ON THE STRUCTURE OF THE SET OF ELEMENTS IN A BANACH ALGEBRA WHICH SATISFY A GIVEN POLYNOMIAL EQUATION, AND THEIR LIFTINGS (PRELIMINARY VERSION)

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ABSTRACT. We generalize earlier results about connected components of idempotents in Banach algebras, due to B. Szőkefalvi Nagy, Y. Kato, S. Maeda, Z. V. Kovarik, J. Zemánek, J. Esterle. Let A be a unital complex Banach algebra, and $p(\lambda) = \prod\limits_{i=1}^{n} (\lambda - \lambda_i)$ a polynomial over $\mathbb{C}$, with all roots distinct. Let $E_p(A) := \{a \in A \mid p(a) = 0\}$. Then all connected components of $E_p(A)$ are pathwise connected (locally pathwise connected) via each of the following three types of paths: 1) similarity via a finite product of exponential functions (via an exponential function); 2) a polynomial path (a cubic polynomial path); 3) a polygonal path (a polygonal path consisting of n segments). If A is a C^*-algebra, $\lambda_i \in \mathbb{R}$, let $S_p(A) := \{a \in A \mid a = a^*, p(a) = 0\}$. Then all connected components of $S_p(A)$ are pathwise connected (locally pathwise connected), via a path of the form $e^{-ic_m t} \ldots e^{-ic_1 t} a e^{ic_1 t} \ldots e^{ic_m t}$, where $c_i = c_i^*$, and $t \in [0, 1]$ (of the form $e^{-ict} a e^{ict}$, where $c = c^*$, and $t \in [0, 1]$). For (self-adjoint) idempotents we have by these old papers that the distance of different connected components of them is at least 1. For $E_p(A)$, $S_p(A)$ we pose the problem if the distance of different connected components is at least $\min\{|\lambda_i - \lambda_j| \mid 1 \leq i, j \leq n, i \neq j\}$. For the case of $S_p(A)$, we give a positive lower bound for these distances, that depends on $\lambda_1, \ldots, \lambda_n$. We show that several local and global lifting theorems for analytic families of idempotents, along analytic families of surjective Banach algebra homomorphisms, from our recent paper with B. Aupetit and M. Mbekhta, have analogues for elements of $E_p(A)$ and $S_p(A)$.

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1. INTRODUCTION

In this paper all Banach spaces and Banach algebras are over $\mathbb{C}$, and all Banach algebras have units whose norms are 1, and all Banach algebra homomorphisms preserve the units. For X a Banach space, $B(X)$ denotes the Banach algebra of all bounded linear operators $X \to X$. For a Banach algebra A,

$$E(A) := \left\{ a \in A \mid a^2 = a \right\},$$

and for a Banach algebra A with an involution,

$$S(A) := \left\{ a \in A \mid a = a^2 = a^* \right\}.$$

We begin with a terminology. An *arc* in a Hausdorff space X is a homeomorphic image of $[0, 1]$ in X. A *path* in a Hausdorff space X is a continuous image of $[0, 1]$. Since we only consider metric, thus Hausdorff spaces, (locally) arcwise connectedness is equivalent to (locally) pathwise connectedness. Namely, Hausdorff continuous images of $[0, 1]$, called *Peano continua*, are arcwise connected, cf. Ku], ... Further in this paper we will deal with (locally) pathwise connectedness, and will look for "nice" connecting paths.

The subject began with an observation of B. Szőkefalvi Nagy ([SzN42], Ch. ... § 3, Hilfssatz, p. 58, [SzN47], Ch. ... , § 1, 3, p. 350 and [RSzN] § 105, Théorème, p. 266) that two orthogonal (i.e., self-adjoint) projections on a Hilbert space H, of distance less than 1, are similar via a unitary. Later C. Davis [Dav], ... , Y. Kato [YKa75], Theorem, p. 257, [YKa76], Theorem 2, p. 367 gave simpler proofs for this theorem.

Y. Kato [YKa76], Ch. I, § 4, 6, p. 33 proved the analogous statement for X a Banach space, for two projections, i.e., idempotent operators in $B(X)$, of distance less than 1, with similarity via an invertible operator. [YKa76], Ch. I, § 4, 6, Problem 4.13, p. 34 showed that under the same hypothesis there is an analytic path connecting the two idempotent operators.

S. Maeda [Ma] investigated the set $S(A)$ of self-adjoint projections in C^*-algebras A. He showed in his Theorem 2 and its Corollary that its connected components are arcwise connected and locally arcwise connected. Thus are relatively open among all self-adjoint projections. He showed in his Lemma 2 that for $e, f \in S(A)$, with $\|f - e\| < 1$, e and f are similar via a self-adjoint involution. In his corollary to Theorem 2 he showed that two self-adjoint idempotents belong to the same connected component if and only if they are similar via a finite product of self-adjoint involutions. He showed in his Theorem 1 that if $e, f \in S(A)$, are similar via a finite product of self-adjoint involutions, then they can be connected by a self-adjoint projection valued path.

Later Z. V. Kovarik [Ko], § 6, Theorem 1 proved, for X a Banach space, and $E_0, E_1 \in B(X)$ projections at distance less than 1 that

1) E_0, E_1 can be connected by a projection-valued analytic path of the form $e^{-itw} E_0 e^{itw}$, $t \in [0, 1]$ and

2) E_0, E_1 can be also connected by a polygonal path consisting of two segments and

3) E_0, E_1 are similar via an involution.

A consequence of 3) is [Ko], § 8, Theorem 2: If two projections are connected by a continuous projection valued path, then they are similar via a finite product of involutions.

J. Zemánek [Ze] investigated the idempotents in Banach algebras. He obtained that

1) $e, f \in E(A)$, $\|f - e\| < 1$ implies similarity of e and f via an exponential, i.e., $f = e^{-c}ee^{c}$, cf. his Lemma 3.1;

2) local arcwise connectedness of $E(A)$ and arcwise connectedness of each connected component of $E(A)$, cf. his Theorem 3.2;

3) $e, f \in E(A)$ are in the same connected component of $E(A)$ if and only if f is of the form
$$e^{-c_m} \ldots e^{-c_1} e e^{c_1} \ldots e^{c_m};$$

4) an idempotent e lies in the centre of A if and only if $\{e\}$ is isolated in $E(A)$ (i.e., is a connected component of $E(A)$); if an idempotent e does not lie in the centre of A, then the connected component of $E(A)$ containing e contains a (complex) line, hence is unbounded.

