CHARACTERIZATION OF FUNCTION CLASSES $C(Y)|X$

Á. Császár, E. Makai, Jr.

Abstract. Let X be a topological space and let $\Phi \subset C(X)$. Then there exists a topological space Y containing X as a subspace and such that $\Phi = C(Y)|X$, if and only if Φ is weakly composition closed, i.e., for any index set I, any $f_i \in \Phi$ $(i \in I)$ and any continuous map $k : \mathbb{R}^I \to \mathbb{R}$ we have $k \circ \langle f_i \rangle \in \Phi$, where $\langle f_i \rangle : X \to \mathbb{R}^I$ is the map with i-th coordinate f_i. The analogous statement is valid for functions to any T_1 space, rather than to $\mathbb{R}$, and even we can consider functions to any set of T_1 spaces, and then a generalization of the above statement is valid, with a suitably defined weak composition closedness property. We also show that some earlier results on characterization of function classes $\Phi \subset C(X)$ of the form $C(Y)|X$, with Y some extension of a given topological space X, and on the characterization of function classes $C(\langle X, \mathcal{T} \rangle)$, with $\mathcal{T}$ some topology on a given set X, respectively, can be generalized in an analogous way as above, by means of composition properties analogous to the above one or by filter closedness (for functions to any set of T_3 spaces, or to any set of topological spaces, respectively).

§1

J. R. Isbell ([I], p.114) proved in a special case, later the first author ([Cs 77], Theorem 3) in generality the following

Theorem 1.1. *(Isbell-Császár). Let X be a set and $\Phi \subset \mathbb{R}^X$ a class of real functions on X. Then the following three statements are equivalent:*

(1) *There exists a topological space Z, $Z \supset X$, such that $\Phi = C(Z)|X$;*
(2) *there exist a topological space Y and a set map $p : X \to Y$, such that $\Phi = C(Y) \circ p$;*
(3) *Φ is weakly composition closed, i.e., for any index set I, any $f_i \in \Phi$ $(i \in I)$ and any continuous map $k : \mathbb{R}^I \to \mathbb{R}$ we have $k \circ \langle f_i \rangle \in \Phi$.* □

The first author ([Cs 77], Theorem 6) also proved the following

Theorem 1.2. *(Császár). Let X be a topological space and let $\Phi \subset C(X)$. Then the following are equivalent:*

(1) *There exists a completely regular space Z (T_0 not assumed) containing X as a subspace, such that $\Phi = C(Z)|X$;*
(2) *Φ is weakly composition closed and $\{f^{-1}(0) \mid f \in \Phi\}$ is a closed base in X.* □

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By the methods of the proofs of the above mentioned theorems of Isbell [I] and the first author [Cs 77] it is easy to establish the following theorem, which has not been explicitly formulated by them. (Cf. also our Theorem 2.1, (2) $\Leftrightarrow$ (3), for $\mathcal{R} = \{\mathbb{R}\}$.)

Theorem 1.3. *(Isbell-Császár). Let X be a topological space and let $\Phi \subset C(X)$. Then the following are equivalent:*

(1) *There exists a topological space Y and a continuous map $p : X \to Y$ such that $\Phi = C(Y) \circ p$;*
(2) *there exists a completely regular space Y and a continuous map $p : X \to Y$ such that $\Phi = C(Y) \circ p$;*
(3) *Φ is weakly composition closed.* $\square$

The first author [Cs 77], Problem 2 posed the problem of characterizing the function classes $\Phi \subset C(X)$ of the form $C(Z)|X$, where Z is an arbitrary superspace of X. We will prove that the evident necessary condition of weak composition closedness (cf. Theorem 1.3, (1) $\Rightarrow$ (3), applied to the injection $p : X \hookrightarrow Z$) also is sufficient for this purpose. Actually, we will prove this for functions to any T_1 space rather than to $\mathbb{R}$, and even we can consider functions to any set of T_1 spaces.

We note that the same Problem 2 in [Cs 77] also contains the analogous question with Z an extension of X. This has been answered by the first author ([Cs 84], Theorem 2.3), and also in this case the evident necessary condition of composition closedness (i.e., for any index set I, any $f_i \in \Phi$ $(i \in I)$ and any continuous map $k : \overline{\langle f_i \rangle(X)} \to \mathbb{R}$ ($\overline{\langle f_i \rangle(X)}$ considered as a subspace of $\mathbb{R}^I$) we have $k \circ \langle f_i \rangle' \in \Phi$, where $\langle f_i \rangle' : X \to \overline{\langle f_i \rangle(X)}$ is the codomain restriction of $\langle f_i \rangle : X \to \mathbb{R}^I$), or of filter closedness (i.e., [$f \in \mathbb{R}^X$ and for each Φ-filter $\mathcal{F}$ in X we have that $f(\mathcal{F})$ is convergent in $\mathbb{R}$] implies $f \in \Phi$, where a filter $\mathcal{F}$ in X is a Φ-filter if for each $f \in \Phi$ we have that $f(\mathcal{F})$ is convergent in $\mathbb{R}$) has been proved to be sufficient. (For a continuous map $f : X \to Y_0$ and $f(X) \subset Y \subset Y_0$ the *codomain restriction* $f' : X \to Y$ *of* f is the continuous map defined by $\forall x \in X \;\; f'(x) = f(x)$, where Y is considered with the subspace topology inherited from Y_0.) We will extend this result to functions to any set of T_3 spaces.

[Cs 74], (2.6) characterized function classes $C(\langle X, \mathcal{T} \rangle)$, with $\mathcal{T}$ some topology on a set X. This has been extended to functions, essentially to a class of topological spaces, closed under products and subspaces, rather than to $\mathbb{R}$, by [G]. We will extend these results too, to functions to any set of topological spaces, and also to certain classes of topological spaces, including those considered in [G].

For related work on function classes we refer, beside the above mentioned papers, to [CI], [HIJ], [Cia], [DM], [Cs 71], [L-C 84] and to [ŽB] who among others has proved analogous theorems for classes of bounded real functions (which amounts about the same as functions to $[0, 1]$). Some uniform analogues of the above questions are treated e.g. in [F], [Cs Cz], [Cs 69], [Cs 71], [Ha 74], [R], [Ha 78], [ŽB]. In the framework of Cauchy spaces, analogous problems are treated in [L-C 84] and [L-C 91].

In the following, unless stated otherwise, a *map* between topological spaces is always assumed to be *continuous*, and *subsets* of topological spaces are always considered with the *subspace topology*. For $f_i : X \to Y_i$ $(i \in I)$, $\langle f_i \rangle : X \to \prod_I Y_i$ denotes the map with i-th coordinate f_i.

