner symmetrization to convex polygons. Let therefore $L \subset \mathbb{R}^2$ be a convex polygon $p_1 p_2 \cdots p_m$. Let us suppose $m > 3$, and let us consider the diagonal $p_2 p_m$ of L, which we may suppose to be parallel to the x_2-axis. Keeping the other vertices fixed, let us move the vertex p_1 on the vertical line l containing it, between $l \cap \operatorname{aff}\{p_2, p_3\}$ and $l \cap \operatorname{aff}\{p_{m-1}, p_m\}$. Let the vertical coordinate of this moving point p_1 on l be denoted by τ, and correspondingly we denote this moving point by $p_1(\tau)$, and the (not strictly) convex polygon $p_1(\tau) p_2 \cdots p_m$ by $L(\tau)$. (By this we mean that each inner angle of the polygon is at most π.) Suppose now $m = 2r > 4$ even and that L is centred at 0. Then the (not strictly) convex polygon with vertices $p_1(\tau), p_2, \cdots, p_r, p_{r+1}(\tau) = -p_1(\tau), p_{r+2}, \cdots, p_{2r}$ will be denoted by $L^*(\tau)$. With these notations we state

Lemma 10.1. *Let $L \subset \mathbb{R}^2$ be a convex polygon $p_1 p_2 \cdots p_m$. Furthermore, let $n \geq 2$ be an integer and $\varphi : [0, \infty) \to \mathbb{R}$ be a non-decreasing convex function. Then, defining $L(\tau)$ and $L^*(\tau)$ as above (for $m > 3$ and $m > 4$, respectively), $\beta_{n,\varphi}(L(\tau))$ and $\beta_{n,\varphi}(L^*(\tau))$ are convex functions of τ. Moreover, they are strictly convex if φ is strictly increasing.*

This lemma is the consequence of the following lemma, for which we need some more notations. Let a convex polygon $L = p_1 p_2 \cdots p_m$ be given, with some distinct diagonals $p_{i'(j)}p_{i''(j)}$ parallel to the x_2-axis, $p_{i''(j)}$ lying above $p_{i'(j)}$, where $1 \le j \le k$. Let the lines $l_j = \text{ aff } \{p_{i'(j)}, p_{i''(j)}\}$ follow each other from left to right, and let l_0 and l_{k+1} be the left and right hand side vertical supporting lines of $p_1p_2\cdots p_m$. Let $\tau_0, \cdots, \tau_{k+1} \in \mathbb{R}$, and move the vertices of L lying on l_j upwards, through the signed distance τ_j. The vertices of L lying between l_j and l_{j+1} are transformed the same time by an affinity T_j, coinciding on l_j and l_{j+1} by the above translations through τ_j and τ_{j+1}, respectively. Then, for a certain non-empty closed convex subset $F \subset \mathbb{R}^{k+2}$ of the parameters $(\tau_0, \cdots, \tau_{k+1})$ (and only on it), the transformed polygon remains convex, in the non-strict sense, with the original cyclic order of the vertices in positive orientation. Let the piecewise affine transformation of the strip between l_0 and l_{k+1} onto itself, coinciding with T_j in the strip between l_j and l_{j+1}, be denoted by $T(\tau_0, \cdots, \tau_{k+1})$.

Lemma 10.2. *Let $L \subset \mathbb{R}^2$ be a convex polygon $p_1 \cdots p_m$, let $n \ge 2$ be an integer and $\varphi : [0, \infty) \to \mathbb{R}$ be a non-decreasing convex function. Then, for the convex polygon $T(\tau_0, \cdots, \tau_{k+1})(L)$ and the set $F \subset \mathbb{R}^{k+2}$ defined above, $\beta_{n,\varphi}(T(\tau_0, \cdots, \tau_{k+1})(L))$ is a convex function of $(\tau_0, \cdots, \tau_{k+1})$ on F. Moreover, it is strictly convex on any segment $S \subset F$, for whose direction vector $(\tau_0^1, \cdots, \tau_{k+1}^1)$ the transformation $T(\tau_0^1, \cdots, \tau_{k+1}^1)$ is not an affinity, provided φ is strictly increasing.*

Proof. We use the notations from the proofs of Lemmas 5.7 and 9.1. Let us have points $x_i, y_i \in L$, $x_i = (x_i', \xi_i)$, $y_i = (y_i', \eta_i)$, $1 \le i \le n$, and let us consider their images $T(\tau_0, \cdots, \tau_{k+1})(x_i', \xi_i)$, $T(\tau_0, \cdots, \tau_{k+1})(y_i', \eta_i)$. These images have first coordinates x_i', y_i', and the second coordinates are linear functions of the parameters $\tau_0, \cdots, \tau_{k+1}$ (actually, for $(x_i', \xi_i), (y_i', \eta_i)$ between l_j and l_{j+1}, the second coordinates of their images are linear functions of τ_j and τ_{j+1}). Hence, for $(x_i', \xi_i), (y_i', \eta_i)$ fixed, $V[D(T(\tau_0, \cdots, \tau_{k+1})(x_i', \xi_i), T(\tau_0, \cdots, \tau_{k+1})$
$(y_i', \eta_i); 1 \le i \le n)]$ is a convex function of $\tau_0, \cdots, \tau_{k+1}$ on the set F, by Lemma 5.4.

