Studia Scientiarum Mathematicarum Hungarica 9 (1974) 191—193.

ON THE THINNEST NON-SEPARABLE LATTICE OF CONVEX PLATES

by

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Suggested by results of A. HEPPES [1] and J. HORVÁTH [2] (see also [3]) concerning transillumination of a lattice-packing of balls, G. FEJES TÓTH proposed the following problem: Find the thinnest n-dimensional lattice of closed n-dimensional balls such that any k-dimensional subspace $(0 \leq k \leq n-1)$ has a point in common with at least one ball.

The case when $k=0$ is a well known problem: the problem of the thinnest lattice-covering of the space with balls [4]. Special attention is due also to the case when $k=n-1$. A family of n-dimensional bodies is said to be *separable* if there is an $(n-1)$-dimensional plane not intersecting any of the bodies but containing on both sides a body. Thus the case of the above problem when $k=n-1$ can be formulated as follows: Find the thinnest non-separable lattice of n-dimensional balls.

Let $d_n(b)$ be the density of a thinnest non-separable lattice of an n-dimensional convex body b. Then we can ask: What is the infimum of $d_n(b)$ if b ranges over all convex bodies? A similar problem concerning connected lattices instead of non-separable ones was considered by GROEMER [5] (see also [6]).

In what follows we solve the above problems for $n=2$. We shall use the terms "disc" and "plate" for a closed 2-dimensional ball and a closed 2-dimensional convex body, respectively.

THEOREM 1. *The density of a non-separable lattice of discs is at least $\sqrt{3}\pi/8$. Equality holds only for a lattice generated by the excircles of a regular triangle* (Fig. 1).

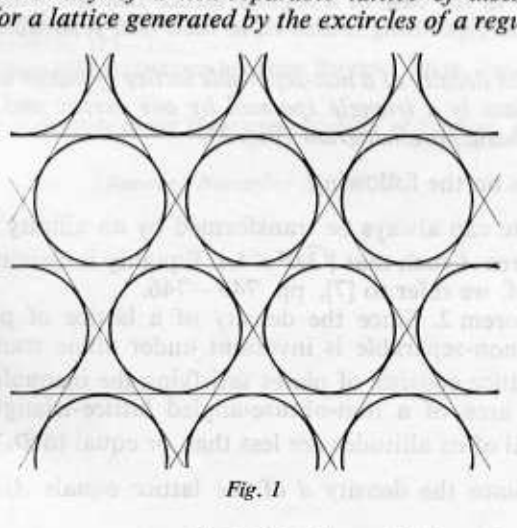

Fig. 1

PROOF. In a non-separable lattice of unit discs let A and B be two nearest centers. Consider a further center outside the line AB nearest to it. Among these centers there is one, say, C lying "above" the segment AB. Since in the triangle ABC AB is a shortest side, we have $\sphericalangle C \leqq 90°$. On the other hand, since C lies above the side AB, we have $\sphericalangle A \leqq 90°$ and $\sphericalangle B \leqq 90°$. Thus the triangle ABC is non-obtuse-angled. Obviously, it has the further property that any two of its side-vectors generate the lattice, in short, it is a lattice-triangle.

We claim that all altitudes of the triangle ABC are less than or equal to 2. For, assuming that for instance the altitude issuing from C is greater than 2, the disc with center C could be separated from the discs centered in A and B by a line parallel to AB. But this line would separate the whole lattice, in contradiction to the assumption that the lattice is non-separable.

We continue to show that among the non-obtuse-angled triangles ABC with altitudes $\leqq 2$ the equilateral triangle $A^*B^*C^*$ with altitude 2 has the greatest area. Let the feet of the altitudes issuing from A, B and C be A', B' and C'. We may suppose that one of the altitudes, say, CC' is equal to 2. It suffices to show that $\overline{AB} \leqq \overline{A^*B^*}$ with equality only if ABC is congruent to $A^*B^*C^*$.

Supposing that $\overline{AB} > \overline{A^*B^*}$ and, say, $\overline{AC} > \overline{CB}$, we move C in the direction BA until we have $\overline{AC} = \overline{CB}$. Meanwhile $\overline{AA'}$ obviously decreases. If in the new position $\sphericalangle C \geqq 90°$, then we have in the new position, and consequently in the original too, $\overline{AA'} > 2$, in contradiction to our assumption. Thus we assume that $\sphericalangle C < 90°$, $\overline{AC} = \overline{CB}$ and $\overline{AB} > \overline{A^*B^*}$. Fixing the position of C, we displace A towards B and B towards A through the same distance until ABC becomes congruent to $A^*B^*C^*$. Observing that during this operation in the right-angled triangle $AA'C$ the hypotenuse AC decreases and $\sphericalangle A'AC$ increases, we see that $\overline{AA'}$ decreases, showing that the original altitude $\overline{AA'}$ was greater than 2.

Thus, denoting the area of ABC with Δ and that of $A^*B^*C^*$ with Δ^*, we have $\Delta \leqq \Delta^* = 4/\sqrt{3}$, yielding for the density $d = \pi/2\Delta$ the desired inequality $d \geqq \pi\sqrt{3}/8$. Equality holds only in the case of a regular lattice-triangle with altitude 2. It is easy to check that the corresponding lattice is, in fact, non-separable.

THEOREM 2. *The density of a non-separable lattice of plates is at least* 3/8. *Equality holds only if the plate is a triangle spanned by one vertex and the midpoints of the opposite sides of a basic parallelogram* (Fig. 2).

The proof rests on the following

LEMMA. A plate can always be transformed by an affinity in a new one having a diameter D and area A such that $\sqrt{3}D^2 \leqq 4A$. Equality is claimed only for a triangle.

As to the proof, we refer to [7], pp. 745—746.

PROOF of Theorem 2. Since the density of a lattice of plates, as well as the property of being non-separable is invariant under affine transformation, we may suppose that the lattice consists of plates satisfying the inequality $\sqrt{3}D^2 \leqq 4A$.

Let Δ be the area of a non-obtuse-angled lattice-triangle. This triangle has the property that all of its altitudes are less than or equal to D. Therefore $\Delta \leqq D^2/\sqrt{3}$, whence $\Delta \leqq \frac{4}{3} A$. Since the density d of the lattice equals $A/2\Delta$, we have $d \geqq 3/8$.

Equality holds only if both the lattice-triangle and the plate are regular triangles, the diameter (i.e. the side-length) of the latter being equal to the altitude of the former. A glance to Fig. 2 shows that this lattice is non-separable. This completes the proof of Theorem 2.

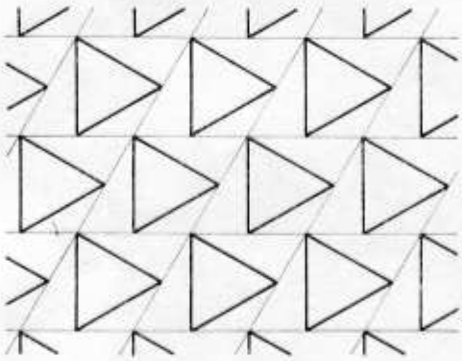

Fig. 2

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(Received November 16, 1973.)