§2

Like in [G], we will consider, rather than real valued functions, functions to any given topological space, and, in fact, to several topological spaces. Let $\mathcal{R} = \{R_\alpha \mid \alpha \in A\}$ denote a set or proper class of topological spaces, and let X be a topological space. Let $\Phi_\alpha \subset C(X, R_\alpha)$ $(\alpha \in A)$. We say that *the system* $\{\Phi_\alpha \mid \alpha \in A\}$ *is weakly composition closed* if for any set I, any $\alpha \in A$ and any $\alpha_i \in A$ $(i \in I)$, any $f_i \in \Phi_{\alpha_i}$ and any $k : \prod_I R_{\alpha_i} \to R_\alpha$ we have $k \circ \langle f_i \rangle \in \Phi_\alpha$.

We say that *the system* $\{\Phi_\alpha \mid \alpha \in A\}$ *is weakly nice* if there is a set $B \subset A$, such that for any $\alpha \in A$ and any $f_\alpha \in \Phi_\alpha$ there exist $g_{\beta\gamma} \in \Phi_\beta$ $(\beta \in B,\ \gamma \in \Gamma_\beta$, where Γ_β is some set) and $h : \prod_B R_\beta^{\Gamma_\beta} \to R_\alpha$, such that $f_\alpha = h \circ \langle g_{\beta\gamma} \rangle$.

For example, if $\mathcal{R}$ is either hereditary or both consists of T_2 spaces and is closed hereditary, respectively, and, for each $\alpha \in A$, Φ_α consists of all $f_\alpha \in C(X, R_\alpha)$ for which $f_\alpha(X)$ or $\overline{f_\alpha(X)}$, respectively, has some topological property, independent of α — e.g. $\overline{f_\alpha(X)}$ is compact — then $\{\Phi_\alpha \mid \alpha \in A\}$ is weakly nice.

Namely, in these cases we may choose a set $B \subset A$ such that $\{R_\beta \mid \beta \in B\}$ contains a homeomorphic copy of each $R_\alpha \in \mathcal{R}$ satisfying $|R_\alpha| \le |X|$ or $|R_\alpha| \le 2^{2^{|X|}}$, respectively. Then we may choose some $\beta_0 \in B$, and we may let $|\Gamma_{\beta_0}| = 1$, and $\Gamma_\beta = \emptyset$ for $\beta \in B \setminus \{\beta_0\}$, thus $\prod_B R_\beta^{\Gamma_\beta} = R_{\beta_0}$, and then h can be chosen as a homeomorphic embedding $\prod_B R_\beta^{\Gamma_\beta} = R_{\beta_0} \to R_\alpha$.

In fact, first suppose that $\mathcal{R}$ is hereditary, and for each $\alpha \in A$ we have $\Phi_\alpha = \{f_\alpha \in C(X, R_\alpha) \mid f_\alpha(X) \in \mathcal{P}\}$, where $\mathcal{P}$ is a class of topological spaces closed under homeomorphisms. Then for any $\alpha \in A$ and any $f_\alpha \in \Phi_\alpha \subset C(X, R_\alpha)$ we have $f_\alpha = h_0 \circ g_0$, where $g_0 : X \to f_\alpha(X)$ is the codomain restriction of f_α, and $h_0 : f_\alpha(X) \hookrightarrow R_\alpha$ is the injection. Then by hypothesis $R_\alpha \in \mathcal{R}$ implies $f_\alpha(X) \in \mathcal{R}$, and then $|f_\alpha(X)| \le |X|$ implies that $f_\alpha(X) \cong R_{\beta_0}$ for some R_{β_0}, $\beta_0 \in B$. Therefore $f_\alpha = h_0 \circ h_1 \circ g$, with $g : X \to R_{\beta_0}$, and $h_1 : R_{\beta_0} \to f_\alpha(X)$ a homeomorphism. Moreover, $g \in \Phi_{\beta_0}$, since by hypothesis $g(X) \cong f_\alpha(X) \in \mathcal{P}$ implies $g(X) \in \mathcal{P}$, thus $g \in \{g_{\beta_0} \in C(X, R_{\beta_0}) \mid g_{\beta_0}(X) \in \mathcal{P}\} = \Phi_{\beta_0}$. Then, choosing $\Gamma_\beta = \emptyset$ for $\beta \in B \setminus \{\beta_0\}$, $|\Gamma_{\beta_0}| = 1$, $g_{(\beta_0)\gamma} = g$ for the unique $\gamma \in \Gamma_{\beta_0}$, and $h = h_0 \circ h_1$, we have $g = \langle g_{\beta\gamma} \rangle$ and $f_\alpha = h \circ g = h \circ \langle g_{\beta\gamma} \rangle$.

The case when $\mathcal{R}$ consists of T_2 spaces and is closed hereditary, is handled analogously, by using a factorization $f_\alpha = h_0 \circ g_0$, where $g_0 : X \to \overline{f_\alpha(X)}$ is the codomain restriction of f_α and $h_0 : \overline{f_\alpha(X)} \hookrightarrow R_\alpha$ the injection, noting that in this case $|\overline{f_\alpha(X)}| \le 2^{2^{|f_\alpha(X)|}} \le 2^{2^{|X|}}$.

The property of weak niceness is some analogue of the solution set condition (cf. [HS]).

A topological space R is S_1 (or *symmetrical*) if $r_1, r_2 \in R$, $r_1 \in \overline{\{r_2\}}$ imply that $r_2 \in \overline{\{r_1\}}$.

We prove the following theorem, that is an extension of Theorems 1.1, 1.3 and [ŽB], Theorem 3.3.

