ON AN INTEGRO-DIFFERENTIAL TRANSFORM ON THE SPHERE

E. Makai, Jr.*, H. Martini, T. Ódor**

Abstract. In a recent paper the authors have proved that a convex body $K \subset \mathbb{R}^d$, $d \geq 2$, containing the origin 0 in its interior, is symmetric with respect to 0 if and only if $V_{d-1}(K \cap H') \geq V_{d-1}(K \cap H)$ for all hyperplanes H, H' such that H and H' are parallel and $H' \ni 0$ (V_{d-1} is $(d-1)$–measure). For the proof the authors have employed a new type of integro–differential transform, that lets to correspond to a sufficiently nice function f on S^{d-1} the function $R^{(1)}f$, where $(R^{(1)}f)(\xi) = \int_{S^{d-1}\cap\xi^\perp} (\partial f/\partial\psi) d\eta$ — with $\xi \in S^{d-1}$ as pole and ψ as geographic latitude — and have determined the null–space of the operator $R^{(1)}$. In this paper we extend the definition to any integer $m \geq 1$, defining $(R^{(m)}f)(\xi)$ analogously as for $m = 1$, but using $\partial^m f/\partial\psi^m$ rather than $\partial f/\partial\psi$. (The case $m = 0$ is the spherical Radon transformation (Funk transformation).) We investigate the null–space of the operator $R^{(m)}$: up to a summand of finite dimension, it consists of the even (odd) functions in the domain of the operator, for m odd (even). For the proof we use spherical harmonics.

1. Introduction

In [Makai–Martini–Ódor] we have proved that a convex body $K \subset \mathbb{R}^d$, $d \geq 2$, containing the origin 0 in its interior, has the property that for every hyperplane H the hyperplane H' parallel to H and passing through 0 satisfies $V_{d-1}(K \cap H') \geq V_{d-1}(K \cap H)$ (V_{d-1} is $(d-1)$–volume), if and only if this convex body K is symmetric with respect to 0. The case $d = 2$ was proved by [Hammer, 1954]. (In fact he proved the analogous statement for 1–dimensional sections for $K \subset \mathbb{R}^d$, and we have proved the analogous statement for k–dimensional sections for $K \subset \mathbb{R}^d$, $1 \leq k \leq d-1$.)

We have proved our result by using a new Radon–type transformation, which can be considered as a common generalization of partial differential operators and Radon–type transformations. Moreover we have proved our theorem by using spherical harmonics and the Funk–Hecke theorem.

1991 *Mathematics Subject Classification.* 1991 *Mathematics Subject Classification.* Primary: 44A12; Secondary: 33C55.

Key words and phrases. integro–differential operator, spherical harmonics, Funk–Hecke theorem, Radon–type transformation.

* Partially supported by Hungarian National Foundation for Scientific Research, Grant No. T-031931 and FKFP Grant No. 0391/1997.

** Partially supported by Hungarian National Foundation for Scientific Research, Grant number 4427

Our integro–differential transformation assigns the function $R^{(1)}f\colon S^{d-1}\to\mathbb{R}$ to a sufficiently nice function $f\colon S^{d-1}\to\mathbb{R}$, where $(R^{(1)}f)(\omega)$ is the integral of the derivative of f in the direction ω over the $S^{d-2}\subset S^{d-1}$ whose spherical center is ω. In order to prove our theorem, we have established that the null–space of this operator $R^{(1)}$ consists of the even functions in the domain of $R^{(1)}$.

Here we investigate certain generalization of this transformation where for $m\geq 0$, the m'th derivative of f with respect to the geographic latitude (ω considered as the north pole) is integrated over the same S^{d-2}. The case $m=0$ is the spherical Radon transformation (Funk transformation), the case $m=1$ is the transformation $R^{(1)}$ above. We obtain that for $m\geq 1$, the null–space of this operator is the direct sum of the even functions (for m odd) or odd functions (for m even) in the domain of this operator, respectively, and a subspace of finite dimension. (We note that the domain of $R^{(1)}$ in this paper is smaller than that in [Makai–Martini–Ódor].)

For general analytical background, we refer to the books [Tricomi, 1957], [Adams, 1975] and [Ziemer, 1989].

2. Preliminaries

As usual, $\mathbb{R}^d$ denotes the d–dimensional Euclidean space which is endowed with the standard inner product and norm $|\cdot|$ structure. We will suppose $d\geq 2$. The origin is denoted by 0 and V_{d-1} is the $(d-1)$–dimensional volume on the hyperplanes.

Let S^{d-1} denote the unit sphere with center 0; its variable point will be denoted by ω, ξ or η. For $\omega\in S^{d-1}$ and $t\in\mathbb{R}$ let $H(\omega,t)$ be the hyperplane given by the equation $\langle x,\omega\rangle=t$. We write $\omega^\perp$ for $H(\omega,0)$. Often we will use a polar coordinate system on S^{d-1}, with ("north") pole some $\xi\in S^{d-1}$. That is, any $\omega\in S^{d-1}$ can be written as

$$(1)\qquad \omega=\xi\sin\psi+\eta\cos\psi,\ \text{ where } \eta\in S^{d-1}\cap\xi^\perp \ \text{ and } \ -\pi/2\leq\psi\leq\pi/2$$

(thus ψ is the *geographic latitude*, which will be more convenient for us than the costumarily used $\varphi=\pi/2-\psi$); then we write

$$(2)\qquad \omega=(\eta,\psi).$$

A real function defined on S^{d-1} is called *even (odd)*, if for all $\omega\in S^{d-1}$ we have $f(-\omega)=f(\omega)$ $(f(-\omega)=-f(\omega))$.