J. Esterle [Es], Theorem and its proof proved refinements of J. Zemánek's results. He obtained that

1) $e, f \in E(A)$, $\|f - e\| < 1$ implies similarity of e and f or the form

$$f = e^{-c''} e^{-c'} e e^{c'} e^{c''} = (1 - c'')(1 - c')ee(1 + c')(1 + c''), \text{ where } (c')^2 = (c'')^2 = 0,$$

where even one of the factors containing c' can be omitted, leaving this formula valid;

2) pathwise connectedness of each connected component of $E(A)$, by polynomial paths (and at the same time similarities via a finite product of exponential functions) of the form

$$e^{-c''_n t} e^{-c'_n t} \ldots e^{-c''_1 t} e^{-c'_1 t} e e^{c'_1 t} e^{c''_1 t} \ldots e^{c'_n t} e^{c''_n t}$$
$$= (1 - c''_n t)(1 - c'_n t) \ldots (1 - c''_1 t)(1 - c'_1 t) e(1 + c'_1 t)(1 + c''_1 t) \ldots (1 + c'_n t)(1 + c''_n t),$$

where $(c'_i)^2 = (c''_i)^2 = 0$, for $1 \le i \le n$, where one of the factors containing c'_1 can be omitted leaving this formula valid. Here, for $\|f - e\| < 1$ we have $n = 1$, thus we have a connection via a cubic polynomial path.

For some further results about $E(A)$ cf. the references in this paper.

Now we turn to the other subject of our paper. For unital complex Banach algebras A we write
$$E(A) := \{a \in A \mid a^2 = a\}.$$

If A is a unital complex Banach algebra with continuous involution *, then we write

$$S(A) := \{a \in A \mid a^2 = a = a^*\}.$$

Let $\pi : B \to A$ be a unit preserving homomorphism between two unital complex Banach algebras B and A. If A and B are Banach algebras with continuous involutions *, then we additionally suppose that π is involution-preserving. Then clearly
$$\pi E(B) \subset E(A),$$

and for unital complex Banach algebras with continuous involutions

$$\pi S(B) \subset S(A).$$

From now on suppose that π is surjective. We say that the *lifting property holds for* $\pi : B \to A$, if

$$\pi E(B) = E(A), \quad \text{or} \quad \pi S(B) = S(A), \quad \text{respectively.}$$

We write for Banach spaces X, Y $\mathcal{B}(X, Y)$ for the Banach space of all bounded linear operators $X \to Y$. We write $\mathcal{B}(X) := \mathcal{B}(X, X)$, and we write $\mathcal{K}(X)$ for the Banach space of compact linear operators in $\mathcal{B}(X)$.

For H a Hilbert space, and $\pi : \mathcal{B}(H) \to \mathcal{B}(H)/\mathcal{K}(H)$ the canonical mapping, we have $\pi S(\mathcal{B}(H)) = S(\mathcal{B}(H)/\mathcal{K}(H))$, cf. [Ca], Theorem 2.4 and [dlH], Proposition 7. For any Banach algebra A and π the canonical mapping $A \to A/\mathrm{rad}\, A$ ($\mathrm{rad}(\cdot)$ being the radical) we have $\pi E(A) = E(A/\mathrm{rad}\, A)$, cf. [Ri], Theorem 2.3.9 and [IKa], p. 125. An analogue of the above mentioned first result is $\pi E(\mathcal{B}(H)) = E(\mathcal{B}(H)/\mathcal{K}(H))$, which is due to [La]; we are grateful to Prof. J.-Ph. Labrousse for personally explaining the difficult passages of his paper. In fact [La] proved more. Suppose $U \subset \mathbb{C}$ with $0 \in U$ is open, and let us have an analytic map $q : U \to E(\mathcal{B}(H)/\mathcal{K}(H))$. Then there exist $V \subset \mathbb{C}$ open, such that $0 \in V \subset U$, and an analytic map $p : V \to E(\mathcal{B}(H))$, such that $\pi(p(\lambda)) = q(\lambda)$ for each $\lambda \in V$. This is called a *local lifting theorem*. If we can choose $V = U$, then we speak about a *global lifting theorem*.

In [AMMZ03] there are proved several further local and global lifting theorems, under hypotheses that the spectra of all elements of $\mathrm{Ker}\, \pi$ are "small". Observe that the spectra of compact operators on a Banach space X are either finite or are of the form $\{0\} \cup \{\lambda_n \mid n \in \mathbb{N}\}$, where $\lambda_n \to 0$, thus they are "small". So no compactness of operators is necessary, but only "small spectra".

All these results of [AMMZ03] were strengthened in [AMMZ14]. There not only the idempotents were analytic functions of $\lambda \in U$ (for $u \subset \mathbb{C}$ open), but also the surjective unit-preserving Banach algebra homomorphisms, written as $\pi(\lambda)$. Under the strongest spectral hypothesis that the spectra of all elements of all kernels $\mathrm{Ker}\, \pi(\lambda)$ are $\{0\}$, we proved global lifting theorems, both for $E(\cdot)$ and for $S(\cdot)$. For the case of $S(\cdot)$, both the idempotents $q(\lambda)$ and the *-homomorphisms $\pi(\lambda)$ were real analytic maps from some open subset G of $\mathbb{R}$ with $0 \in G$. Actually in [AMMZ14] not only a single analytic family of idempotents could be lifted, but even a mutually orthogonal sequence of analytic families of idempotents could be lifted to a mutually orthogonal sequence of analytic families of idempotents. Here two idempotents e, f in some Banach algebra are *orthogonal*, if $ef = fe = 0$.