Theorem 2.1. *Let X be a topological space and $\mathcal{R} = \{R_\alpha \mid \alpha \in A\}$ a class of topological spaces. Further let $\Phi_\alpha \subset C(X, R_\alpha)$ $(\alpha \in A)$. Then we have the implications* (1) $\Rightarrow$ (2) $\Rightarrow$ (3)*, where*

(1) *there exists a topological space Z, $X \subset Z$, such that for each $\alpha \in A$ we have $\Phi_\alpha = C(Z, R_\alpha)|X$;*

(2) *there exist a topological space* Y *and a map* $p : X \to Y$*, such that for each* $\alpha \in A$ *we have* $\Phi_\alpha = C(Y, R_\alpha) \circ p$*;*

(3) $\{\Phi_\alpha \mid \alpha \in A\}$ *is weakly composition closed.*

If $\{\Phi_\alpha \mid \alpha \in A\}$ *is weakly nice, then* $(3) \Rightarrow (2)$*, and if* $\mathcal{R}$ *is a class of* T_1 (S_1) *spaces then* $(2) \Rightarrow (1)$*. Moreover, in the implication* $(3) \Rightarrow (2)$*, if* $\{R_\beta \mid \beta \in B\} \subset \mathcal{P}$ *(with* $B \subset A$ *as in the definition of weak niceness), where* $\mathcal{P}$ *is a class of topological spaces closed under products, then we may assume* $Y \in \mathcal{P}$ *as well. Further, in the implication* $(2) \Rightarrow (1)$*, we may assume additionally that* X *is open in* Z*, and also that if* X, Y *are* T_0 *then* Z *is* T_0 *as well. Alternatively, for* $\mathcal{R}$ *a set of* T_1 (S_1) *spaces, we may assume that* X *is closed in* Z*, and if* X, Y *are* T_0*,* T_1*,* T_2 *or* T_3*, then* Z *is* T_0*,* T_1*,* T_2 *or* T_3*, respectively.*

Proof. **A)** $(1) \Rightarrow (2)$ is evident.

B) $(2) \Rightarrow (3)$ is proved like in [I]. Namely if, with the notations of (2), $\Phi_\alpha = C(Y, R_\alpha) \circ p$, and I, α, α_i, f_i and k are as in the definition of weak composition closedness, then $f_i = g_i \circ p$, $g_i \in C(Y, R_{\alpha_i})$ and $k \circ \langle f_i \rangle = k \circ \langle g_i \rangle \circ p \in C(Y, R_\alpha) \circ p = \Phi_\alpha$.

C) Now, assuming that $\{\Phi_\alpha \mid \alpha \in A\}$ is weakly nice, we prove $(3) \Rightarrow (2)$, again following [I]. Assuming (3), let $Y = \prod_{\beta \in B} R_\beta^{\Phi_\beta}$ and let $p : X \to Y$ be the mapping whose (β, f_β)-th coordinate (with $f_\beta \in \Phi_\beta$) is just f_β. Then the projection of Y to the (β, f_β)-th coordinate space is a map $\pi : Y \to R_\beta$, whose composition with $p : X \to Y$ is $(C(Y, R_\beta) \circ p \ni)\, \pi \circ p = f_\beta \in \Phi_\beta$. So $C(Y, R_\beta) \circ p$ contains all above f_β-s, therefore $C(Y, R_\beta) \circ p \supset \Phi_\beta$, for all $\beta \in B$.

For arbitrary $\alpha \in A$ and $f_\alpha \in \Phi_\alpha$ we have by the weak niceness, for some $g_{\beta\gamma} \in \Phi_\beta \subset C(Y, R_\beta) \circ p$ $(\beta \in B,\ \gamma \in \Gamma_\beta)$ and $h : \prod_B R_\beta^{\Gamma_\beta} \to R_\alpha$, that $f_\alpha = h \circ \langle g_{\beta\gamma} \rangle$. Thus we obtain $f_\alpha \in h \circ C(Y, \prod_B R_\beta^{\Gamma_\beta}) \circ p \subset C(Y, R_\alpha) \circ p$. Therefore $\Phi_\alpha \subset C(Y, R_\alpha) \circ p$.

Now let $f : Y \to R_\alpha$. By the weak composition closedness we have $f \circ p \in \Phi_\alpha$, establishing $C(Y, R_\alpha) \circ p \subset \Phi_\alpha$. Hence $C(Y, R_\alpha) \circ p = \Phi_\alpha$, i.e., (2) holds. The additional assertion about $\mathcal{P}$ is clear by construction.

D) Now we prove $(2) \Rightarrow (1)$, first assuming that $\mathcal{R}$ is a class of T_1 spaces. With the notations of (2) we have $p : X \to Y$, such that for each $\alpha \in A$ we have $\Phi_\alpha = C(Y, R_\alpha) \circ p$. We are going to construct another topological space Z, such that $X \subset Z$, and for each $\alpha \in A$ we have $\Phi_\alpha = C(Z, R_\alpha)|X$. This will be done on the lines of [He] and [K].

Let us consider the disjoint union Z of X and Y. We define the topology of Z as follows. For $x \in X$ a neighbourhood base is $\{U \mid U \ni x$ is open in the original topology of $X\}$. For $y \in Y$ a neighbourhood base is $\{V \cup p^{-1}(V) \mid V \ni y$ is open in the original topology of $Y\}$. Thus the injections $X \hookrightarrow Z$, $Y \hookrightarrow Z$ become homeomorphic embeddings from the original topologies of X and Y. Moreover, $p \in C(X, Y)$ has an extension $q \in C(Z, Y)$, i.e., $p = q|X$, where $q(x) = p(x)$, $q(y) = y$ $(x \in X,\ y \in Y)$.

Now let us consider a map $f \in C(Z, R)$, where R is a T_1 space. Then $g = f|Y \in C(Y, R)$. Here g determines f uniquely, since $x \in X$ implies $\overline{\{x\}} \ni p(x)$, hence $\overline{\{f(x)\}} \ni f(p(x))$, $f(x) = f(p(x))$. Moreover, each $g \in C(Y, R)$ can be extended to an $f \in C(Z, R)$, namely to $f = g \circ q$. Thus the correspondences $f \mapsto f|Y$, $g \mapsto g \circ q$ yield a bijection of $C(Z, R)$ to $C(Y, R)$.

We have by (2) for each $\alpha \in A$ that $\Phi_\alpha = \{g \circ p \mid g \in C(Y, R_\alpha)\}$, that further equals $\{(g \circ q)|X \mid g \in C(Y, R_\alpha)\} = \{f|X \mid f \in C(Z, R_\alpha)\}$. Therefore for each $\alpha \in A$ we have $\Phi_\alpha = C(Z, R_\alpha)|X$, as asserted.

If $\mathcal{R}$ is a class of S_1 spaces, then we consider the T_0-reflections $r_0 R_\alpha$ of R_α. From the T_1 case we have $\{g_0 \circ q \mid g_0 \in C(Y, r_0 R_\alpha)\} = \{f_0 \mid f_0 \in C(Z, r_0 R_\alpha)\}$, that implies $\{g \circ q \mid g \in C(Y, R_\alpha)\} = \{f \mid f \in C(Z, R_\alpha)\}$, and, like above, this implies for each $\alpha \in A$ that $\Phi_\alpha = \{f|X \mid f \in C(Z, R_\alpha)\}$.