We turn to spherical harmonics, which are higher–dimensional generalizations of the trigonometric functions $\cos(nx)$, $\sin(nx)$ (these are obtained for $d=2$). Standard references are [Müller, 1966], [Seeley, 1966], [Erdélyi et al., 1953] and, for $d=3$ in more detail, [Sansone, 1959]; further references, with some geometrical applications, are e.g. [Blaschke, 1956], §23, Anhang, [Gardner, 1995], Appendix C, and also the survey paper [Groemer, 1993] as well as the books [Schneider, 1993], pp. 428–432 and [Groemer, 1996], which contain ample further bibliography.

A polynomial $f\colon\mathbb{R}^d\to\mathbb{R}$ is *harmonic*, if $\sum_{i=1}^{d}(\partial/\partial x_i)^2 f=0$. (This is invariant under the choice of an orthonormal base.) For an integer $n\geq 0$ a *spherical* *harmonic (of degree n)* in d dimensions is the restriction of a homogeneous harmonic polynomial $f\colon \mathbb{R}^d \to \mathbb{R}$ (of degree n) to S^{d-1}. (Since d will be fixed, later we will not refer to the dimension.) The spherical harmonics of degree n form a finite dimensional vector space. Choosing from this subspace an orthonormal base $\{Y_{ni} \mid 1 \le i \le N(d,n)\}$ (orthonormality meant in the space $L^2(S^{d-1})$, for the Lebesgue measure on S^{d-1}), their union for each $n \ge 0$ is a complete orthonormal system in $L^2(S^{d-1})$. Thus each $f \in L^2(S^{d-1})$ has a Fourier expansion $\sum_{n=0}^{\infty}\left(\sum_{i=1}^{N(d,n)} c_{ni}Y_{ni}\right)$. Here we will write $\sum_{i=1}^{N(d,n)} c_{ni}Y_{ni} = Y_n(f)$, thus the Fourier expansion of f is $\sum_{n=0}^{\infty} Y_n(f)$.

The spherical harmonics are the eigenfunctions of many linear operators commuting with rotations. For example, the Funk–Hecke theorem ([Seeley, 1966], Theorem 3) says the following. Let F be measurable on $[-1,1]$, with $\int_{-1}^{1} |F(t)|(1-t^2)^{(d-3)/2}dt < \infty$. Then any spherical harmonic Y_n of n–th degree is an eigenfunction of the integral operator $f \mapsto g = g(\xi) = \int_{S^{d-1}} F(\langle \xi, \eta \rangle) f(\eta) d\eta$, that is

$$\int_{S^{d-1}} F(\langle \xi, \eta \rangle) Y_n(\eta) d\eta = \lambda_n Y_n(\xi)\,,$$

where the eigenvalue λ_n equals

$$\lambda_n = V_{d-2}(S^{d-2}) C_n(1)^{-1} \int_{-1}^{1} F(t) C_n(t)(1-t^2)^{(d-3)/2} dt\,.$$

Here V_{d-2} means $(d-2)$–dimensional volume, and $C_n(t) = C_n^{(d-2)/2}(t)$ is the n'th Gegenbauer polynomial, of order $(d-2)/2$, that is a non–zero polynomial of degree n, satisfying for $0 \le n < m$ the orthogonality relations

$$\int_{-1}^{1} C_n(t) C_m(t)(1-t^2)^{(d-3)/2} dt = 0\,.$$

There holds $C_n(1) \ne 0$, [Seeley, 1966], (3). For n odd (even) C_n is an odd (even) function [Erdélyi et al., 1953], §10.9, (16). References to Gegenbauer polynomials are [Erdélyi et al., 1953] and [Tricomi, 1955].

For suitable measures or distributions on $[-1,1]$ a formula similar to the Funk–Hecke theorem holds, cf. for example Lemma 3.2.

3. The operators $R_\psi^{(m)}$ and R_ψ^P

Now we in essence generalize the definition of the operator $R_\psi^{(1)}$ from [Makai–Martini–Ódor]. (We observe that although $R_\psi^{(m)}$ for $m = 1$ in this paper is defined by the same formula as $R_\psi^{(1)}$ in [Makai–Martini–Ódor], however the domain of the operator $R_\psi^{(m)}$ for $m = 1$ in this paper is smaller than that of the operator $R_\psi^{(1)}$ from [Makai–Martini–Ódor]: namely it is $C^1(S^{d-1})$ rather than $Lip(S^{d-1})$.) Let $d \geq 2$, and let $m \geq 1$ be an integer. Let $-\pi/2 \leq \psi \leq \pi/2$, $f \in C^m(S^{d-1})$ and $\xi \in S^{d-1}$. Using polar coordinates with pole ξ (cf. (1) and (2)), we define the integro–differential transform $R_\psi^{(m)} f$ of f by

$$(R_\psi^{(m)} f)(\xi) = \int\limits_{S^{d-1}\cap\xi^\perp} \frac{\partial^m f}{\partial\psi^m}(\eta,\psi)d\eta\,. \tag{3}$$

Here $(\partial^m f/\partial\psi^m)(\eta,\psi)$ is the m–th angular derivative of f at (η,ψ) along the meridian passing through η. For $\psi = 0$ we drop the lower index.

The case $m = 1$ of the following lemma is essentially contained in Lemma 3.6 of [Makai–Martini–Ódor] (there $Lip(S^{d-1})$ and $L^\infty(S^{d-1})$ appear rather than $C^1(S^{d-1})$ and $C(S^{d-1})$).