Under weaker spectral hypotheses, [AMMZ14] proved local lifting theorems, both for $E(\cdot)$ and $S(\cdot)$. For the unital complex Banach algebra case it was sufficient to suppose that the spectra of all elements of $\mathrm{Ker}\, \pi(0)$ did not disconnect $\mathbb{C}$. For the case of unital complex Banach algebras with continuous involutions (with real analycity of $q(\cdot)$ and the *-homomorphisms $\pi(\cdot)$ like above) it was sufficient to suppose that the spectra of all elements of $\mathrm{Ker}\, \pi(0)$ was totally disconnected (i.e., they did not contain any connected subsets consisting of more than one points; this property implies that they did not disconnect $\mathbb{C}$). Observe that in both of these cases we had no hypotheses for the spectra of elements of $\mathrm{Ker}\, \pi(\lambda)$, for $\lambda \neq 0$.

A large part of the results of this paper have been announced in [MZ].

2. THEOREMS

Let F be a commutative field, and A a unital F-algebra. In general we will write 0 or 1 both for the zero or unit in F and A, but it will be clear which is meant;

but sometimes we will write 0_F and 0_A, and 1_F and 1_A. For $n \geq 2$ an integer, and $\lambda_1, \ldots, \lambda_n \in F$ distinct we write

$$p(\lambda) := \prod_{i=1}^{n} (\lambda - \lambda_i),$$

that is a polynomial over F, and

$$E(A) := \{a \in A \mid a = a^2\} \text{ and}$$
$$E_p(A) := \{a \in A \mid p(a) = 0_A\}.$$

We consider n and $\lambda_1, \ldots, \lambda_n$ as fixed, in the whole paper. A particular case is when A is the algebra of all bounded linear operators on a complex vector space X (with $F = \mathbb{C}$).

We say that

$$\{e_1, \ldots, e_n\} \subset E(A) \text{ forms a } partition \text{ of } unity.$$

If $e_i e_j = 0$ for $1 \leq i, j \leq n$, $i \neq j$, and $\sum_{i=1}^{n} e_i = 1$. In other words, we have an $E(A)$-valued measure on F, concentrated on $\{\lambda_1, \ldots, \lambda_n\} \subset F$, with total mass 1_A, with the measure of $\{\lambda_i\}$ being e_i.

Proposition 1. *With the above notations, we have for $a \in A$ that*

$$a \in E_p(A)$$

if and only if a is of the form

$$a = \sum_{i=1}^{n} \lambda_i e_i, \text{ where } \{e_1, \ldots, e_n\} \subset E(A) \text{ is a partition of unity.}$$

Here, for each $1 \leq i \leq n$, the function $a \mapsto e_i = e_i(a)$ is a polynomial, with coefficients in F. These polynomials only depend on $\lambda_1, \ldots, \lambda_n$.

If $a = \sum_{i=1}^{n} \lambda_i e_i = \sum_{j=1}^{m} \mu_j f_j$, for $\{e_1, \ldots, e_n\}, \{f_1, \ldots, f_m\} \subset E(A)$ partitions of unity, with distinct λ_i's and distinct μ_j's, and with $e_i \neq 0$, $f_j \neq 0$ for $1 \leq i \leq n$ and $1 \leq j \leq m$, then $m = n$, and, after a permutation of the indices, $\lambda_i = \mu_i$ and $e_i = f_i$, for $1 \leq i \leq n$. In particular, for $\{e_1, \ldots, e_n\}$, $\{f_1, \ldots, f_n\} \subset E(A)$ partitions of unity, with distinct λ_i's, $a = \sum_{i=1}^{n} \lambda_i e_i = \sum_{i=1}^{n} \lambda_i f_i$ implies $e_i = f_i$ for $1 \leq i \leq n$.

Corollary 2. *Let $A = B(X)$, where X is a complex Banach space. Then for $T \in B(X)$ we have $T \in E_p(A)$ if and only if there exists a direct sum decomposition $X_1 \oplus \ldots \oplus X_n$ of X, where each X_i is a closed subspace of X, such that*

$$T \mid X_i = \lambda_i \cdot id_{X_i}, \text{ for each } 1 \leq i \leq n.$$

■

Remark A. Now we show by an example that our considerations do not work if some λ_i's are equal, even in the "simplest" case of $A = B(H)$, the algebra of bounded linear operators of a Hilbert space. We consider the simplest such polynomial: $p(\lambda) := \lambda^2$. Then we can write operators $T \in B(H \oplus H)$ in 2×2 block matrix form. If T is superdiagonal, i.e.,

$$T = \begin{pmatrix} 0 & T_{12} \\ 0 & 0 \end{pmatrix},$$

we have $T^2 = 0$, and the structure of T can be as complicated as the structure of $T_{12} \in B(H)$ can be.

If moreover F and A have involutions *, we always suppose $\lambda_i^* = \lambda_i$, $1 \leq i \leq n$, and we write

$$S(A) := \{a \in A \mid a = a^2 = a^*\}, \quad \text{and}$$
$$S_p(A) := \{a \in A \mid a = a^*, \ p(a) = 0_A\}.$$

For the C^*-algebra case, $\lambda_i \in \mathbb{R}$ is a natural hypothesis. In fact, if $q(\lambda) := \pi\{(\lambda - \lambda_i) \mid 1 \leq i \leq n, \lambda_i \in \mathbb{R}\}$, then $p(a) = 0$ and $a = a^*$ imply $q(a) = 0$. Thus we could use $q(\cdot)$ rather than $p(\cdot)$.

A particular case is when A is the algebra of all bounded linear operators on a complex Hilbert space H (with $F = \mathbb{C}$).

Then we have the analogue of Proposition 1.

Proposition 3. *With the above notations, we have for $a \in A$ that*

$$a \in S_p(A)$$

if and only if a is of the form

$$a = \sum_{i=1}^{n} \lambda_i e_i, \ \text{where} \ \{e_1, \ldots, e_n\} \subset S(A) \ \text{is a partition of unity}.$$

Here, for each $1 \leq i \leq n$, the function $a \mapsto e_i = e_i(a)$ is a polynomial, with coefficients in F, all of which are self-adjoint. These polynomials only depend on $\lambda_1, \ldots, \lambda_n$, and coincide with those from Proposition 1 (except that here $\lambda_i^* = \lambda_i$).