Now we present an alternative construction, first for the case that $\mathcal{R}$ is a set of T_1 spaces. By H. Herrlich [He], Satz for any set $\mathcal{S}$ of T_1 spaces there exists a T_3 space $C_\mathcal{S}$ with at least two points such that for each $S \in \mathcal{S}$ each $f : C_\mathcal{S} \to S$ is constant. (Actually this is stated there only for a single T_1 space. However there is a single T_1 space S_0, e.g. a one with cofinite topology, such that each $S \in \mathcal{S}$ admits an injective map to S_0. Then $C_\mathcal{S}$ can be chosen as the space constructed by [He] for this T_1 space S_0.) Let $C = C_\mathcal{R}$. Let $a, b \in C$, $a \neq b$. Let us consider the disjoint union Z' of X, Y and $X \times C$, and then let us identify the pairs of points $\{x, (x, a)\}$ ($x \in X$, $(x, a) \in X \times C$), for each $x \in X$, and $\{p(x), (x, b)\}$ ($p(x) \in Y$, $(x, b) \in X \times C$), for each $x \in X$. Performing all these identifications, we obtain a set Z. (This is analogous to a mapping cylinder.) The points of Z will be denoted the same way as those of Z', noting that this notation is sometimes ambiguous. Similarly, the images of X, Y, $\{x\} \times C$, etc. by the quotient map $Z' \to Z$ will be denoted by X, Y, $\{x\} \times C$, etc.

Now we define the topology on Z. For $x \in X$ a neighbourhood base is $\{U \times W \mid U \ni x$ is open in the original topology of X, $W \ni a$ is open in the original topology of C, $W \not\ni b\}$. For $y \in Y$ a neighbourhood base is $\{V \cup (p^{-1}(V) \times (W \setminus \{b\})) \mid V \ni y$ is open in the original topology of Y, $W \ni b$ is open in the original topology of C, $W \not\ni a\}$. For $(x, c) \in X \times C$, $c \neq a, b$ a neighbourhood base is $\{\{x\} \times W \mid W \ni c$ is open in the original topology of C, $W \not\ni a, b\}$. Thus the injections $X \hookrightarrow Z$, $Y \hookrightarrow Z$, $i_x : C \hookrightarrow Z$, where $i_x(c) = (x, c)$ ($x \in X$), become homeomorphic embeddings from the original topologies of X, Y and C. Moreover, $p \in C(X, Y)$ has an extension $q \in C(Z, Y)$, i.e., $p = q|X$, where $q(x) = p(x)$, $q((x, c)) = p(x)$, $q(y) = y$ ($x \in X$, $c \in C$, $y \in Y$).

Now let us consider a map $f \in C(Z, R_\alpha)$ ($\alpha \in A$). Then $g = f|Y \in C(Y, R_\alpha)$. By construction g determines f uniquely, since, by the choice of C, f is constant on each subset $\{x\} \times C$ ($x \in X$). Moreover, each $g \in C(Y, R_\alpha)$ can be extended to an $f \in C(Z, R_\alpha)$, namely to $f = g \circ q$. Thus the correspondences $f \mapsto f|Y$, $g \mapsto g \circ q$ yield a bijection of $C(Z, R_\alpha)$ to $C(Y, R_\alpha)$.

Like above, by (2) for each $\alpha \in A$ we have $\Phi_\alpha = \{g \circ p \mid g \in C(Y, R_\alpha)\} = \{(g \circ q)|X \mid g \in C(Y, R_\alpha)\} = \{f|X \mid f \in C(Z, R_\alpha)\} = C(Z, R_\alpha)|X$, as asserted.

The case when $\mathcal{R}$ is a set of S_1 spaces is reduced to the case when $\mathcal{R}$ is a set of T_1 spaces like above. We consider the set $\mathcal{R}_0 = \{r_0 R \mid R \in \mathcal{R}\}$, and then for the space Z constructed for the set $\mathcal{R}_0$ we have for each $\alpha \in A$ that $\Phi_\alpha = C(Z, R_\alpha)|X$.

E) The additional assertions about Y, about openness or closedness of X in Z, and about the T_0, T_1, T_2 and T_3 property of Z, under the respective hypotheses, are immediate consequences of the constructions of Y and Z. □

Remark. Even in the combined implication (3) ⇒ (1) in Theorem 2.1 we cannot assert the $T_{3.5}$ property of Z, if X is $T_{3.5}$, even if $\mathcal{R} = \{\mathbb{R}\}$. Namely then by Theorem 1.2 another necessary condition for this is that $\{f^{-1}(0) \mid f \in \Phi\}$ be a closed base of X, which does not follow from the weak composition closedness (e.g. $\Phi = \{\text{constant maps } X \to \mathbb{R}\}$).

§3

Also several other results of [Cs 74], [Cs 77] and [Cs 84] about characterization of subsets $\Phi \subset C(X)$ of the form $C(Y)|X$, where Y is some extension of a given topological space X, or characterization of $C(\langle X, \mathcal{T}\rangle)$, with $\mathcal{T}$ some topology on a given set X, or its generalization in [G], mentioned in §1, can be put in our slightly more general setting. (The case when X is a set, can be considered as a special case of X being a topological space, namely that X is a discrete topological space.) Since here the proofs are just slight modifications of the original proofs and of part **D)** of the proof of Theorem 2.1, we will present here only two theorems. For the reader's convenience, we will give complete proofs.

An extension Z of a topological space X is called a *loose (or simple) extension* if X is open in Z and, for any $z \in Z \setminus X$, $X \cup \{z\}$ is open in Z.

Let X be a topological space, $\mathcal{R} = \{R_\alpha \mid \alpha \in A\}$ a class of topological spaces, and let $\Phi_\alpha \subset C(X, R_\alpha)$ $(\alpha \in A)$. We say that $\{\Phi_\alpha \mid \alpha \in A\}$ is *composition closed (strongly composition closed)* if for any set I, any $\alpha \in A$ and any $\alpha_i \in A$ $(i \in I)$, any $f_i \in \Phi_{\alpha_i}$ and any $k : \overline{\langle f_i\rangle(X)} \to R_\alpha$ $(k : \langle f_i\rangle(X) \to R_\alpha)$ we have $k \circ \langle f_i\rangle' \in \Phi_\alpha$, where closure is taken in $\prod_I R_{\alpha_i}$, and $\langle f_i\rangle' : X \to \overline{\langle f_i\rangle(X)}$ $(\langle f_i\rangle' : X \to \langle f_i\rangle(X))$ is the codomain restriction of $\langle f_i\rangle : X \to \prod_I R_{\alpha_i}$.