Now $\beta_{n,\varphi}(L)$ is the integral of $\varphi\left(V[D((x_i', \xi_i), (y_i', \eta_i); 1 \le i \le n)]\right)$, for (x_i', ξ_i), (y_i', η_i) varying in L. Further, $\beta_{n,\varphi}(T(\tau_0, \cdots, \tau_{k+1})(L))$ is the integral of $\varphi(V[D(T$ $(\tau_0, \cdots, \tau_{k+1})(x_i', \xi_i), T(\tau_0, \cdots, \tau_{k+1})(y_i', \eta_i); 1 \le i \le n)])$, for $(x_i', \xi_i), (y_i', \eta_i)$ varying in L. Since here the integrand is a convex function of the parameters $\tau_0, \cdots, \tau_{k+1}$ on the set F, the integral is a convex function of them on the set F as well.

Now suppose that φ is strictly increasing. Like in Lemma 9.1, it suffices to establish that, for any segment S in F satisfying the property stated in the lemma, for some $(x_i', \xi_i), (y_i', \eta_i)$ we have that $V[D(T(\tau_0, \cdots, \tau_{k+1})(x_i', \xi_i), T(\tau_0, \cdots, \tau_{k+1})$ $(y_i', \eta_i); 1 \le i \le n)]$ is not a linear function on S. Let the midpoint of S be $(\tau_0^0, \cdots, \tau_{k+1}^0)$, and its direction vector be $(\tau_0^1, \cdots, \tau_{k+1}^1)$. Then $T(\tau_0^1, \cdots, \tau_{k+1}^1)$ is not an affinity, thus for some j, $1 \le j \le k$, its restrictions to the strips between l_{j-1} and l_j, and between l_j and l_{j+1}, respectively, are different affinities. Now let us choose x_i', y_i' so that $(x_1', 0), (y_1', 0)$ lie between l_{j-1} and l_j, all other $(x_i', 0), (y_i', 0)$ lie between l_j and l_{j+1}, and $x_i' < y_i'$ for $i = 1, \cdots, n$. Further on, let us connect a point of $l_0 \cap (T(\tau_0^0, \cdots, \tau_{k+1}^0)(L))$ and $l_{k+1} \cap (T(\tau_0^0, \cdots, \tau_{k+1}^0)(L))$ by a segment

Q, and let us choose ξ_i, η_i so that $T(\tau_0^0, \cdots, \tau_{k+1}^0)(x_i', \xi_i), T(\tau_0^0, \cdots, \tau_{k+1}^0)(y_i', \eta_i)$ lie on Q. Then $V[D(T(\tau_0, \cdots, \tau_{k+1})(x_i', \xi_i), T(\tau_0, \cdots, \tau_{k+1})(y_i', \eta_i); 1 \leq i \leq n)]$ equals 0 for $(\tau_0, \cdots, \tau_{k+1}) = (\tau_0^0, \cdots, \tau_{k+1}^0)$, but is positive for $(\tau_0, \cdots, \tau_{k+1}) \in S \setminus \{(\tau_0^0, \cdots, \tau_{k+1}^0)\}$, since then $T(\tau_0, \cdots, \tau_{k+1})\ (x_i', \xi_i), T(\tau_0, \cdots, \tau_{k+1})(y_i', \eta_i) \in T(\tau_0, \cdots, \tau_{k+1}) T(\tau_0^0, \cdots, \tau_{k+1}^0)^{-1}(Q) = T(\tau_0 - \tau_0^0, \cdots, \tau_{k+1} - \tau_{k+1}^0)(Q)$, and the portions of this last set between l_{j-1} and l_j, and between l_j and l_{j+1}, are two non-collinear segments, by the choice of j. Therefore, $V[D(T(\tau_0, \cdots, \tau_{k+1})(x_i', \xi_i), T(\tau_0, \cdots, \tau_{k+1})(y_i', \eta_i); 1 \leq i \leq n)]$ is not a linear function of $\tau_0, \cdots, \tau_{k+1}$ on S, as we needed to prove. Thus we have proved Lemma 10.2 and, consequently, Lemma 10.1. $\square$

Now, following [DL], we prove a theorem, that by $b(L) = \inf_n b_n(L)$ implies the lower estimates in Theorem 5.2, without the discussion of the cases of equality.