Of course, the last statement of Proposition 1 remains valid if we replace $E(A)$ by $S(A)$, and require $\lambda_i^* = \lambda_i$, $\mu_j^* = \mu_j$, for $1 \leq i \leq n$, $1 \leq j \leq m$.

Corollary 4. *Let $A = B(H)$, where H is a complex Hilbert space. Then for $T \in B(H)$ we have $T \in S_p(A)$ if and only if there exists an orthogonal direct sum decomposition $H_1 \oplus \ldots \oplus H_n$ of H, where each H_i is a closed subspace of H, such that*

$$T \mid H_i = \lambda_i \cdot id_{H_i}, \quad \text{for each} \ 1 \leq i \leq n.$$

∎

Corollary 5. *Let F be a commutative topological field, and A a unital topological F-algebra. Then the function $a \mapsto e_i(a)$ from Propositions 1 and 3, being a polynomial, is continuous, for each $1 \leq i \leq n$.* ∎

Remark B. For an ordered partition of unity $(e_1, ..., e_n)$, the function $(e_1, ..., e_n) \mapsto \sum_{i=1}^{n} \lambda_i e_i \in E_p(A)$ is clearly continuous. By Corollary 5, for $a \in E_p(A)$, the function

$a \mapsto (e_1(a), \ldots, e_n(a))$ is continuous. Their composition $a \mapsto (e_1(a), \ldots, e_n(a)) \mapsto \sum \lambda_i e_i(a)$ is identity by Proposition 1. We show that the other composition $(e_1, \ldots, e_n) \mapsto \sum \lambda_i e_i \mapsto \left(e_1\left(\sum \lambda_i e_i\right), \ldots, e_n\left(\sum \lambda_i e_i\right)\right)$ is also identity. We calculate, e.g., $e_1\left(\sum \lambda_i e_i\right)$. Analogously, as in the proof of Proposition 1, **2**, we have

$$e_1\left(\sum \lambda_i e_i\right) = \prod_{j=2}^{n}\left(\sum_{i=1}^{n}(\lambda_i - \lambda_j)e_i\right) \bigg/ \prod_{j=2}^{n}(\lambda_1 - \lambda_j)$$

like in that proof, this product equals

$$e_1^{n-1}\prod_{j=2}^{n}(\lambda_i - \lambda_j) \bigg/ \prod_{j=2}^{n}(\lambda_1 - \lambda_j) + \sum_{i=2}^{n}e_i^{n-1}\prod_{j=2}^{n}(\lambda_i - \lambda_j) \bigg/ \prod_{j=2}^{n}(\lambda_1 - \lambda_j) = e_1.$$

Hence these maps constitute two homeomorphisms, inverse to each other.

The analogous statement holds for ordered self-adjoint partitions of unity, and $a \in S_p(A)$, by Proposition 3, and the above arguments.

These statements mean that when investigating local connectedness or connectedness of components of $E_p(A)$ or $S_p(A)$ via analytic or polynomial paths, we obtain the same answers for connected components of ordered (self-adjoint) partitions of unity. For this observe only that the maps $(e_1, \ldots, e_n) \mapsto \sum_{i=1}^{n} \lambda_i e_i$ and $a \mapsto (e_1(a), \ldots, e_n(a))$ are polynomial (hence analytic) maps. Of course, for polygonal maps we do not have an equivalence, but as we will see in the prooofs of Theorems 12, 13, 14, we will deduce polygonal connections in $E_p(A)$ and $S_p(A)$ from polygonal connections in ordered partitions of unity, and ordered self-adjoint partitions of unity. For this observe that the map $(e_1, \ldots, e_n) \mapsto \sum_{i=1}^{n} \lambda_i e_i$ is linear.

Later in this paper we will only consider $E_p(A)$ and $S_p(A)$.

Apart from Theorems ... and ... , from now on, in the whole paper we restrict our attention to the case when A is a unital complex Banach algebra, or sometimes a C^*-algebra, or more generally a unital complex Banach algebra with a continuous involution. For $x \in A$ we write $\sigma(x)$ for the *spectrum of x* in A.

Remark C. For unital complex Banach algebras A Corollary 5 can be shown also in other ways.

1) For $a \in E_p(A)$, by the spectral mapping theorem, we have

$$\emptyset \neq \sigma(a) \subset \{\lambda_1, \ldots, \lambda_n\}.$$

Then e_i, for $1 \leq i \leq n$, can be obtained as a Riesz idempotent

$$e_i := \frac{1}{2\pi i} \int_{\Gamma_i} (a - \lambda)^{-1} d\lambda, \tag{C.1}$$

where Γ_i is a small circle with centre λ_i, and for $i \neq j$ we have $\Gamma_i \cap \Gamma_j = \emptyset$. Then also $a = \sum_{i=1}^{n} \lambda_i e_i$. Now (C.1) also implies continuity of the function $a \mapsto e_i = e_i(a)$ (and self-adjointness of e_i for C^*-algebras, and λ_i's real).

2) Let a, $a + \varepsilon \in E_p(A)$, where $\|\varepsilon\| \leq 1$, and $1 \leq i \leq n$. Then

$$\|e_i(a + \varepsilon) - e_i(a)\| = \|p_i(a + \varepsilon) - p_i(a)\|$$

$$= \left\| \left[\prod_{\substack{j=1 \\ j \neq i}}^{n} (a + \varepsilon - \lambda_j) - \prod_{\substack{j=1 \\ j \neq i}}^{n} (a - \lambda_j) \right] \Big/ \prod_{\substack{j=1 \\ j \neq i}}^{n} (\lambda_i - \lambda_j) \right\|$$

$$\leq \left[\prod_{\substack{j=1 \\ j \neq i}}^{n} (\|a\| + \|\varepsilon\| + |\lambda_j|) - \prod_{\substack{j=1 \\ j \neq i}}^{n} (\|a\| + |\lambda_j|) \right] \Big/ \prod_{\substack{j=1 \\ j \neq i}}^{n} |\lambda_i - \lambda_j|$$

$$= \sigma_{\|a\|}(\|\varepsilon\|),$$

where $\sigma_{\|a\|}$ means that the constant in the σ sign depends on $\|a\|$ (observe that $\lambda_1, \ldots, \lambda_n$ are fixed in the whole paper).