We say that $\{\Phi_\alpha \mid \alpha \in A\}$ is *nice (strongly nice)* if there is a set $B \subset A$, such that for any $\alpha \in A$ and any $f_\alpha \in \Phi_\alpha$ there exist $g_{\beta\gamma} \in \Phi_\beta$ $(\beta \in B,\ \gamma \in \Gamma_\beta$, where Γ_β is some set) and $h : \overline{\langle g_{\beta\gamma}\rangle(X)} \to R_\alpha$ $(h : \langle g_{\beta\gamma}\rangle(X) \to R_\alpha)$, such that $f_\alpha = h \circ \langle g_{\beta\gamma}\rangle'$, where $\langle g_{\beta\gamma}\rangle' : X \to \overline{\langle g_{\beta\gamma}\rangle(X)}$ $(\langle g_{\beta\gamma}\rangle' : X \to \langle g_{\beta\gamma}\rangle(X))$ is the codomain restriction of $\langle g_{\beta\gamma}\rangle : X \to \prod_B R_\beta^{\Gamma_\beta}$. (Observe however that weak niceness $\Rightarrow$ niceness $\Rightarrow$ strong niceness, that also shows that the examples of weakly nice systems $\{\Phi_\alpha \mid \alpha \in A\}$ given after the definition of weak niceness are examples for niceness and strong niceness as well. In particular, the classes of spaces considered in [G], being hereditary, are strongly nice.)

We say that a filter $\mathcal{F}$ in X is a $\{\Phi_\alpha \mid \alpha \in A\}$*-filter*, if for each $\alpha \in A$ and for each $f_\alpha \in \Phi_\alpha$ we have that $f_\alpha(\mathcal{F})$ is convergent in R_α. We say that $\{\Phi_\alpha \mid \alpha \in A\}$ is *filter-closed*, if $\alpha \in A$, $f_\alpha \in C(X, R_\alpha)$ and [for each $\{\Phi_\alpha \mid \alpha \in A\}$-filter $\mathcal{F}$ in X we have that $f_\alpha(\mathcal{F})$ is convergent in R_α] imply that $f_\alpha \in \Phi_\alpha$. (For the case that $\{R_\alpha \mid \alpha \in A\} = \{\mathbb{R}\}$ and $\Phi \subset C(X, \mathbb{R})$, our definition of filter-closedness reduces to the definition in [Cs 74] (cf. also our §1), in view of the T_1-property of $\mathbb{R}$. Namely, [Cs 74] requires in this definition $f \in \mathbb{R}^X$ rather than $f \in C(X, \mathbb{R})$. However, for $\Phi \subset C(X)$, any convergent filter in X is a Φ-filter. Therefore, for $f \in \mathbb{R}^X$, [for each Φ-filter $\mathcal{F}$ in X we have that $f(\mathcal{F})$ is convergent in $\mathbb{R}$] implies that for each $x \in X$ there exists $g(x) \in \mathbb{R}$, such that $f(\mathcal{V}(x)) \to g(x)$, where $\mathcal{V}(x)$ is the neighbourhood filter of x in X. By $f(x) \in \cap f(\mathcal{V}(x))$ this implies $\overline{\{f(x)\}} \ni g(x)$, so, by the T_1-property of $\mathbb{R}$, $f(\mathcal{V}(x)) \to g(x) = f(x)$, hence $f \in C(X, \mathbb{R})$.)

The following theorem is an extension of [Cs 74], (2.1), (2.4), [Cs 77], Theorem 4, [Cs 84], Theorem 2.3 and [ŽB], Theorem 3.4.

Theorem 3.1. *Let X be a topological space and $\mathcal{R} = \{R_\alpha \mid \alpha \in A\}$ a class of topological spaces. Further let $\Phi_\alpha \subset C(X, R_\alpha)$ $(\alpha \in A)$. Then we have the implications* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4)*, where*

(1) *$\{\Phi_\alpha \mid \alpha \in A\}$ is filter-closed;*
(2) *there exists an extension Z of X, such that for each $\alpha \in A$ we have $\Phi_\alpha = C(Z, R_\alpha)|X$;*

(3) *there exist a topological space* Y *and a map* $p: X \to Y$*, such that* $\overline{p(X)} = Y$ *and for each* $\alpha \in A$ *we have* $\Phi_\alpha = C(Y, R_\alpha) \circ p$*;*

(4) $\{\Phi_\alpha \mid \alpha \in A\}$ *is composition closed.*

If $\{\Phi_\alpha \mid \alpha \in A\}$ *is nice then* (4) $\Rightarrow$ (3)*, if* $\mathcal{R}$ *consists of* T_1 *(*S_1*) spaces then* (3) $\Rightarrow$ (2)*, and if* $\mathcal{R}$ *consists of regular spaces then* (2) $\Rightarrow$ (1)*. Moreover, in the implication* (4) $\Rightarrow$ (3)*, if* $\{R_\beta \mid \beta \in B\} \subset \mathcal{P}$ *(with* $B \subset A$ *as in the definition of niceness), where* $\mathcal{P}$ *is a class of topological spaces closed under taking products and closed subspaces, then we may assume* $Y \in \mathcal{P}$ *as well. Further, in the implication* (3) $\Rightarrow$ (2)*, we may assume additionally that* X *is open in* Z *and also that if* X, Y *are* T_0 *then* Z *is* T_0 *as well. Similarly, in the implication* (1) $\Rightarrow$ (2)*, we may assume additionally that* Z *is a loose extension of* X*, hence* Z *is* T_0 *for* X T_0*, and, if each* $R \in \mathcal{R}$ *is* T_2*, then we may assume also that distinct points of* $Z \setminus X$ *have disjoint neighbourhoods in* Z*.*

Proof. **A)** First we prove (1) $\Rightarrow$ (2), on the lines of [Cs 84], Theorem 2.3.

We begin by proving an analogue of [Cs 84], Corollary 2.2: (a) $\{\Phi_\alpha \mid \alpha \in A\}$ ($\Phi_\alpha \subset C(X, R_\alpha)$) is filter-closed iff $\alpha \in A$, $f_\alpha \in C(X, R_\alpha)$ and [for each open $\{\Phi_\alpha \mid \alpha \in A\}$-filter $\mathcal{F}$ in X we have that $f_\alpha(\mathcal{F})$ is convergent in R_α] imply that $f_\alpha \in \Phi_\alpha$. (An open filter is a filter having an open base.) For this it clearly suffices to prove an analogue of [Cs 84], Lemma 2.1 (due to E. Lowen-Colebunders): (b) if $\alpha \in A$, $f_\alpha \in C(X, R_\alpha)$ and [for each open $\{\Phi_\alpha \mid \alpha \in A\}$-filter $\mathcal{F}$ in X we have that $f_\alpha(\mathcal{F})$ is convergent in R_α], then [for each $\{\Phi_\alpha \mid \alpha \in A\}$-filter $\mathcal{F}$ in X we have that $f_\alpha(\mathcal{F})$ is convergent in R_α].