Theorem 10.3. *Let $L \subset \mathbb{R}^2$ be any planar convex body or any centred planar convex body, respectively. Let T be a triangle and P be a parallelogram, respectively, of area $V(L)$. Let $n \geq 2$ be any integer and $\varphi : [0, \infty) \to \mathbb{R}$ a non-decreasing convex function. Then*

$$b_n(L) \geq b_n(T) \text{ and } \beta_{n,\varphi}(L) \leq \beta_{n,\varphi}(T)$$

and

$$b_n(L) \geq b_n(P) \text{ and } \beta_{n,\varphi}(L) \leq \beta_{n,\varphi}(P),$$

respectively. In particular, for $1 \leq p < \infty$,

$$\beta_n^p(L) \leq \beta_n^p(T)$$

and

$$\beta_n^p(L) \leq \beta_n^p(P),$$

respectively.

In any of these inequalities, equality holds for example if L is a triangle or a parallelogram, respectively. For φ strictly increasing and L a polygon, which is not a triangle or a parallelogram, respectively, strict inequality holds in the inequalities concerning $\beta_{n,\varphi}$. In particular, this holds for the inequalities concerning β_n^p.

Proof. The inequalities concerning b_n follow, like in Remark 9.4, from those concerning β_n^p, which in turn are special cases of the inequalities concerning $\beta_{n,\varphi}$. Therefore it suffices only to deal with $\beta_{n,\varphi}$.

Recall from the proof of Theorem 9.3 that $\beta_{n,\varphi}(L)$ is a continuous functional of L. Hence we may suppose that L is a polygon, or a centred polygon, respectively. By equiaffine invariance of $\beta_{n,\varphi}(L)$ we have equalities for L a triangle and a parallelogram, respectively.

Now let L be an m-gon or a centred $2r$-gon, respectively, where $m > 3$ and $r > 2$, respectively. Let us consider the polygons $L(\tau)$ and $L^*(\tau)$, respectively, from Lemma 10.1, which have area $V(L)$. Since $\beta_{n,\varphi}(L(\tau))$ and $\beta_{n,\varphi}(L^*(\tau))$ are convex functions of τ, the maximum of any of these functions is attained at some endpoint τ_0 of the interval of τ's under consideration, i.e., when $\{p_1\} = l \cap \text{aff}\,\{p_2, p_3\}$, or $\{p_1\} = l \cap \text{aff}\,\{p_{m-1}, p_m\}$. Let us denote by L_1 the polygon $L(\tau_0)$ and $L^*(\tau_0)$, respectively. Then $V(L_1) = V(L)$, L_1 is a convex $(m-1)$-gon or a centred $2(r-1)$-gon, respectively, and

$$\beta_{n,\varphi}(L) \le \beta_{n,\varphi}(L_1).$$

Repeating this procedure $m-3$ or $r-2$ times, respectively, we obtain a sequence $L_1, L_2, \cdots$ of convex polygons and centred convex polygons, respectively, ending with a triangle T and a parallelogram P, respectively, such that $V(T) = V(L)$, and $V(P) = V(L)$, respectively, and

$$\beta_{n,\varphi}(L) \le \beta_{n,\varphi}(T)$$

and

$$\beta_{n,\varphi}(L) \le \beta_{n,\varphi}(P),$$

respectively.

For φ strictly increasing and L a polygon which is not a triangle or a parallelogram, respectively, by Lemma 10.1, $\beta_{n,\varphi}(L(\tau))$ and $\beta_{n,\varphi}(L^*(\tau))$ are strictly convex, hence $\beta_{n,\varphi}(L) < \beta_{n,\varphi}(L_1)$, hence

$$\beta_{n,\varphi}(L) < \beta_{n,\varphi}(T)$$

and

$$\beta_{n,\varphi}(L) < \beta_{n,\varphi}(P),$$

respectively. $\square$

[DL] investigated the expected value of the area of the convex hull of $n\,(\ge 3)$ points in a planar convex body and proved that, for fixed area of the body, it attains its maximum for example for a triangle. Its minimum has been investigated in [Gro 2, Gro 3], where actually its pth moments $(1 \le p < \infty)$ and, more generally, the expected value of $\varphi(V(\text{ conv }\{x_1, \cdots, x_n\}))$ $(n \ge d+1)$, respectively, were estimated from below for φ a non-decreasing convex function, in the analogous d-dimensional problem. The minimizing convex bodies are for example the ellipsoids (of fixed volume) and, for φ strictly increasing, only these.