Remark D. For A a C^*-algebra (which is the only case of *-algebras that we will be able to handle when investigating paths) it is no restriction of generality that we restricted our attention to real λ_i's. In fact, let $a = a^* \in A$. Then $\sigma(a) \subset \mathbb{R}$, cf. [En], p. 1. Therefore for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ $a - \lambda$ is invertible. So if we had admitted also non-real λ_i's, we could have omitted all the factors $a - \lambda_i$ with $\lambda_i \in \mathbb{C} \setminus \mathbb{R}$ from $p(\lambda)$, thus obtaining a polynomial $q(\lambda)$ with $q(a) = 0$.

Theorem 6. *Let A be a unital complex Banach algebra. Then $E_p(A)$ is locally polynomially connected, via cubic polynomial paths, for $t \in [0, 1]$.*

Moreover, suppose that $a_0, a_1 \in E_p(A)$, a_0 is fixed, and $\|a_1 - a_0\|$ is sufficiently small. Then a_0 and a_1 are similar, via some exponential (thus invertible element), i.e.,

$$a_1 = e^{-c} a_0 e^c.$$

This implies an analytic path $a(t) \in E_p(A)$ for $t \in [0, 1]$, from a_0 to a_1, namely

$$a(t) = e^{-ct} a_0 e^{ct}.$$

The distance of this path from a_0 tends to 0, if $\|a_1 - a_0\| \to 0$.

Hence the connected component of a_0 in $E_p(A)$ is locally pathwise connected, via similarity with an exponential function.

Theorem 7. *Let A be a unital complex Banach algebra, and let c be a connected component of $E_p(A)$. Then C is a relatively open subset of $E_p(A)$. Let $a_0, a_1 \in C$. Then a_0 and a_1 are similar, via a finite product of exponentials (that is invertible), i.e.,*

$$a_1 = e^{-c_m} \ldots e^{-c_1} a_0 e^{c_1} \ldots e^{c_m},$$

for some integer $m \geq 1$, where $c_i \in A$. This implies an analytic path $a(t) \in C$ from a_0 to a_1, for $t \in [0, 1]$, namely

$$a(t) = e^{-c_m t} \ldots e^{-c_1 t} a_0 e^{c_1 t} \ldots e^{c_m t}.$$

Additionally, we may suppose

$$c_1^2 = \cdots = c_m^2 = 0,$$

which implies a polynomial path

$$\tilde{a}(t) = e^{-c_m t} \ldots e^{-c_1 t} a_0 e^{c_1 t} \ldots e^{c_m t} = (1 - c_m t) \ldots (1 - c_1 t) a_0 (1 + c_1 t) \ldots (1 + c_m t),$$

from a_0 to a_1, for $t \in [0,1]$. Here one of the factors containing c_1 can be deleted, without changing the value of the right-hand side in the last equation.

Hence c is pathwise connected via similarities with finite products of exponential functions, and also with polynomial paths. Moreover, there is a single path satisfying both properties.

Corollary 8. *Let A be a unital complex Banach algebra. Let $a_0 \in E_p(A)$. Then a_0 is in the centre of A if and only if the connected component of a_0 in $E_p(A)$ is $\{a_0\}$ (i.e., a_0 is isolated in $E_p(A)$).*

Theorem 9. *Let A be a unital complex Banach algebra. If some connected component of $E_p(A)$ does not intersect the centre of A, then any element of C is contained in a complex line entirely contained in C. In particular, C is unbounded.*

Corollary 10. *Let A be a unital complex Banach algebra. Then $E_p(A)$ is a union of its isolated point and of complex lines.*

Theorem 11. *Let $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Let A be a unital complex C^*-algebra. Suppose that $a_0, a_1 \in S_p(A)$, a_0 is fixed, and $\|a_1 - a_0\|$ is sufficiently small. Then a_0 and a_1 are similar via a unitary,*

$$a_1 = e^{-ic} a_0 e^{ic}, \quad \text{where} \quad c = c^* \in A, \quad \text{and} \quad \|c\| \text{ is small.}$$

This implies a self-adjoint analytic path $a(t) \in S_p(A)$ from a_0 to a_1, for $t \in [0,1]$, namely

$$a(t) = a(t)^* = e^{-ict} a_0 e^{ict}.$$

The distance of this path from a_0 tends to 0, if $\|a_1 - a_0\| \to 0$.

Hence the connected component of a_0 in $S_p(A)$ is locally pathwise connected via a similarity with a unitary valued exponential function.

Theorem 12. *Let $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Let A be a unital complex C^*-algebra, and let C be a connected component of $S_p(A)$. Then C is a relatively open subset of $S_p(A)$. Let $a_0, a_1 \in C$. Then a_c and a_1 are similar. Via some unitary $s \in A$, i.e.,*

$$a_1 = s^{-1} a_0 s = s^* a_0 s_s \quad \text{where} \quad s^{-1} = s^*.$$

Here, for some integer $m \geq 1$, s is of the form

$$s = e^{ic_1} \ldots e^{ic_m}, \quad \text{where} \quad c_i = c_i^* \in A.$$

This implies a self-adjoint analytic path $a(t) = a(t)^* \in C$ from a_0 to a_1, for $t \in [0,1]$, namely

$$a(t) = e^{-ic_m t} \ldots e^{-ic_1 t} a_0 e^{ic_1 t} \ldots e^{ic_m t}.$$

Corollary 13. *Let $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Let A be a unital complex C^*-algebra. Let $a_0 \in S_p(A)$. Then a_0 is in the centre of A if and only if the connected component of a_0 in $S_p(A)$ is $\{a_0\}$ (i.e., a_0 is isolated in $S_p(A)$).*

Remark E. For the C^*-algebra case, local connectedness of $S_p(A)$ via cubic polynomial paths (analogously to Theorem 6 about $E_p(A)$) and connectedness of connected components of $S_p(A)$ via polygonal paths (analogously to Theorem 7 about $E_p(A)$) are false, already for $S(A)$.