We let for a filter $\mathcal{F}$ in X $\mathcal{F}^o$ ($\subset \mathcal{F}$) be the open filter in X generated by $\{U \subset X \text{ open} \mid U \in \mathcal{F}\}$. Then, in order to prove (b), it suffices to establish: (c) $\mathcal{F}$ is a $\{\Phi_\alpha \mid \alpha \in A\}$-filter in X implies that $\mathcal{F}^o$ is a $\{\Phi_\alpha \mid \alpha \in A\}$-filter in X. Namely, if (c) holds, then for any $\{\Phi_\alpha \mid \alpha \in A\}$-filter $\mathcal{F}$ in X we have that $f_\alpha(\mathcal{F}^o)$ is convergent in R_α, hence, by $f_\alpha(\mathcal{F}^o) \subset f_\alpha(\mathcal{F})$, also $f_\alpha(\mathcal{F})$ is convergent in R_α.

Now we establish claim (c). Suppose that $\mathcal{F}$ is a $\{\Phi_\alpha \mid \alpha \in A\}$-filter in X, i.e., for each $\alpha \in A$ and for each $f_\alpha \in \Phi_\alpha$ there exists $r_\alpha \in R_\alpha$, such that $f_\alpha(\mathcal{F}) \to r_\alpha$. Let us fix an $\alpha \in A$ and an $f_\alpha \in \Phi_\alpha$. Let $V_\alpha \subset R_\alpha$ be open, $r_\alpha \in V_\alpha$. Then there exists $F \in \mathcal{F}$ such that $f_\alpha(F) \subset V_\alpha$, so $F \subset f_\alpha^{-1}(V_\alpha)$, implying $f_\alpha^{-1}(V_\alpha) \in \mathcal{F}$, so even $f_\alpha^{-1}(V_\alpha) \in \mathcal{F}^o$. Moreover, $f_\alpha(f_\alpha^{-1}(V_\alpha)) \subset V_\alpha$, so $f_\alpha(\mathcal{F}^o) \to r_\alpha$. Hence $\mathcal{F}^o$ is a $\{\Phi_\alpha \mid \alpha \in A\}$-filter in X, as claimed. Thus we have proved our claims (c), (b), (a).

By (a) we have for each $\alpha \in A$ that $\Phi_\alpha = \{f_\alpha \in C(X, R_\alpha) \mid$ for each open $\{\Phi_\alpha \mid \alpha \in A\}$-filter $\mathcal{F}$ in X we have that $f_\alpha(\mathcal{F})$ is convergent in $R_\alpha\}$. We define Z as the loose extension of X, corresponding to the trace filter system $\{\mathcal{F} \mid \mathcal{F}$ is an open $\{\Phi_\alpha \mid \alpha \in A\}$-filter in $X\}$. The point of $Z \setminus X$ corresponding to an above $\mathcal{F}$ will be denoted by $z_{\mathcal{F}}$. (That is, Z is the disjoint union of X and $\{z_{\mathcal{F}} \mid \mathcal{F}$ is an open $\{\Phi_\alpha \mid \alpha \in A\}$-filter in $X\}$, $\mathcal{F} \mapsto z_{\mathcal{F}}$ being a bijection; for $x \in X$ a neighbourhood base of x in Z is $\{U \ni x \mid U \subset X$ is open in the topology of $X\}$, for $z_{\mathcal{F}}$ a neighbourhood base in Z is $\{F \cup \{z_{\mathcal{F}}\} \mid F \in \mathcal{F}\}$.) Then for any topological space R we have $C(Z, R) = \{g \in R^Z \mid g|X \in C(X, R)$ and for each open $\{\Phi_\alpha \mid \alpha \in A\}$-filter $\mathcal{F}$ in X we have $g(\mathcal{F}) \to g(z_{\mathcal{F}})\}$, hence $C(Z, R)|X = \{f \in C(X, R) \mid$ for each open $\{\Phi_\alpha \mid \alpha \in A\}$-filter $\mathcal{F}$ in X we have that $f(\mathcal{F})$ is convergent in $R\}$. Applying this for $R = R_\alpha \in \mathcal{R}$, we obtain $C(Z, R_\alpha)|X = \Phi_\alpha$.

Of course, to have the same conclusion $C(Z, R_\alpha)|X = \Phi_\alpha$, it is sufficient to consider, rather than the set $\{\mathcal{F}_\lambda \mid \lambda \in \Lambda\}$ of all open $\{\Phi_\alpha \mid \alpha \in A\}$-filters $\mathcal{F}$ in X, only a subset $\{\mathcal{F}_{\lambda'} \mid \lambda' \in \Lambda'\}$ ($\Lambda' \subset \Lambda$), where for each $\lambda \in \Lambda$ there exists a

$\lambda' \in \Lambda'$ such that $\mathcal{F}_\lambda \supset \mathcal{F}_{\lambda'}$, and then consider the loose extension with trace filters $\{\mathcal{F}_{\lambda'} \mid \lambda' \in \Lambda'\}$. Namely then, analogously as above, for any topological space R we have $C(Z,R)|X = \{f \in C(X,R) \mid$ for each $\lambda' \in \Lambda'$ we have that $f(\mathcal{F}_{\lambda'})$ is convergent in $R\} = \{f \in C(X,R) \mid$ for each $\lambda \in \Lambda$ we have that $f(\mathcal{F}_\lambda)$ is convergent in $R\}$, which yields for $R = R_\alpha \in \mathcal{R}$ that $C(Z,R_\alpha)|X = \Phi_\alpha$.