Lemma 3.1. *Let $d \geq 2$, $m \geq 1$, $-\pi/2 \leq \psi \leq \pi/2$ and $f, g \in C^m(S^{d-1})$. Then we have $R_\psi^{(m)} f \in C(S^{d-1})$, and $R_\psi^{(m)}$ is symmetric, i.e., $\int\limits_{S^{d-1}} (R_\psi^{(m)} f)(\xi)g(\xi)d\xi = \int\limits_{S^{d-1}} f(\xi)(R_\psi^{(m)} g)(\xi)d\xi$.*

Proof. The relation $R_\psi^{(m)} f \in C(S^{d-1})$ is proved in a standard way.

Further, we have by Lebesgue's dominated convergence theorem

$$\int\limits_{S^{d-1}} (R_\psi^{(m)} f)(\xi)g(\xi)d\xi =$$

$$\lim_{\varepsilon\to 0} \int\limits_{S^{d-1}} \left[\int\limits_{S^{d-1}\cap\xi^\perp} \left(\frac{\partial^{m-1} f}{\partial\psi^{m-1}}(\eta,\psi+\varepsilon) - \frac{\partial^{m-1} f}{\partial\psi^{m-1}}(\eta,\psi) \right) \varepsilon^{-1} d\eta \right] g(\xi)d\xi\,.$$

Induction on m shows that this equals $\int\limits_{S^{d-1}} f(\xi)(R_\psi^{(m)} g)(\xi)d\xi$. The induction basis is $m = 1$, which follows from Lemma 3.6 of [Makai–Martini–Ódor]. □

Defining $R_\psi^{(0)}$ in the analogous way, i.e., by

$$(R_\psi^{(0)} f)(\xi) = \int\limits_{S^{d-1}\cap\xi^\perp} f(\eta,\psi)d\eta\,,$$

we have the following lemma, that generalizes Lemma 3.7 of [Makai–Martini–Ódor]. Its statement in the case $m = 0$ is due to [Radon, 1917] (for $d = 3$), and to [Schneider, 1969], formula (5) (an alternative proof cf. in [Makai–Martini–Ódor], Lemma 3.7), while in the case $m = 1$ it is due to [Makai–Martini–Ódor], Lemma 3.7.

Lemma 3.2. *Let $d \geq 2$, $m \geq 0$, $-\pi/2 \leq \psi \leq \pi/2$, and let $Y_n\colon S^{d-1} \to \mathbb{R}$ be a spherical harmonic of degree n. Then Y_n is an eigenfunction of $R_\psi^{(m)}$, i.e., $R_\psi^{(m)} Y_n = \lambda_n Y_n$, with*

$$\lambda_n = V_{d-2}(S^{d-2}) C_n^{(d-2)/2}(1)^{-1} \left(\frac{d}{d\psi}\right)^m C_n^{(d-2)/2}(\sin\psi) \ .$$

Proof. By the case $m = 0$ of our statement (referred to above) we have

$$\int_{S^{d-1}\cap\xi^\perp} Y_n(\eta,\psi)d\eta = V_{d-2}(S^{d-2}) C_n(1)^{-1} C_n(\sin\psi) \cdot Y_n(\xi)\,.$$

From this case subsequent differentiations with respect to ψ prove the statement for any $m \geq 1$. □

The following theorem is more or less a generalization of Theorem 3.8 of [Makai–Martini–Ódor] (there $Lip(S^{d-1})$ and $L^\infty(S^{d-1})$ appear rather than $C^1(S^{d-1})$ and $C(S^{d-1})$). The statement corresponding to the case $m = 0$, $\psi = 0$ in this theorem is the Funk integral theorem (for $f \in C(S^{d-1})$), cf. [Minkowski, 1905], [Funk, 1913], Kap. 2, [Bonnesen–Fenchel, 1934], pp. 136–138, [Lifshitz–Pogorelov, 1954], [Petty, 1961], [Schneider, 1969], [Helgason, 1980], Ch. 3, §1.B and [Helgason, 1984], the case $m = 0$, ψ arbitrary is essentially contained in [Schneider, 1969], while the case $m = 1$ is contained in [Makai–Martini–Ódor], Theorem 3.8.

Theorem 3.3. *Let $d \geq 2$, $m \geq 1$ and $-\pi/2 \leq \psi \leq \pi/2$. Then the null–space of the operator $R_\psi^{(m)}\colon C^m(S^{d-1}) \to C(S^{d-1})$ equals $\{f \in C^m(S^{d-1}) \mid$ the Fourier expansion $\sum_{n=0}^{\infty} Y_n(f)$ of f satisfies that $(d/d\psi)^m\, C_n^{(d-2)/2}(\sin\psi) \neq 0$ implies $Y_n(f) = 0\}$. In particular, for $\psi = 0$ and m odd (even) the null–space of $R^{(m)} = R_0^{(m)}$ contains $\{f \in C^m(S^{d-1}) \mid f$ is even (f is the sum of an odd function and a constant)$\}$.*