Actually, this happens in the simplest case $A := B(\mathbb{C}^2)$. There the connected components of $S_p(A)$ are those with given rank, cf. [AMMZ03]. Thus one connected component is

$$C := \{T \in S(B(\mathbb{C}^2)) \mid \operatorname{rank} T = 1\}.$$

For idempotent elements T, rank $T = 1$ means that the eigenvlaues are 0, 1 (with multiplicities 1), or equivalently that the sum and product of the eigenvalues are 1, and 0, respectively. Equivalently, we have

$$T_r T = 1 \quad \text{and} \quad \det T = 0. \tag{E.1}$$

Self-adjointness of $T =: (a_{ij})_{i,j=1}^2$ means $a_{11}, a_{22} \in \mathbb{R}$, $a_{21} = \overline{a_{12}}$. That is,

for some $a, b, c, d \in \mathbb{R}$ we have $a_{11} = a$, $a_{22} = b$, $a_{12} = c + id$, $a_{21} = c - id$.

Rewriting (E.1) for T self-adjoint, we have

$$a + b = 1 \quad \text{and} \quad ab - (c^2 + d^2) = 0. \tag{E.2}$$

Eliminating b, we obtain $a(1 - a) - (c^2 + d^2) = 0$, or, equivalently

$$\frac{1}{4} = \left(a - \frac{1}{2}\right)^2 + c^2 + d^2. \tag{E.3}$$

Now suppose that for $t \in [0, 1]$ $T = T(t) =: (a_{ij}(t))_{i,j=1}^2$ is a polynomial of t. Then, with the above notations, for $t \in [0, 1]$

$$a = a(t), \quad b = b(t), \quad c = c(t), \quad d = d(t)$$

are polynomials of t as well, and (E.3) holds for all $t \in [0, 1]$. Now observe that if a polynomial of t vanishes at more values of t than its degree, then it is identically 0. This implies that equality (E.3) continues to hold for all $t \in \mathbb{R}$. Then each of $a(t) - 1/2$, $a(t)$, $b(t)$, $(c(t)$, $d(t)$ is a polynomial bounded on $\mathbb{R}$, i.e., is constant.

This shows that C contains no non-constant polynomial path, while $|c| > 1$. Hence C is neither locally connected, nor connected via any polynomial paths.

Moreover, we cannot prove Theorem 6 and the part of Theorem 7 concerning similarities via finite products of exponentials for general Banach *-algebras. Namely, we used in their proofs S. Maeda [Ma], who already in the proof of his Lemma 1, display, first equality uses the C^*-algebra property. Probably the mentioned statements are false, but we have no counterexamples.

The following Theorem 13 ?? is a particular case of the next Theorem 14. We separated these two statements because the more general Theorem 14 will be proved by reducing its statement to its particular case dealt with in Theorem 13.

Theorem 14. *Let $A = B(X)$, where X is a complex Banach space. Let $a_0, a_1 \in E_p(A)$, with a_0 fixed and $\|a_1 - a_0\|$ sufficiently small. Then there exists a polygonal path in $E_p(A)$, connecting a_0 and a_1, consisting of n segments. The distance of this path from a_0 tends to 0, if $\|a_1 - a_0\| \to 0$.*

Hence the connected component of a_0 in $E_p(A)$ is locally pathwise connected via paths of n segments.

Theorem 15. *Let A be a unital complex Banach algebra. Let $a_0, a_1 \in E_p(A)$, with a_0 fixed, and $\|a_1 - a_0\|$ small. Then there exists a polygonal path in $E_p(A)$, connecting a_0 and a_1, consisting of n segments. The distance of this path from a_0 tends to 0, if $\|a_1 - a_0\| \to 0$.*

Hence the connected component of a_0 in $E_p(A)$ is locally pathwise connected via paths of n segments.

Theorem 16. *Let A be a unital complex Banach algebra, let C be a connected component of $E_p(A)$, and let $a_0, a_1 \in C$. Then there exists a polygonal path in $E_p(A)$, connecting a_0 and a_1.*

Remark F. For C^*-algebras, in general, polygonal connection between elements of a connected component of $S_p(A)$ is impossible. This holds already for $p(x) = x(x-1)$, i.e., for $S(A)$, even in the simplest case $A = B(H)$, for H a Hilbert space. Although this follows from Remark E, we give here another argument, which yields infinitely many examples for this special case.

The connected components of $S(A) = S(B(H))$ are

$$\big\{ e \in S(A) \mid \dim N(E) = \alpha, \; \dim R(E) = \beta,$$
$$\text{for some cardinalities } \alpha, \beta \geq 0, \; \text{with } \alpha + \beta = \dim H \big\}.$$

(Here dim is the dimension in Hilbert space sense.) Let $\dim H \geq 2$, and let $0 < \beta = k < \dim H$ an integer. Let C_k be the connected component of $S(A)$, consisting of self-adjoint (i.e., orthogonal) projections of rank k. We claim that C_k contains no non-trivial segment (while it consists of more than one element).

Let $a_0, a_1 \in C_k$, such that the segment $[a_0, a_1]$ lies in C_k. For $a \in [a_0, a_1]$ a is a compact operator, with singular values $\|a\| = s_0(a) \geq s_1(a) \geq s_2(a) \geq \dots$. These $s_i(a)$'s are the eigenvalues of the non-negative square root $\sqrt{a^*a} = a$, with multiplicities, in decreasing order. (For $\dim H$ finite, $s_i(a) = 0$ for $i \geq \dim H$.) Therefore $1 = s_0(a) = \dots = s_{k-1}(a) > 0 = s_k(a) = s_{k+1}(a) = \dots$. Thus all singular numbers are constant on the segment $[a_0, a_1]$. This implies

$$a_0 = a_1,$$

by B. Aupetit, E. Makai, Jr., J. Zemánek [AMZ], Theorem, p. 517.

Corollary 17. *Let $A = B(H)$, where H is a (complex) Hilbert space. Then the (pathwise) connected components of $E_p(A)$ are of the form*

$\{a \in E_p(A) \mid$ for each $1 \leq i \leq n$ the Hilbert space dimension of the eigensubspace corresponding to the eigenvalue λ_i is $\alpha_i\}$, where $\alpha_1, \dots, \alpha_n \geq 0$ are any cardinalities whose sum is the Hilbert space dimension of H.