In the papers [Mac], [Gro 3], [DL] (more precisely, Lemma 3 in [Gro 3] and Lemma 2 in [DL]) the convexity of $V(\text{ conv }\{(x_1', \xi_1), \cdots, (x_n', \xi_n)\})$ as a function of $\xi_1, \cdots, \xi_n$ is stated (with the notations of our Lemma 5.7, i.e., $(x_i', \xi_i) \in \mathbb{R}^d$ is the point whose projection to the linear subspace spanned by the first $d-1$ basic vectors is x_i', and whose dth coordinate is ξ_i). Using this instead of our Lemma 5.4, we get analogues of our Lemmas 10.1 and 10.2 – with $\beta_{n,\varphi}(L)$ replaced by

$$\alpha_{n,\varphi}(L) = \int\limits_L \cdots \int\limits_L \varphi(V(\text{ conv } \{x_1, \cdots, x_n\}))dx_1 \cdots dx_n,$$

where $n \geq d+1$ – namely

LEMMA 10.4. *Let $L, \varphi, L(\tau), L^*(\tau), T(\tau_0, \cdots, \tau_{k+1})(L), F, S$ be defined as in Lemmas 10.1 and 10.2, and let $n \geq 3$. Then $\alpha_{n,\varphi}(L(\tau)), \alpha_{n,\varphi}(L^*(\tau))$ and $\alpha_{n,\varphi}(T(\tau_0, \cdots, \tau_{k+1})(L))$ have, under the same assumptions as in Lemmas 10.1 and 10.2, the same convexity and strict convexity properties, respectively, on the same domains, as $\beta_{n,\varphi}(L(\tau)), \beta_{n,\varphi}(L^*(\tau))$ and $\beta_{n,\varphi}(T(\tau_0,\cdots,\tau_{k+1})(L))$ have in Lemmas 10.1 and 10.2.*

Analogously to the proof of Theorem 10.3, we obtain from this

THEOREM 10.5. *Let L, T, P, φ be as in Theorem 10.3, and let $n \geq 3$. Then*

$$\alpha_{n,\varphi}(L) \leq \alpha_{n,\varphi}(T)$$

and

$$\alpha_{n,\varphi}(L) \leq \alpha_{n,\varphi}(P),$$

respectively. Equality occurs for example if L is a triangle or a parallelogram, respectively, and for φ strictly increasing and L a polygon only in this case.

PROOF. The proofs of Lemma 10.4 and Theorem 10.5 follow from the remarks preceding them. □

REMARK 10.6. Theorem 10.3 does not have a generalization in $\mathbb{R}^d$ $(d \geq 2)$ to the case of $\beta_n(L_1,\cdots,L_n) = \sup\{V(D(x_1, y_1,\cdots,x_n, y_n)) \mid x_1, y_1 \in L_1, \cdots, x_n, y_n \in L_n\}$, or $\beta_{n,\varphi}(L_1,\cdots, L_n)$, as investigated in sections 5 and 9, 10, respectively. For $2 \leq d \leq n$, for any given values of $V(L_1), \cdots, V(L_n)$, we can let $L_1, \cdots, L_d$ be very oblong ellipsoids of revolution, with axes of revolution mutually perpendicular, and $L_{d+1}, \cdots, L_n$ arbitrary, and then $\beta_n(L_1, \cdots, L_n)$ and, for φ strictly increasing, $\beta_{n,\varphi}(L_1, \cdots, L_n)$ can be arbitrarily large. Probably in Theorems 10.3 and 10.5 the only cases of equality, for $b_n(L)$ and, for φ strictly increasing, for $\beta_{n,\varphi}(L)$ and $\alpha_{n,\varphi}(L)$ are those of the triangle and the parallelogram, respectively. Using the notations of Theorems 10.3 and 10.5, for $\alpha_{n,\varphi}(L)$, with $\varphi(t) = t$, the inequality $\alpha_{n,\varphi}(L) < \alpha_{n,\varphi}(T)$ for L not a triangle has been proved in [Gi]. By $b_n(L) \geq b(L)$ and Theorem 5.2, the above conjectures about the cases of equality for $b_n(L)$ are true whenever $b_n(T) = b(T)$ and $b_n(P) = b(P)$, respectively, which hold for $3|n$ and $2|n$, respectively (taking n segments in T and P, respectively, which are the three sides, with multiplicity $n/3$ and the two diagonals, with multiplicity