Proof. For the first statement we proceed analogously to [Alexandroff, 1937], [Petty, 1961], [Schneider, 1969], [Schneider, 1970], [Falconer, 1983]. Let $f \in C^m(S^{d-1})$. Then, by 3.1, we have $R_\psi^{(m)} f \in C(S^{d-1}) \subset L^2(S^{d-1})$. Moreover, by completeness of spherical harmonics, $R_\psi^{(m)} f = 0$ holds (a.e.) if and only if for each $n \geq 0$, and each spherical harmonic Y_n of degree n we have $0 = \langle R_\psi^{(m)} f, Y_n\rangle$, where $\langle\ ,\ \rangle$ now denotes scalar product in $L^2(S^{d-1})$. Letting $\sum_{n=0}^{\infty} Y_n(f)$ be the Fourier expansion of f, we have by 3.1 and 3.2 that

$$\langle R_\psi^{(m)} f, Y_n\rangle = \langle f, R_\psi^{(m)} Y_n\rangle = \lambda_n \langle f, Y_n\rangle =$$

$$V_{d-2}(S^{d-2}) C_n(1)^{-1} \left(\frac{d}{d\psi}\right)^m C_n(\sin\psi) \cdot \langle Y_n(f), Y_n\rangle.$$

For fixed n and Y_n arbitrary this is 0 if and only if $(d/d\psi)^m\, C_n(\sin\psi) \cdot Y_n(f) = 0$. This implies the first statement.

For the second statement observe that for f constant we have $R_\psi^{(m)} f = 0$. Furthermore, for $\psi = 0$, m odd (even) and f even (odd) by 3.1 we have for all $\xi \in S^{d-1}$ that for all $\eta \in S^{d-1} \cap \xi^\perp$ there holds $(\partial^m f / \partial \psi^m)(\eta, 0) + (\partial^m f / \partial \psi^m)(-\eta, 0) = 0$, thus $R^{(m)} f = 0$. $\square$

More generally, let $d \ge 2$, $m \ge 1$ an integer, and P a real polynomial of degree m. Further let $-\pi/2 \le \psi \le \pi/2$, $f \in C^m(S^{d-1})$ and $\xi \in S^{d-1}$. Using polar coordinates with pole ξ, we define the integro–differential transform $R_\psi^P f$ of f by

$$(R_\psi^P f)(\xi) = \int\limits_{S^{d-1} \cap \xi^\perp} \left(P \left(\frac{\partial}{\partial \psi} \right) \right) f(\eta, \psi) d\eta \,. \tag{4}$$

For $\psi = 0$ we drop the lower index. Then, analogously like above, we have

Theorem 3.4. *Let $d \ge 2$ be a fixed integer. Let $m \ge 1$ be an integer, P a real polynomial of degree m, and $-\pi/2 \le \psi \le \pi/2$. Furthermore, let $f, g \in C^m(S^{d-1})$. Then we have $R_\psi^P f \in C(S^{d-1})$, and R_ψ^P is symmetric, i.e., $\int\limits_{S^{d-1}} (R_\psi^P f)(\xi) g(\xi) d\xi = \int\limits_{S^{d-1}} f(\xi)(R_\psi^P g)(\xi) d\xi$. The null–space of the operator $R_\psi^P \colon C^m(S^{d-1}) \to C(S^{d-1})$ equals $\{f \in C^m(S^{d-1}) \mid$ the Fourier expansion $\sum\limits_{n=0}^{\infty} Y_n(f)$ of f satisfies that $(P(d/d\psi)) C_n^{(d-2)/2}(\sin \psi) \neq 0$ implies $Y_n(f) = 0\}$.*

Proof. Let $P(t) = \sum_{r=0}^m c_r t^r$. The statement $R_\psi^P f \in C(S^{d-1})$ and symmetry of R_ψ^P follow from 3.1. Then 3.2 implies its analogue for R_ψ^P, with eigenvalue

$$\lambda_n = V_{d-2}(S^{d-2}) C_n^{(d-2)/2}(1)^{-1} \left(P \left(\frac{d}{d\psi} \right) \right) C_n^{(d-2)/2}(\sin \psi) \,.$$

Then, like in 3.3, we obtain that we have $R_\psi^P f = 0$ if and only if

$$\left(P \left(\frac{d}{d\psi} \right) \right) C_n^{(d-2)/2}(\sin \psi) \neq 0$$

implies $Y_n(f) = 0$, where $\sum\limits_{n=0}^{\infty} Y_n(f)$ is the Fourier expansion of f. $\square$

Remark. By Lemma 3.1 and Theorem 3.4 $R_\psi^{(m)}$ and R_ψ^P are symmetric operators. Actually it is easily seen that their closures are self–adjoint operators in $L^2(S^{d-1})$, with domain $\{\sum Y_n \mid Y_n$ is a spherical harmonic of degree n, $\sum (1 + \lambda_n^2) \|Y_n\|^2 < \infty\}$, and are given by $\sum Y_n \mapsto \sum \lambda_n Y_n$, where the λ_n are the corresponding eigenvalues.

4. The null–spaces of the operators $R^{(m)}$ and R^P

Now, for $d \ge 2$, we turn to the case of general $m \ge 1$, and investigate the null–space of $R^{(m)} \colon C^m(S^{d-1}) \to C(S^{d-1})$.

For $d = 2$ the complete answer is given by

Proposition 4.1. *Let $d = 2$ and $m \geq 1$. Then the null–space of the operator $R^{(m)}: C^m$
$(S^{d-1}) \to C(S^{d-1})$ equals for m odd (even) $\{f \in C^m(S^{d-1}) \mid f$ is even (f is the sum of an odd function and a constant)$\}$.*

Proof. By 3.3, we have that $f \in C^m(S^{d-1})$ satisfies $R^{(m)} f = 0$ if and only if $(d/d\psi)^m C_n^0$
$(\sin\psi)|_{\psi=0} \neq 0$ implies $Y_n(f) = 0$. However, by [Müller, 1966], p. 33, we have for some $a_n \neq 0$ that $C_n^0(\sin\psi) = C_n^0[\cos(\pi/2 - \psi)] = a_n \cos[n(\pi/2 - \psi)]$. A straightforward calculation gives then that

$$\left(\frac{d}{d\psi}\right)^m \cos[n(\pi/2 - \psi)]\Big|_{\psi=0}$$

equals 0 if $n = 0$ or if $n - m$ is odd, and equals $(-1)^{\lfloor n/2 \rfloor}(-1)^{\lfloor m/2 \rfloor} n^m \neq 0$ if $n - m$ is even and $n \geq 1$. Then 3.3 implies our statement. □