For the same A, considered as a C^*-algebra, and with $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, we have the following. The (pathwise) connected components of $S_p(A)$ are of the same form as given above, where still the respective eigensubspaces are orthogonal.

Therefore the connected components of $E_p(A)$ and $S_p(A)$ are just the similarity classes of operators in $E_p(A)$ and $S_p(A)$ via conjugation with an invertible, or unitary element.

Remark G. For $A = B(H)$, if we consider all self-adjoint elements, with their respective self-adjoint partitions of unity, then unitary similarity of these self-adjoint partitions of unity for $a_0, a_1 \in A$ implies a path joining a_0, a_1 in the set of elements unitarily similar to a_0, a_1. This can be shown as in the second part of the proof of Corollary 16. However it is not clear how to extend the definition of $S_p(A)$ from finite real spectra to more general self-adjoint partitions of unity (and then to investigate their connected components).

Problem. It would be a natural conjecture that the distance of different connected components of $E_p(A)$, for A a unital complex Banach algebra (at least for A a unital complex C^*-algebra) and that the distance of different connected components of $S_p(A)$, for A a unital complex C^*-algebra, and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ is at least

$$\min\{|\lambda_i - \lambda_j| \mid 1 \leq i, j \leq n, \ i \neq j\}.$$

This was the case for $p(\lambda) = \lambda(\lambda - 1)$, i.e., for idempotents, or self-adjoint idempotents. This implies the conjecture for $n = 2$, i.e., for $p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)$ in general. We just have to observe that the set $E_{(\lambda - \lambda_2)(\lambda - \lambda_2)}(A)$, or $S_{(\lambda - \lambda_n)(\lambda - \lambda_2)}(A)$ can be obtained from $E(A)$, or $S(A)$, by the transformation $x \mapsto \lambda_1 \cdot 1_A + (\lambda_2 - \lambda_1)x$, for $x \in A$. For $S_{(\lambda - \lambda_1)(\lambda - \lambda_2)}(A)$, with $\lambda_1, \lambda_2 \in \mathbb{R}$, we have by the same argument that the distance of different connected components is at least $|\lambda_1 - \lambda_2|$.

Clearly this conjecture, if true, would be sharp, for any A. Namely, $\lambda_1 \cdot 1_A, \ldots, \lambda_n \cdot 1_A \in E_p(A)$ (or $\in S_p(A)$), and since they are central, by Corollaries 8 and 12 their connected components in $E_p(A)$ (in $S_p(A)$) are $\{\lambda_1 \cdot 1_A\}, \ldots, \{\lambda_n \cdot 1_A\}$. The minimal pairwise distance of these components is

$$\min\{|\lambda_i - \lambda_j| \mid 1 \leq i, j \leq n, \ i \neq j\}.$$

However, for $n \geq 3$ our proof does not give even that two different connected components of $E_p(A)$ would have a positive distance. There arise several questions.

1) Are these distances positive?

2) Are these distances bounded below by some positive function of $\lambda_1, \ldots, \lambda_n$?

3) Are these distances at least

$$\min\{|\lambda_i - \lambda_j| \mid 1 \leq i, j \leq n, \ i \neq j\}?$$

The same questions arise for the C^*-algebra case, with self-adjoint idempotents, and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, but then questions 1) and 2) are answered positively in the following Theorem 16.

Even the commutative case would be of interest.

Observe that if we had a positive answer for 2) or 3), then this would imply the stronger statement that even the spectral distance of different connected components (i.e., the infimum of the spectral radii of the differences of elements in the

different components of $E_p(A)$ in question) would satisfy the respective inequality. (For the C^*-algebra case the norm equals the spectral radius, so there is no such separate question.) In fact, $E_p(A)$ and its connected components do not depend on the particular norm of A. For $a \in C$, $b \in D$, with C, D distinct components of $E_p(A)$ we can renorm A so that $\|a - b\| < \varrho(a - b) + \varepsilon$, for any $\varepsilon > 0$ (and $\|1_A\|$ remains 1), where $\varrho(\cdot)$ denotes spectral radius. Then any lower bound for $\|a - b\|$ implies the same lower bound for $\varrho(a - b)$.

If question 3) had a positive answer for Banach algebras, then it would have a positive answer for C^*-algebras as well. Namely, different connected components $S_p(A)$ lie in different connected components of $E_p(A)$, by [BFML], § 1, Applications 2). Observe that the considerations there only use the C^*-algebra generated by $\{P(t) \mid t \in [0, 1]\}$. Also, they concern only the case of $E(A)$ and $S(A)$. However, a path connecting $a_0, a_1 \in S_p(A)$ in $E_p(A)$ (cf. Theorem 7 and Theorem 16) yields paths connecting the idempotents $e_i(a_0)$ to the idempotents $e_i(a_1)$, where $e_i(\cdot)$ are the functions in Proposition 1. Moreover, by Corollary 5, $e_i(\cdot)$ from Proposition 1 is a continuous function of its argument.

Since [BFML], § 1, Applications 2) essentially uses the C^*-algebra property, most probably for Banach *-algebras with continuous involutions different connected components of $S_p(A)$, or $S(A)$ may lie in the same connected component of $E_p(A)$, or $E(A)$. However, we do not have a concrete example.

Theorem 18. *Let A be a unital complex C^*-algebra, and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Then the distance of different connected components of $S_p(A)$ is at least*

$$\left(\min_{\substack{1 \le \alpha, \beta \le n \\ \alpha \ne \beta}} |\lambda_\alpha - \lambda_\beta| \right) \min_{1 \le i \le n} \prod_{\substack{j=1 \\ j \ne i}}^n |\lambda_i - \lambda_j| \Bigg/$$

$$\Bigg/ \left[\prod_{\substack{j=1 \\ j \ne i}}^n \left(\max_{k \ne j} |\lambda_k - \lambda_j| + \min_{\substack{1 \le \alpha, \beta \le n \\ \alpha \ne \beta}} |\lambda_\alpha - \lambda_\beta| \right) - \prod_{\substack{j=1 \\ j \ne i}}^n \max_{k \ne j} |\lambda_k - \lambda_j| \right].$$