Such a subset $\{\mathcal{F}_{\lambda'} \mid \lambda' \in \Lambda'\}$ can be obtained in the following way. Let $\Phi_\alpha = \{f_{\alpha\delta} \mid \delta \in \Delta_\alpha\}$ be a labelling of Φ_α, and let $r_{\alpha\delta} \in R_\alpha$ for $\alpha \in A$ and $\delta \in \Delta_\alpha$. Let $\mathcal{F}(\langle r_{\alpha\delta}\rangle)$ be the coarsest filter in X (if such exists), for which for each $\alpha \in A$ and for each $\delta \in \Delta_\alpha$ we have that $f_{\alpha\delta}(\mathcal{F}(\langle r_{\alpha\delta}\rangle)) \to r_{\alpha\delta}$, i.e., $\mathcal{F}(\langle r_{\alpha\delta}\rangle)$ be the filter in X generated by $\cup\{f_{\alpha\delta}^{-1}(\mathcal{V}_{R_\alpha}(r_{\alpha\delta})) \mid \alpha \in A,\ \delta \in \Delta_\alpha\}$, if this filter exists ($\mathcal{V}_{R_\alpha}(r_{\alpha\delta})$ is the neighbourhood filter of $r_{\alpha\delta}$ in R_α). This filter $\mathcal{F}(\langle r_{\alpha\delta}\rangle)$, if it exists, is an open $\{\Phi_\alpha \mid \alpha \in A\}$-filter in X. The collection of all such (existing) filters, with $r_{\alpha\delta} \in R_\alpha$ varying arbitrarily and independently of each other, for each $\alpha \in A$ and $\delta \in \Delta_\alpha$, is a subset $\{\mathcal{F}_{\lambda'} \mid \lambda' \in \Lambda'\}$ as looked for above.

If for each $\alpha \in A$ we have that R_α is T_2, then two distinct filters $\mathcal{F}_1, \mathcal{F}_2$ in X of the form $\mathcal{F}(\langle r_{\alpha\delta}\rangle)$ have disjoint elements. This shows that, in the loose extension with trace filters $\{\mathcal{F}_{\lambda'} \mid \lambda' \in \Lambda'\}$ from the previous paragraph, the corresponding points $z_{\mathcal{F}_1}, z_{\mathcal{F}_2}$ have disjoint neighbourhoods.

B) (2) $\Rightarrow$ (3) is evident.

C) (3) $\Rightarrow$ (4) is proved like [Cs 77], Theorem 4, $(a) \Rightarrow (b)$ (and our Theorem 2.1, (2) $\Rightarrow$ (3)). With the notations of (3), we have $\overline{p(X)} = Y$ and, for each $\alpha \in A$, that $\Phi_\alpha = C(Y,R_\alpha) \circ p$. Now let I, α, α_i, f_i, $\langle f_i\rangle'$ and k be as in the definition of the composition closedness. We have $f_i = g_i \circ p$, where $g_i \in C(Y,R_{\alpha_i})$. Then $Y = \overline{p(X)}$ implies $\langle g_i\rangle(Y) \subset \overline{\langle g_i\rangle(p(X))} = \overline{\langle f_i\rangle(X)}$. Let $\langle g_i\rangle' : Y \to \overline{\langle f_i\rangle(X)}$ be the codomain restriction of $\langle g_i\rangle : Y \to \prod_I R_{\alpha_i}$. Then we have $\langle f_i\rangle = \langle g_i\rangle \circ p$, hence $\langle f_i\rangle' = \langle g_i\rangle' \circ p$, and $k \circ \langle f_i\rangle' = k \circ \langle g_i\rangle' \circ p \in C(Y,R_\alpha) \circ p = \Phi_\alpha$.

D) If $\{\Phi_\alpha \mid \alpha \in A\}$ is nice then (4) $\Rightarrow$ (3) is proved like [Cs 77], Theorem 4, $(b) \Rightarrow (c)$ (and our Theorem 2.1, (3) $\Rightarrow$ (2)). Let $Y_0 = \prod_B R_\beta^{\Phi_\beta}$ and let $p_0 : X \to Y_0$ be the mapping with (β, f_β)-th coordinate ($f_\beta \in \Phi_\beta$) just f_β. We define $Y = \overline{p_0(X)} \subset Y_0$, with injection $i : Y \hookrightarrow Y_0$. Letting $p : X \to Y$ be the codomain restriction of p_0, we have $\overline{p(X)} = Y$ and $p_0 = i \circ p$. Then letting $\pi : Y_0 \to R_\beta$ be the projection to the (β, f_β)-th coordinate space, we have $f_\beta = \pi \circ p_0 = \pi \circ i \circ p \in C(Y,R_\beta) \circ p$. This proves $\Phi_\beta \subset C(Y,R_\beta) \circ p$, for all $\beta \in B$.

For arbitrary $\alpha \in A$ and $f_\alpha \in \Phi_\alpha$, by the niceness we have for some $g_{\beta\gamma} \in \Phi_\beta \subset C(Y,R_\beta) \circ p$ $(\beta \in B,\ \gamma \in \Gamma_\beta)$, where $g_{\beta\gamma} = g^*_{\beta\gamma} \circ p$ with $g^*_{\beta\gamma} \in C(Y,R_\beta)$, and for some $h : \overline{\langle g_{\beta\gamma}\rangle(X)} \to R_\alpha$, that $f_\alpha = h \circ \langle g_{\beta\gamma}\rangle'$. Then $\langle g^*_{\beta\gamma}\rangle : Y \to \prod_B R_\beta^{\Gamma_\beta}$, and $\langle g^*_{\beta\gamma}\rangle(Y) = \langle g^*_{\beta\gamma}\rangle\overline{p(X)} \subset \overline{\langle g^*_{\beta\gamma}\rangle p(X)} = \overline{\langle g_{\beta\gamma}\rangle(X)}$. Thus $\langle g^*_{\beta\gamma}\rangle = j \circ \langle g^*_{\beta\gamma}\rangle'$, where $j : \overline{\langle g_{\beta\gamma}\rangle(X)} \hookrightarrow \prod_B R_\beta^{\Gamma_\beta}$ is the injection and $\langle g^*_{\beta\gamma}\rangle' : Y \to \overline{\langle g_{\beta\gamma}\rangle(X)}$ the codomain restriction of $\langle g^*_{\beta\gamma}\rangle$. Then we have $j \circ \langle g_{\beta\gamma}\rangle' = \langle g_{\beta\gamma}\rangle = \langle g^*_{\beta\gamma}\rangle \circ p = j \circ \langle g^*_{\beta\gamma}\rangle' \circ p$. This implies $\langle g_{\beta\gamma}\rangle' = \langle g^*_{\beta\gamma}\rangle' \circ p$, and thus $f_\alpha = h \circ \langle g^*_{\beta\gamma}\rangle' \circ p \in C(Y,R_\alpha) \circ p$. This proves $\Phi_\alpha \subset C(Y,R_\alpha) \circ p$.