Lemma 4.2. *Let $d \geq 3$ and $m \geq 1$. Then there exists a polynomial $\Gamma_{d,m,n}$ of n, of degree m, such that for $n - m$ even we have*

$$\left(\frac{d}{d\psi}\right)^m C_n^{(d-2)/2}(\sin\psi)\Big|_{\psi=0} = 0$$

if and only if $\Gamma_{d,m,n} = 0$. We have ($\Gamma_{d,0,n} = 1$,) $\Gamma_{d,1,n} = n + d - 3$, $\Gamma_{d,2,n} = n(n + d - 3)$ and $\Gamma_{d,3,n} = (n + d - 3)(n^2 + (d - 2)n - (d - 2))$.

Proof. By [Tricomi, 1955], p. 182, for $d \geq 3$ we have

$$C_n^{(d-2)/2}(\sin\psi) = C_n^{(d-2)/2}[\cos(\pi/2 - \psi)] =$$

$$(-1)^n \sum_{i=0}^{n} \binom{-d/2+1}{i} \binom{-d/2+1}{n-i} \cos[(n-2i)(\pi/2 - \psi)]\,.$$

Using

$$\left(\frac{d}{d\psi}\right)^m \cos\psi = \cos(\psi + m\pi/2)\,,$$

we have for $n - m$ even that

$$\gamma_{d,m,n} := \left(\frac{d}{d\psi}\right)^m C_n^{(d-2)/2}(\sin\psi)\Big|_{\psi=0} =$$

$$(-1)^n \sum_{i=0}^{n} \binom{-d/2+1}{i} \binom{-d/2+1}{n-i} [-(n-2i)]^m \cos[(n-2i)\pi/2 + m\pi/2)] =$$

$$(-1)^n (-1)^{(n+m)/2} \sum_{i=0}^{n} (-1)^i \binom{-d/2+1}{i} \binom{-d/2+1}{n-i} (2i - n)^m.$$

Here $(2i - n)^m = \sum_{p=0}^{m} (-1)^p \binom{m}{p} 2^{m-p} n^p i^{m-p}$, and

$$i^{m-p} = \sum_{q=0}^{m-p} c_{m-p,q} \binom{i}{q} q!\,,$$

for some constants $c_{m-p,q}$, where $c_{m-p,m-p} = 1$. Hence

$$(2i-n)^m = \sum_{q=0}^{m} f_{m,q}(n) \binom{i}{q} q!\,,$$

where

$$f_{m,q}(n) = \sum_{p=0}^{m-q} (-1)^p \binom{m}{p} 2^{m-p} c_{m-p,q} n^p.$$

Therefore

$$\gamma_{d,m,n} = (-1)^n (-1)^{(n+m)/2} \sum_{q=0}^{m} f_{m,q}(n) \delta_{d,n,q}\,,$$

where

$$\delta_{d,n,q} = \sum_{i=0}^{n} (-1)^i \binom{-d/2+1}{i} \binom{-d/2+1}{n-i} \binom{i}{q} q!\,.$$

Here $\delta_{d,n,q}$ is the coefficient of x^n in the power series expansion of

$$x^q \left[\left(\frac{d}{dx} \right)^q (1-x)^{(2-d)/2} \right] (1+x)^{(2-d)/2} =$$

$$(-1)^q \left(-\frac{d}{2}+1 \right) \left(-\frac{d}{2} \right) \dots \left(-\frac{d}{2} - q + 2 \right) x^q (1-x^2)^{-d/2-q+1} (1+x)^q =$$

$$(-1)^q \left(-\frac{d}{2}+1 \right) \left(-\frac{d}{2} \right) \dots \left(-\frac{d}{2} - q + 2 \right) x^q \sum_{k=0}^{\infty} (-1)^k \binom{-d/2-q+1}{k} x^{2k} \sum_{l=0}^{q} \binom{q}{l} x^l,$$

i.e.,

$$\delta_{d,n,q} = (-1)^q \left(-\frac{d}{2}+1 \right) \left(-\frac{d}{2} \right) \dots \left(-\frac{d}{2} - q + 2 \right) \times$$

$$\sum_{k \ge 0,\, 0 \le l \le q,\, 2k+l=n-q} (-1)^k \binom{-d/2-q+1}{k} \binom{q}{l}.$$

Here in the last summation we have $2k = n-q-l \in [n-2q, n-q]$, thus $\lceil n/2 \rceil - q \le k \le \lfloor (n-q)/2 \rfloor \le \lfloor n/2 \rfloor$. Therefore we introduce the notation $k = \lfloor n/2 \rfloor - j$; then $0 \le \lfloor n/2 \rfloor - \lfloor (n-q)/2 \rfloor = q/2 + \delta \le j \le q - \varepsilon$. Here $\delta = 0$ for q even, $\delta = 1/2$ or $-1/2$ for q odd and n even or odd, respectively, and $\varepsilon = 0$ for n even, $\varepsilon = 1$ for n odd. Furthermore, $l = n - 2k - q = \varepsilon + 2j - q$.