$n/2$, respectively). The estimate $b_3(L) \geq 3/2\,(= b_3(T))$, with equality only for a triangle, is implied by [Mak], Lemma 5 (taking in account also the reformulation in our Remark 5.3). It can be conjectured that in $\mathbb{R}^d$, for given $V_d(L)$, and φ a non-decreasing convex function, $\beta_{n,\varphi}(L)$ and $\alpha_{n,\varphi}(L)$ attain their maxima for simplices. For $\alpha_{n,\varphi}(L)$, with $\varphi(t) = t$, this is a well-known conjecture (cf., for example, [DL]), a weaker version of which is proved in [BB], Theorem 1. In [DL] the inequality

$$\alpha_{n,\varphi}(L) \leq \alpha_{n,\varphi}(\textstyle\sum)$$

has been proved, for L a convex d-polytope with $d+2$ vertices, $\sum$ a simplex with $V(L) = V(\sum)$, and $\varphi(t) = t$. Actually their arguments prove (cf. also the proof of our Theorem 10.3) that

$$\alpha_{n,\varphi}(L) \leq \alpha_{n,\varphi}(\textstyle\sum),\ \beta_{n,\varphi}(L) \leq \beta_{n,\varphi}(\textstyle\sum),$$

for L and $\sum$ as just above, and φ a non-decreasing convex function. For $L \subset \mathbb{R}^d$ being a centrally symmetric convex body, the sharp lower bound for $V(L)/\max \{V(\sum) | \sum \subset L \text{ is a simplex}\}$ has been determined by [FR], Theorem 6. Its value is $\binom{d}{\lfloor \frac{d}{2} \rfloor}$, and this value is attained, for example, if $L = \text{conv}\left\{\sum^d \cup (-\sum^d)\right\}$, where $\sum^d$ is a d-simplex with barycentre 0 ($\lfloor x \rfloor$ denotes the integer part of x). ■

§11 Some examples related to cross-section bodies

Let $K \subset \mathbb{R}^d$ be a convex body. Then for a chord $[x, y]$ of K, having the maximum length among all chords of K, both x and y are extremal points of K. With respect to hyperplane sections of convex bodies it has been proved in [W] and [Ph] that, for each $d \geq 5$, there is a simplex in $\mathbb{R}^d$ having a hyperplane section of larger $(d-1)$-volume than that of any of its facets. We settle the case of k-dimensional sections of convex bodies in $\mathbb{R}^d$ by the following proposition, which shows that, for $k \geq 2$, there is no analogous simple method to find the k-dimensional section of maximal k-volume of a convex polytope. This proposition is proved on the lines of [W] and [Ph].

PROPOSITION 11.1. *For any integers k and d, where $1 < k < d$, there exists a convex d-polytope $P \subset \mathbb{R}^d$ such that its intersection with some k-plane L_k has a larger k-volume than its intersection with any k-plane spanned by $k+1$ vertices of P.*

PROOF.
1) First we consider the case $k = 2$, $d = 3$. Then let $P = P_{2,3}$ be the convex hull of a regular triangle of side length 2, lying in the plane $x_3 = \epsilon\,(> 0)$ and of centroid $(0, 0, \epsilon)$, and of its mirror image with respect to the origin. Then its section with the 2-plane $x_3 = 0$ has area $3\sqrt{3}/2$, and for $\epsilon \to 0$ its sections with 2-planes passing through some three vertices have areas $4\sqrt{3}/3 + o(1)$, $\sqrt{3}$, $\sqrt{3}/3 + o(1)$. This settles the case $k = 2$, $d = 3$.

2) Now let $k = 2$ and $d \geq 3$ arbitrary. We consider $\mathbb{R}^3$ as embedded in $\mathbb{R}^d$, as the linear subspace spanned by $\{e_1, e_2, e_3\}$, where e_i is the ith basic unit vector (and analogously for $\mathbb{R}^d$, $\mathbb{R}^{d+1}$, with $\{e_1, \cdots, e_d\}$ instead of $\{e_1, e_2, e_3\}$). We construct the polyhedron $P_{2,d}$ as $\text{conv}\,(P_{2,3} \cup \{\epsilon e_4, \cdots, \epsilon e_d\})$, which is a $(d-3)$-fold pyramid over $P_{2,3}$. We establish that any 2-plane passing through three vertices of $P_{2,d}$ has an intersection of 2-volume less than $3\sqrt{3}/2$, for ϵ small enough. We use induction for d, supposing this statement true for d (≥ 3), and establish it for $d + 1$. The polytope $P_{2,d+1}$ is a pyramid over $P_{2,d}$, with apex ϵe_{d+1}. Choose three vertices v_1, v_2, v_3 of $P_{2,d+1}$. If none of them is ϵe_{d+1}, then each is a vertex of $P_{2,d}$, and the 2-plane L_2 spanned by them lies in the linear subspace spanned by $e_1, \cdots, e_d$, thus $L_2 \cap P_{2,d+1} = L_2 \cap P_{2,d}$, and then the induction works. If, for example, $v_3 = \epsilon e_{d+1}$, then $L_2 \cap P_{2,d+1}$ is the pyramid with apex v_3 and base $L_2 \cap P_{2,d} = (L_2 \cap \mathbb{R}^d) \cap P_{2,d} = [v_1, v_2]$, hence $L_2 \cap P_{2,d+1} = \text{conv}\,\{v_1, v_2, v_3\}$. Now $\| v_3 \| = o(1)$, and each of v_1, v_2 either is a vertex of $P_{2,3}$ or lies close to 0. If some of v_1 and v_2 lies close to 0, then $V_2(\text{conv}\,\{v_1, v_2, v_3\}) = o(1)$, while if both v_1 and v_2 are vertices of $P_{2,3}$, then $V_2(\text{conv}\,\{v_1, v_2, v_3\})$ equals $\sqrt{3}/3 + o(1)$, or $o(1)$. This settles the case $k = 2$, $d > k$ arbitrary.