Conversely, let $\alpha \in A$ and $k : Y \to R_\alpha$. Then the composition closedness of $\{\Phi_\alpha \mid \alpha \in A\}$ implies that $k \circ p \in \Phi_\alpha$, which proves $C(Y,R_\alpha) \circ p \subset \Phi_\alpha$. Hence we have $C(Y,R_\alpha) \circ p = \Phi_\alpha$, thus (3) holds.

E) If $\mathcal{R}$ consists of T_1 (S_1) spaces only, then the implication (3) $\Rightarrow$ (2) is a simple consequence of the proof of the analogous implication (2) $\Rightarrow$ (1) in Theorem 2.1, cf. the first construction there. We have to observe only that, if in that construction we have $\overline{p(X)} = Y$, then X is a dense subspace of Z.

F) If $\mathcal{F}$ consists of regular spaces only, then (2) $\Rightarrow$ (1) is proved like [Cs 74], (2.1). Let $Z \supset X$ be an extension of X, and let $\Phi_\alpha = C(Z, R_\alpha)|X$ for each $\alpha \in A$. Further let $\alpha \in A$, and let $f_\alpha \in C(X, R_\alpha)$ satisfy [for each $\{\Phi_\alpha \mid \alpha \in A\}$-filter $\mathcal{F}$ in X we have that $f_\alpha(\mathcal{F})$ is convergent in R_α]. Then we have to show that $f_\alpha \in \Phi_\alpha$.

For $z \in Z$ let $\mathcal{V}(z)$ be its neighbourhood filter in Z, and $\mathcal{F}(z) = \mathcal{V}(z)|X$ the trace filter in X. Then for each $z \in Z$ we have that $\mathcal{F}(z)$ is a $\{\Phi_\alpha \mid \alpha \in A\}$-filter. In fact, for any $\beta \in A$ and any $g_\beta \in \Phi_\beta = C(Z, R_\beta)|X$, we have $g_\beta = h_\beta|X$ where $h_\beta \in C(Z, R_\beta)$, so $g_\beta(\mathcal{F}(z)) = h_\beta(\mathcal{F}'(z)) \to h_\beta(z)$, where $\mathcal{F}'(z)$ is the filter in Z generated by $\mathcal{F}(z)$.

For our f_α we have that $f_\alpha(\mathcal{F})$ is convergent in R_α, for each $\{\Phi_\alpha \mid \alpha \in A\}$-filter $\mathcal{F}$ in X. In particular, this holds for each filter $\mathcal{F} = \mathcal{F}(z)$, where $z \in Z$. Thus there exists a function $k_\alpha \in R_\alpha^Z$, such that $k_\alpha(x) = f_\alpha(x)$ $(x \in X)$ and $f_\alpha(\mathcal{F}(z)) \to k_\alpha(z)$ $(z \in Z \setminus X)$. By the regularity of R_α, we have $k_\alpha \in C(Z, R_\alpha)$, hence $f_\alpha = k_\alpha|X \in C(Z, R_\alpha)|X = \Phi_\alpha$, as needed to show.

G) The additional assertions about Y, about openness of X in Z, about T_0 property of Z and additional properties of Z, under the respective hypotheses, are either immediate consequences of the constructions, or have been pointed out above. $\square$

The following theorem is an extension of [Cs 74], (2.6), [Cs 77], Theorem 2, [G], Teorema 2.1, (a) (taking in consideration [G], Proposizione 2.1, (b)) and [ŽB], Theorem 4.3.

Theorem 3.2. *Let X be a topological space and $\mathcal{R} = \{R_\alpha \mid \alpha \in A\}$ a class of topological spaces. Further let $\Phi_\alpha \subset C(X, R_\alpha)$ $(\alpha \in A)$. Then we have the implication* (1) $\Rightarrow$ (2), *where*

(1) *there exist a topological space Y and a map $p : X \to Y$, such that $p(X) = Y$ and for each $\alpha \in A$ we have $\Phi_\alpha = C(Y, R_\alpha) \circ p$;*
(2) *$\{\Phi_\alpha \mid \alpha \in A\}$ is strongly composition closed.*

If $\{\Phi_\alpha \mid \alpha \in A\}$ is strongly nice, then (2) $\Rightarrow$ (1). *Moreover, in the implication* (2) $\Rightarrow$ (1), *if $\{R_\beta \mid \beta \in B\} \subset \mathcal{P}$ (with $B \subset A$ as in the definition of strong niceness), where $\mathcal{P}$ is a class of topological spaces closed under taking products and subspaces, then we may assume $Y \in \mathcal{P}$ as well.*

Proof. **A)** (1) $\Rightarrow$ (2) is proved like [Cs 74], (2.6) and [Cs 77], Theorem 2, (a) $\Rightarrow$ (b) (and our Theorem 3.1, (3) $\Rightarrow$ (4)). Compared to this last one, and with analogous notations, we have here $p(X) = Y$ and $\langle g_i \rangle(Y) = \langle f_i \rangle(X)$, and we consider the codomain restriction $\langle g_i \rangle' : Y \to \langle f_i \rangle(X)$ of $\langle g_i \rangle : Y \to \prod_I R_{\alpha_i}$. With these changes, the proof runs analogously.

B) If $\{\Phi_\alpha \mid \alpha \in A\}$ is strongly nice, then (2) $\Rightarrow$ (1) is proved like [Cs 74], (2.6) and [Cs 77], Theorem 2, (b) $\Rightarrow$ (c) (and our Theorem 3.1, (4) $\Rightarrow$ (3)). Compared to this last one, and with analogous notations, we define here $Y = p_0(X)$, yielding that $p : X \to Y$, the codomain restriction of $p_0 : X \to Y_0$, is onto. Then $\langle g^*_{\beta\gamma} \rangle(Y) = \langle g_{\beta\gamma} \rangle(X)$, and we have the codomain restriction $\langle g^*_{\beta\gamma} \rangle' : Y \to \langle g_{\beta\gamma} \rangle(X)$ of $\langle g^*_{\beta\gamma} \rangle : Y \to \prod_B R_\beta^{\Gamma_\beta}$, and the injection $j : \langle g_{\beta\gamma} \rangle(X) \hookrightarrow \prod_B R_\beta^{\Gamma_\beta}$. With these changes, the proof again runs analogously.

C) The additional assertion is evident by construction. $\square$

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L. Eötvös University, Department of Analysis, H-1088 Budapest, Múzeum krt. 6-8,
HUNGARY,
Mathematical Institute of the Hungarian Academy of Sciences, H-1364 Budapest, Pf. 127, HUNGARY

E-mail address: `makai@math-inst.hu`