We have

$$\delta_{d,n,q} \binom{-d/2+1}{\lfloor n/2 \rfloor}^{-1} = \sum_{q/2+\delta \le j \le q-\varepsilon,\ j \le \lfloor n/2 \rfloor} \delta_{d,n,q,j}\,,$$

where

$$\delta_{d,n,q,j} = (-1)^q \left(-\frac{d}{2}+1 \right) \left(-\frac{d}{2} \right) \dots \left(-\frac{d}{2} - q + 2 \right) \times$$

$$(-1)^{\lfloor n/2 \rfloor - j} \binom{-d/2-q+1}{\lfloor n/2 \rfloor - j} \binom{-d/2+1}{\lfloor n/2 \rfloor}^{-1} \binom{q}{\varepsilon + 2j - q} =$$

$$(-1)^{\lfloor n/2 \rfloor}(-1)^{q-j}\left(-\frac{d}{2}+1\right)\left(-\frac{d}{2}\right)\ldots\left(-\frac{d}{2}-q+2\right)\times$$
$$\left(-\frac{d}{2}-q+1\right)\ldots\left(-\frac{d}{2}-q-\left\lfloor\frac{n}{2}\right\rfloor+j+2\right)\times$$
$$\left(\left\lfloor\frac{n}{2}\right\rfloor-j\right)!^{-1}\left(-\frac{d}{2}+1\right)^{-1}\ldots\left(-\frac{d}{2}-\left\lfloor\frac{n}{2}\right\rfloor+2\right)^{-1}\left\lfloor\frac{n}{2}\right\rfloor!\binom{q}{\varepsilon+2j-q}=$$
$$(-1)^{\lfloor n/2 \rfloor}(-1)^{q-j}\left(-\frac{d}{2}-\left\lfloor\frac{n}{2}\right\rfloor+1\right)\ldots\left(-\frac{d}{2}-\left\lfloor\frac{n}{2}\right\rfloor-q+j+2\right)\times$$
$$\left\lfloor\frac{n}{2}\right\rfloor\ldots\left(\left\lfloor\frac{n}{2}\right\rfloor-j+1\right)\binom{q}{\varepsilon+2j-q}=(-1)^{\lfloor n/2 \rfloor}g_{d,q,j}(n)\,,$$

where $g_{d,q,j}(n)$ equals, for n having the parity of m, a polynomial of n, of degree $(q-j)+j=q$, and of leading coefficient $2^{-q}\binom{q}{\varepsilon+2j-q}$. However, for $0\le\lfloor n/2\rfloor<j$ we have $g_{d,q,j}(n)=0$, thus

$$\delta_{d,n,q}\binom{-d/2+1}{\lfloor n/2\rfloor}^{-1}=(-1)^{\lfloor n/2\rfloor}\sum_{j=q/2+\delta}^{q-\varepsilon}g_{d,q,j}(n)\,.$$

Collecting these results, we have

$$(-1)^n(-1)^{(n+m)/2}(-1)^{\lfloor n/2\rfloor}\gamma_{d,m,n}\binom{-d/2+1}{\lfloor n/2\rfloor}^{-1}=$$
$$(-1)^{\lfloor n/2\rfloor}\sum_{q=0}^{m}f_{m,q}(n)\delta_{d,n,q}\binom{-d/2+1}{\lfloor n/2\rfloor}^{-1}=$$
$$\sum_{q=0}^{m}f_{m,q}(n)\sum_{j=q/2+\delta}^{q-\varepsilon}g_{d,q,j}(n):=\mathit{\Gamma}_{d,m,n}\,.$$

Here each $f_{m,q}(n)$ or $g_{d,q,j}(n)$ equals, for n having the parity of m, a polynomial of n, of degree $m-q$ or q, respectively. Moreover, δ and ε in the bounds of the inner summation only depend on the parities of n and q. Therefore $\mathit{\Gamma}_{d,m,n}$ equals, for n having the parity of m, a polynomial of n, of degree $\le m$. Furthermore, we have $\gamma_{d,m,n}=0$ if and only if $\mathit{\Gamma}_{d,m,n}=0$.

Now we show that $\mathit{\Gamma}_{d,m,n}$ equals, for n having the parity of m, a polynomial of n, of degree exactly m. More exactly, we show that its leading coefficient is 1. We have

$$\mathit{\Gamma}_{d,m,n}=\sum_{q=0}^{m}f_{m,q}(n)\sum_{j=q/2+\delta}^{q-\varepsilon}g_{d,q,j}(n)\,.$$

Here, by $c_{q,q}=1$, the leading term of $f_{m,q}(n)$ is $(-1)^{m-q}\binom{m}{q}2^q n^{m-q}$. Similarly, the leading term of $g_{d,q,j}(n)$ is $2^{-q}\binom{q}{\varepsilon+2j-q}n^q$. Hence the coefficient of n^m in $\mathit{\Gamma}_{d,m,n}$ is

$$\sum_{q=0}^{m}(-1)^{m-q}\binom{m}{q}\sum_{j=q/2+\delta}^{q-\varepsilon}\binom{q}{\varepsilon+2j-q}\,.$$

A small discussion, taking into consideration the parities of m and q, shows that this equals

$$\sum_{q=0}^{m}(-1)^{m-q}\binom{m}{q}\sum_{0\le\varepsilon+2j-q\le q}\binom{q}{\varepsilon+2j-q}=$$

$$\sum_{q=1}^{m}(-1)^{m-q}\binom{m}{q}2^{q-1}+\frac{1}{2}+(-1)^m\frac{1}{2}=$$

$$(2-1)^m\frac{1}{2}-(-1)^m\frac{1}{2}+\frac{1}{2}+(-1)^m\frac{1}{2}=1.$$

The above considerations show that, for any given m, we can evaluate $\gamma_{d,m,n}$ for all n. Performing the above calculations for $m\le 3$, we obtain the formulas for $\Gamma_{d,m,n}$, given in the lemma. □

Using the above lemma we prove the following theorem, that is more or less a generalization of the last statement of Theorem 3.8 of [Makai–Martini–Ódor].