3) We turn to the case of k, d arbitrary. By the last result it suffices to prove the following: the validity of the statement of the proposition for k, d implies its validity for $k+1, d+1$. This follows from [Ph], Theorem 2. Namely, there it is proved that if we consider a fixed point p of $P_{k,d}$ and draw in $\mathbb{R}^{d+1}$ a line perpendicular to $\text{aff}\, P_{k,d} = \mathbb{R}^d$ at p, and choose a point $q = p + \lambda e_{d+1}$ on this perpendicular line, then for λ sufficiently large the following holds for $P = \text{conv}\,(P_{k,d} \cup \{q\})$. Taking any $k + 2$ affinely independent vertices of P, and denoting by L_{k+1} the $(k + 1)$-plane spanned by them, we have:

A) If each of the considered vertices is different from q, then $L_{k+1} \cap P = L_{k+1} \cap (\mathbb{R}^d \cap P) = L_{k+1} \cap P_{k,d}$ has $(k + 1)$-volume less than some constant if $\lambda \to \infty$.

B) If q is one of the considered vertices, then $L_{k+1} \cap P$ is a pyramid with apex q and base $L_{k+1} \cap P_{k,d} = (L_{k+1} \cap \mathbb{R}^d) \cap P_{k,d}$, where $L_{k+1} \cap \mathbb{R}^d$ is a k-plane in $\mathbb{R}^d$ spanned by some $k + 1$ vertices of $P_{k,d}$. Moreover,

$$V_{k+1}(L_{k+1} \cap P) = V_k((L_{k+1} \cap \mathbb{R}^d) \cap P_{k,d}) \cdot \lambda \cdot (1 + o(1))$$

for $\lambda \to \infty$.

However, there exists a k-plane L'_k in $\mathbb{R}^d$ such that $V_k(L'_k \cap P_{k,d})$ is greater than $V_k(L_k \cap P_{k,d})$ for any k-plane L_k spanned by $k + 1$ vertices of $P_{k,d}$. Let $L'_{k+1} = \text{aff}\,(L'_k \cup \{q\})$. Then

$$V_{k+1}(L'_{k+1} \cap P) = V_k(L'_k \cap P_{k,d}) \cdot \lambda \cdot (1 + o(1)).$$

Hence, $V_{k+1}(L'_{k+1} \cap P)$ is, for sufficiently large λ, greater than any $V_{k+1}(L_{k+1} \cap P)$, considered either under (A) or under (B). Thus we can take this P as $P_{k+1,d+1}$. $\square$

Now we turn to the question of convexity of the *cross-section body* CK for any convex body $K \subset \mathbb{R}^d$. We recall from section 1 that $CK = \{\mu u \mid u \in S^{d-1},\, 0 \le \mu \le \underline{V}_{d-1}(K,u)\}$, where $S^{d-1} \subset \mathbb{R}^d$ is the unit sphere, $\underline{V}_{d-1}(K,u) = \max\{V_{d-1}(K \cap (u^\perp + \lambda u)) \mid \lambda \in \mathbb{R}\}$, and $u^\perp = \{x \in \mathbb{R}^d \mid \langle u, x\rangle = 0\}$. If moreover $0 \in \text{ int } K$, the *intersection body* IK of K is defined by $IK = \{\mu u \mid u \in S^{d-1},\, 0 \le \mu \le V_{d-1}(K \cap u^\perp)\}$; see also [Ga], § 8.1 and § 8.3, and [Sch], § 7.4. Recall from § 1 that for $d = 2$ the body CK is obtained from $K + (-K)$ by rotation about 0 through 90^o; thus for $d = 2$ the body CK is convex. Furthermore, for any d and any K being centred (i.e., symmetric about the origin), we have $CK = IK$ (cf. § 1), and IK is convex for any centred K by [Bu 1], see also [MP], Theorem 3.9. An "extreme example" of an asymmetric convex body is a simplex. We investigate its cross-section body for $d = 3$. Denoting for a non-singular linear transformation T the transpose of its inverse by T^{-1*}, and for a set A its closure by cl A, first we prove