Theorem 4.3. *Let $d\ge 3$ be a fixed integer. Then, for any integer $m\ge 1$, there exists a set $A_{(m)}$ of non–negative integers of the same parity as m, with cardinality $|A_{(m)}|\le m$, such that the following holds. The null–space of the operator $R^{(m)}\colon C^m(S^{d-1})\to C(S^{d-1})$ equals $\{f\in C^m(S^{d-1})\mid f$ is of the form $f=g+\sum_{n\in A_{(m)}}Y_n$, where g is even (odd) for m odd (even), and Y_n is a spherical harmonic of degree $n\}$. In particular, for $m=2,\,3$ we have $A_{(2)}=\{0\}$, $A_{(3)}=\emptyset$.*

Proof. By 3.3, we have that $f\in C^m(S^{d-1})$ satisfies $R^{(m)}f=0$ if and only if, for the Fourier expansion $\sum_{n=0}^{\infty}Y_n(f)$ of f, we have that $\gamma_{d,m,n}:=\left.(d/d\psi)^m\,C_n^{(d-2)/2}(\sin\psi)\right|_{\psi=0}\ne 0$ implies $Y_n(f)=0$. Since C_n is odd (even) for n odd (even), therefore for $n-m$ odd we have $\gamma_{d,m,n}=0$. So we only need to consider the case $n-m$ even. Then by 4.2 we have $\gamma_{d,m,n}=0$ if and only if $\Gamma_{d,m,n}=0$, and $\Gamma_{d,m,n}$ equals, for n having the parity of m, a polynomial of n, of degree m.

We let $A_{(m)}=\{n\mid n\ge 0$ is an integer, $n-m$ is even, $\Gamma_{d,m,n}=0\}$. Then $\{f\in C^m\ (S^{d-1})\mid R^{(m)}f=0\}=\big\{f\in C^m(S^{d-1})\mid f=\sum\{Y_n(f)\mid n\ge 0$ is an integer, and either $2\nmid(n-m)$, or $(2\mid(n-m)$ and $n\in A_{(m)})\}\big\}$.

For the cases $m=2,\,3$, we consider the equation $\Gamma_{d,m,n}=0$ from 4.2. For $m=2$ its only non–negative even root is $n=0$. Now let $m=3$ and $n\ge 0$ odd. Then we have $n+d-3\ge 1$. Furthermore, the discriminant of $n^2+(d-2)n-(d-2)$ is d^2-4, that is not a perfect square for $d\ge 3$, thus the roots of this polynomial are irrational. □

Theorem 4.4. *Let $d\ge 2$ be a fixed integer. Let $m\ge 1$ be an integer, P a polynomial of degree m, and $\psi=0$. If P is odd (even) for m odd (even), then there exists a set A_P of non–negative integers of the same parity as m, with cardinality $|A_P|\le m$, such that the following holds. The null–space of the operator $R^P=R_0^P$ equals $\{f\in C^m(S^{d-1})\mid f$ is of the form $f=g+\sum_{n\in A_P}Y_n$, where g is even (odd) for m odd (even), and Y_n is a spherical harmonic of degree $n\}$.*

Proof. Suppose that $P=P(t)=\sum_{r=0}^{m}c_rt^r$ is such as given in the theorem. Then we have $(P\,(d/d\psi))\,C_n^{(d-2)/2}\,(\sin\psi)|_{\psi=0}=0$ for $n-m$ odd. Now let us suppose that $n-m$ is even. First we suppose that $d \geq 3$. We have, like at 4.3 and 4.2,

$$\left(P\left(\frac{d}{d\psi}\right)\right) C_n^{(d-2)/2}(\sin\psi)\bigg|_{\psi=0} = \sum_{r=0}^{m} c_r \gamma_{d,r,n} =$$

$$(-1)^n(-1)^{\lfloor n/2 \rfloor}\binom{-d/2+1}{\lfloor n/2 \rfloor}\sum_{r=0}^{m} c_r(-1)^{(n+r)/2}\Gamma_{d,r,n}$$

(observe that $n \equiv r \pmod 2$). This expression equals 0 if and only if the last sum equals 0. Furthermore, this last sum equals an m–th degree polynomial of n, if $n-m$ is even. Now let A_P be the set of those non–negative integer roots n of this polynomial, for which $n-m$ is even. Then, like in 4.3, the statement of the theorem holds for this set A_P.

Now let $d=2$. Then we have from the proof of 4.1, with the same $a_n \neq 0$,

$$\left(P\left(\frac{d}{d\psi}\right)\right) C_n^{0}(\sin\psi)\bigg|_{\psi=0} = a_n(-1)^{\lfloor n/2 \rfloor}\sum_{r=0}^{m} c_r(-1)^{\lfloor r/2 \rfloor} n^r.$$

Then we define A_P using this last sum, like above, and again the statement of the theorem holds for this set A_P. □

Remark. Possibly the space $C^m(S^{d-1})$ can be replaced by a suitable Sobolev class or by a suitable space of distributions, still yielding similar results.

Acknowledgment. The authors express their gratitude to R. J. Gardner for his several valuable remarks concerning this paper.

References

[Adams, 1975] R. A. Adams, *Sobolev Spaces*, Academic Press Inc., New York–London, 9247.

[Alexandroff, 1937] A. D. Alexandroff, *Zur Theorie der gemischten Volumina von konvexen Körpern II., Neue Ungleichungen zwischen den gemischten Volumina und ihre Anwendungen*, Mat. Sb. **2** (1937), 1205–1238 (Russian with German summary), Zbl. **18**,276.

[Blaschke, 1956] W. Blaschke, *Kreis und Kugel*, 2-e durchges. verbess. Aufl., de Gruyter, Berlin, 1123.

[Bonnesen–Fenchel, 1934] T. Bonnesen–W. Fenchel, *Theorie der konvexen Körper*, Springer, Berlin, 1934, Zbl. **8**,77.

[Erdélyi et al., 1953] A. Erdélyi–W. Magnus–F. Oberhettinger–F. G. Tricomi, *Higher transcendental functions II.*, McGraw–Hill, New York–Toronto–London, 419.

[Falconer, 1983] K. J. Falconer, *Applications of a result on spherical integration to the theory of convex sets*, Amer. Math. Monthly **90** (1983), 52012.

[Funk, 1913] P. Funk, *Über Flächen mit lauter geschlossenen geodätischen Linien*, Math. Ann. **74** (1913), 278–300, Jahrbuch Fortschr. Math. **44**,692.

[Gardner, 1995] R. J. Gardner, *Geometric Tomography*, Encyclopedia of Math. and its Appls. **58**, Cambridge Univ. Press, Cambridge, 52006.

[Groemer, 1993] H. Groemer, *Fourier series and spherical harmonics in convexity*, in: Handbook of Convex Geometry (eds. P. M. Gruber, J. M. Wills), North Holland, Amsterdam, 1993, 52001.

[Groemer, 1996] H. Groemer, *Geometric Applications of Fourier Series and Spherical Harmonics*, Encyclopedia of Math. and its Appls. **61**, Cambridge Univ. Press, Cambridge, 52001.

[Hammer, 1954] P. C. Hammer, *Diameters of convex bodies*, Proc. Amer. Math. Soc. **5** (1954), 819.

[Helgason, 1980] S. Helgason, *The Radon Transform*, Birkhäuser, Boston, Mass., 43012.

[Helgason, 1984] S. Helgason, *Groups and Geometric Analysis. Integral Geometry, Invariant Differential Operators, and Spherical Functions*, Academic Press, Inc., Orlando, Fla., 22017.

[Lifshitz–Pogorelov, 1954] I. M. Lifshitz–A. V. Pogorelov, *On the determination of Fermi surfaces and electron velocities in metals by the oscillation of magnetic susceptibility (in Russian)*, Dokl. Akad. Nauk SSSR **96** (1954), 1143–1145, Zbl. **57**.448.

[Makai–Martini–Ódor] E. Makai, Jr.–H. Martini–T. Ódor, *Maximal sections and centrally symmetric bodies*, Mathematika, accepted for publication.

[Minkowski, 1905] H. Minkowski, *Über die Körper konstanter Breite*, Mat. Sb. **25** (1905), 505–508 (in Russian), Ges. Abh. 2, 277–279, Jahrbuch Fortschr. Math. **36**,526.

[Müller, 1966] C. Müller, *Spherical Harmonics*, Lecture Notes in Math. 17, Springer, Berlin, 7593.

[Petty, 1961] C. M. Petty, *Centroid surfaces*, Pacific J. Math. **11** (1961), A3558.

[Radon, 1917] J. Radon, *Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten*, Ber. Verh. Sächs. Akad. Wiss. Leipzig, Math. Nat. Kl. **69** (1917), 262–277, Jahrbuch Fortschr. Math. **46**,436.

[Sansone, 1959] G. Sansone, *Orthogonal Functions*, Rev. ed., Interscience, London–New York, 2140.

[Schneider, 1969] R. Schneider, *Functions on a sphere with vanishing integrals over certain subspheres*, J. of Math. Anal. and Appl. **26** (1969), 6004.

[Schneider, 1970] R. Schneider, *Über eine Integralgleichung in der Theorie der konvexen Körper*, Math. Nachr. **44** (1970), 1043.

[Schneider, 1993] R. Schneider, *Convex Bodies: the Brunn–Minkowski Theory*, Encyclopedia of Math. and its Appls. **44**, Cambridge Univ. Press, Cambridge, 52007.

[Seeley, 1966] R. T. Seeley, *Spherical harmonics*, Amer. Math. Monthly **73** (1966), 1577.

[Tricomi, 1955] F. G. Tricomi, *Vorlesungen über Orthogonalreihen*, Grundlehren Math. Wiss. **76**, Springer– Verlag, Berlin–Göttingen–Heidelberg, 30.

[Tricomi, 1957] F. G. Tricomi, *Integral Equations*, Interscience, New York–London, 1177.

[Whitney, 1957] H. Whitney, *Geometric Integration Theory*, Princeton Univ. Press, Princeton, N.J., 309.

[Ziemer, 1989] W. P. Ziemer, *Weakly Differentiable Functions. Sobolev Spaces and Functions of Bounded Variation*, Springer–Verlag, New York, 46046.

*, **: Alfréd Rényi Mathematical Institute, Hungarian Academy of Sciences, P.O.B. 127, H-1364 Budapest, Hungary;
Technische Universität Chemnitz, Fakultät für Mathematik, D-09107 Chemnitz, Germany

E-mail address: makai@renyi.hu, martini@mathematik.tu-chemnitz.de, odor@renyi.hu