Lemma 11.2. *Let $K \subset \mathbb{R}^d$ be a convex body, and let $T : \mathbb{R}^d \to \mathbb{R}^d$ be a volume-preserving affinity satisfying $T(0) = 0$. Then we have*

$$C(T(K)) = T^{-1*}(CK).$$

Proof. Recall from [L 1], § 2 (cf. also [L 2], § 7) that for the intersection body IK with $0 \in \operatorname{int} K$ and T like in the lemma we have

$$I(T(K)) = T^{-1*}(IK).$$

Then we have

$$CK = \text{ cl } \left(\bigcup\{I(K-x) | x \in \text{ int } K\}\right),$$

hence by the above cited result

$$\begin{aligned} C(T(K)) &= \text{ cl } \left(\bigcup\{I(T(K) - T(x)) | x \in \text{ int } K\}\right) \\ &= \text{ cl } \left(\bigcup\{T^{-1*} I(K-x) | x \in \text{ int } K\}\right) = T^{-1*} CK. \quad \square \end{aligned}$$

Example 11.3. Let $K \subset \mathbb{R}^3$ be a regular tetrahedron of edge-length $2\sqrt{2}$, with vertices at every second vertex of the cube Q with vertex set $\{(\pm 1, \pm 1, \pm 1)\}$. Then $CK = 2Q$. Hence the cross-section body of any tetrahedron is a parallelepiped.

Proof. By [A], Theorem 2, we have for $\sum^d = \text{ conv } \{e_1, \cdots, e_d\} \subset \mathbb{R}^d$ (e_i are the basic unit vectors) the following. For any d real numbers $a_1 < \cdots < a_d$, and for any $b \in \mathbb{R}$,

$$\frac{V_{d-1}\left(\{(x_1, \cdots, x_d) \in \sum^d \mid \sum_{i=1}^d a_i x_i \le b\}\right)}{V_{d-1}\left(\sum^d\right)} = \sum_{i=1}^d \frac{(\min(a_i - b, 0))^{d-1}}{\prod_{\substack{j=1 \\ j \ne i}}^d (a_i - a_j)}.$$

Applying this for $d = 4$, we find by differentiation that the section area

$$V_{d-2}\left(\{(x_1, \cdots, x_d) \in \sum^{d} | \sum_{i=1}^{d} a_i x_i = b\}\right)$$

is a quadratic polynomial of b for $a_2 \leq b \leq a_3$ ($b < a_2$ or $b > a_3$ cannot give maximal section area). Then a straightforward calculation leads to the result stated in this example, for sections with planes defined by different a_i's. A continuity argument then finishes the proof of the first statement. The second statement follows from the first one by Lemma 11.2. $\square$

All the above facts seem to support the conjecture that for any convex body $K \subset \mathbb{R}^d$ we have that CK is convex. However, recently U. Brehm [B] has informed the authors that he has found a counterexample, namely the n-simplex ($n > 3$). This leaves open at least the following

PROBLEM 11.4 Is CK convex, for any convex body $K \subset \mathbb{R}^3$?

ACKNOWLEDGEMENT: We would like to express our gratitude to I. Bárány for several stimulating conversations.

NOTE ADDED IN PROOF. Since the submission of this paper, we have been kindly informed by the following authors about the following facts: Conjecture 3.3 from [MM] has been solved in the positive by M. Fradelizi [F]; even one can choose the point x as the barycentre of the convex body K. And Problem 11.4 above has been solved in the positive, by M. Meyer [Me].

References

[A] M. M. Ali, Content of the frustum of a simplex. *Pacific J. Math.* 48 (1973), 313-322, MR 52#15232.

[BB] I. Bárány, Ch. Buchta, Random polytopes in a convex polytope, independence of shape and concentration of vertices, *Math. Ann.* 297 (1993), 467-497, MR 95g:52005.

[B] U. Brehm, Convex bodies with non-convex cross-section bodies, Preprint MATH-GE-06-97, TU Dresden, 1997, submitted.

[BMMP] J. Bourgain, M. Meyer, V. Milman, A. Pajor, On a geometric inequality, Geom. Aspects of Funct. Anal. (1986/87), 217-282, *Lect. Notes in Math.* 1317, Springer, Berlin-New York 1988, MR 89h:46012.

[Bu 1] H. Busemann, A theorem on convex bodies of the Brunn-Minkowski type. *Proc. Nat. Acad. Sci. USA* 35 (1949), 27-31, MR 10-395.

[Bu 2] H. Busemann, Volume in terms of concurrent cross-sections, *Pacific J. Math.* 3 (1953), 1-12, MR 14-1115.

[Ca] J.W.S. Cassels, *An Introduction to the Geometry of Numbers.* Grundlehren Math. Wiss. 99, Springer, Berlin-G"ottingen-Heidelberg 1959, MR 28#1175.

[DL] L. Dalla, D.G. Larman, Volumes of a random polytope in a convex set. DIMACS Series in Discr. Math. and Theor. Comp. Sci. 4, 1991. *Applied Geom. and Discr. Math., The Victor Klee Festschrift,* Amer. Math. Soc., Providence, R.I., 1991, 175-180, MR 92m:52008.

[D] W. J. Davis: Remarks on finite rank projections. *J. Appr. Th.* 9 (1973), 205-211, MR 50#947.

[F] M. Fradelizi: Sections of convex bodies through their centroids, *Arch. Math.,* to appear.

[FR] I. Fáry, L. Rédei, Der zentralsymmetrische Kern und die zentralsymmetrische Hülle von konvexen Körpern. *Math. Ann.* 122 (1950), 205-220, MR 12-526.

[Ga] R. J. Gardner, *Geometric Tomography.* Cambridge University Press, Cambridge 1995, MR 96j:52006.

[Gi] A.A. Giannopoulos, On the mean value of the area of a random polygon in a plane convex body. *Mathematika* 39 (1992), 279-290. MR 93m:52004.

[Gro 1] H. Groemer, Zusammenhängende Lagerungen konvexer K"orper. *Math. Z.* 94 (1966), 66-78, MR 33#7935.

[Gro 2] H. Groemer, On some mean values associated with a randomly selected simplex in a convex set. *Pacific J. Math.* 45 (1973), 525-533, MR 47#5916.

[Gro 3] H. Groemer, On the mean value of the volume of a random polytope in a convex set. *Arch. Math.* (Basel) 25 (1974), 86-90, MR 49#6036.

[Gro 4] H. Groemer, On the average size of polytopes in a convex set. *Geom. Ded.* 13 (1982), 47-62, MR 84d:52013.

[J] F. John, Extremum problems with inequalities as subsidiary conditions. *Studies and Essays, presented to R. Courant.* Interscience, New York, 1948, 187-204, MR 10-719.

[KLL] H. König, D.R. Lewis, P.K. Lin, Finite dimensional projection constants. *Studia Math.* 75 (1983), 341-358, MR 84k:46014.

[KT-J] H. König, N. Tomczak-Jaegermann, Norms of minimal projections, *J. Funct. Anal.* 119 (1994), 253-280, MR 94m:46024.

[L 1] E. Lutwak, On some ellipsoid formulas of Busemann, Furstenberg and Tzkoni, Guggenheimer, and Petty. *J. Math. Anal. Appl.* 159 (1991), 18-26, MR 92k:52014.

[L 2] E. Lutwak, Selected affine isoperimetric inequalities. In: *Handbook of Convex Geometry,* Eds. P.M. Gruber and J.M. Wills, North-Holland, Amsterdam-Oxford-New York 1993, 151-176, MR 94h:52014.

[Mac] A.M. Macbeath, An extremal property of the hypersphere. *Proc. Camb. Phil. Soc.* 47 (1951), 245-247, MR 12-526.

[Mak] E. Makai, Jr., Five-neighbour packing of convex plates. Intuitive Geom. (Siófok, 1985), *Colloq. Math. Soc. J. Bolyai* 48, North-Holland, Amsterdam-New York, 1987, MR 88k:52019.

[MM] E. Makai, H. Martini, The cross-section body, plane sections of convex bodies and approximation of convex bodies, I. *Geom. Dedicata* 63 (1996), 267-296.

[Me] M. Meyer, Lecture on the 1997 Oberwolfach conference on convexity.

[MP] V.D. Milman, A. Pajor, Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space. Geometric Aspects of Functional Analysis, Israel Seminar (GAFA), 1987-88, *Lecture Notes Math.* 1376, Springer, Berlin-Heidelberg 1989, 64-104, MR 90g:52003.

[Ph] J. Philip, Plane sections of simplices. *Math. Prog.* 3 (1972), 312-325, MR 47#7598.

[Sch] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, Cambridge University Press, Cambridge 1993, MR 94d:52007.

[W] D.W. Walkup, A simplex with a large cross-section. *Amer. Math. Monthly* 75 (1968), 34-36, MR 37#2095.

E. Makai, Jr.
Mathematical Institute of the
Hungarian Academy of Sciences
Pf. 127
H-1364 Budapest
HUNGARY
makai@math-inst.hu

H. Martini
Technische Universität Chemnitz-Zwickau
Fakultät für Mathematik
Postfach 964
D-09009 Chemnitz
GERMANY
martini@mathematik.tu-chemnitz.de

æ