For $n = 2$ this gives back that the distance between different connected components of $S(A)$ is at least 1, and the distance between different connected components of $S_p(A)$ is at least $|\lambda_1 - \lambda_2|$, that is sharp, cf. the following problem. However, for $n \ge 3$ this estimate is probably far from the conjecturable value

$$\min_{i \ne j} |\lambda_i - \lambda_j|.$$

E.g., for the case $\lambda_i = e^{(2\pi i)(i/n)}$ the expression in Theorem 17 is

$$e^{n??\left(\int_0^\pi \log(2 \sin t) dt / \pi + o(1) \right)} \Big/ 2^n,$$

while

$$\min_{i \ne j} |\lambda_i - \lambda_j| = |\lambda_2 - \lambda_1| = 2 \sin(\pi/n) = (2\pi/n) \cdot (1 + o(1)).$$

Still observe that the estimate in Theorem 17 is invariant under simultaneous translation of $\lambda_1, \ldots, \lambda_n$, and their simultaneous multiplication by a number of

absolute value 1, and is homogeneous of first degree under their simultaneous multiplication by a positive number. These properties are shared by the (yet unknown) expression of the exact infimum.

Now we turn to another subject. We begin with a notation. For Banach spaces X, Y, we denote by $\mathcal{B}(X, Y)$ the Banach space of all bounded linear operators $X \to Y$. In our papers B. Aupetit, E. Makai, Jr., M. Mbekhta, J. Zemánek [AMMZ03], [AMMZ14] we investigated the following situation. Let A and B be Banach algebras, and let $0 \in U \subset \mathbb{C}$ be an open set. We investigated analytic families of idempotents $q : U \to E(B) \subset B$, and analytic familes $\pi : U \to \mathcal{B}(A, B)$, whose values are surjective Banach algebra homomorphisms $A \to B$. We looked for conditions which assure that there exists an analytic family of idempotents $p : U \to E(A) \subset A$, or $p : V \to E(A) \subset A$, where $0 \in V \subset U$ is some open set of $\mathbb{C}$, such that

$$q(\lambda) = \pi(\lambda) \text{ for all } \lambda \in U, \text{ or } \lambda \in V, \text{ respectively.}$$

(We called them global, or local liftings of analytic families of idempotents along analytic families of surjective Banach algebra homomorphisms.) We had analogous theorems for $S(A)$ and $S(B)$ as well.

It is a natural question, posed by L. W. Marcoux on the conference on linear algebra in Ljubljana, Slovenia, 2014 (and also earlier by some participant of a conference on operator theory at the Banach centre, Warsaw, some years ago), whether these theorems have extensions for $E_p(A)$, or $S_p(A)$.

A *real analytic map* from an open subset G of $\mathbb{R}$ ($\mathbb{R}^n$) to a Banach space is a map f which for each $x_0 \in G$ has locally a power series expansion

$$f(x) = \sum_0^\infty a_i (x - x_0)^i$$

(with the analogous formula for $\mathbb{R}^n$). When we write in the following theorems spectrum of an element of Ker $\pi(\lambda) \subset A$ or Ker $\pi(0) \subset A$, we mean spectra in A. Two idempotents e, f in some Banach algebra are *orthogonal* if $ef = fe = 0$. For examples where the hypotheses of the following theorems are satisfied cf. [AMMZ14], § 3.

We remark that the global lifting Theorems 19, 20 are generalizations of [AMMZ14], Theorems 3, 4, if these are restricted to single idempotents, rather than for sequences of idempotents. However, the local lifting Theorems 21, 22 have stronger counterparts for idempotents, cf. [AMMZ14], Theorems 1, 2, inasmuch there the spectral hypotheses are weaker: the spectrum of each element of Ker $\pi(0)$ does not disconnect $\mathbb{C}$, or is totally disconnected, respectively.

Theorem 19. *Let U be an open subset of $\mathbb{C}$. Let A and B be unital complex Banach algebras, and let $\pi : U \to \mathcal{B}(A, B)$ be an analytic map, whose values are surjective unit-preserving homomorphisms $A \to B$. Suppose that the spectrum of each element of Ker $\pi(\lambda)$, for each $\lambda \in U$, is $\{0\}$. Let $b(\cdot) : U \to E_p(B)$ ($\subset B$) be an analytic map. Then there exists an analytic map $a(\cdot) : U \to E_p(A)$ ($\subset A$) such that*

$$\pi(\lambda)a(\lambda) = b(\lambda) \text{ for each } \lambda \in U.$$

Theorem 20. *Let G be an open subset of $\mathbb{R}$. Let A and B be unital complex Banach algebras with continuous involutions, and let $\pi : G \to \mathcal{B}(A, B)$ be a real analytic map, whose values are surjective unit-preserving *-homomorphisms $A \to B$. Suppose that the spectrum of each element of $\mathrm{Ker}\,\pi(\lambda)$, for each $\lambda \in G$, is $\{0\}$. Let $b(\cdot) : G \to S_p(B)$ $(\subset B)$ be a real analytic map. Then there exists a real analytic map $a(\cdot) : G \to S_p(A)$ $(\subset A)$ such that*

$$\pi(\lambda)a(\lambda) = b(\lambda) \ \ for\ each\ \ \lambda \in G.$$

The following Theorems 20 and 21 are localized versions of Theorems 18 and 19. It is interesting that we have in Theorems 20 and 21 the spectral hypothesis of Theorems 18 and 19 for $\lambda = 0$ only, and still we have local versions of Theorems 18 and 19 for some open set (V or H) containing 0.

Theorem 21. *Let U be an open subset of $\mathbb{C}$, containing 0. Let A and B be unital complex Banach algebras, and let $\pi : U \to \mathcal{B}(A, B)$ be an analytic map, whose values are unit-preserving homomorphisms $A \to B$, such that $\pi(0)$ is surjective. Suppose that the spectrum of each element of $\mathrm{Ker}\,\pi(0)$ is $\{0\}$. Let $b(\cdot) : U \to E_p(B)$ $(\subset B)$ be an analytic map. Then there exists an open set $V \subset \mathbb{C}$, such that $0 \in V \subset U$, and an analytic map $a(\cdot) : V \to E_p(A)$ $(\subset A)$, such that*

$$\pi(\lambda)a(\lambda) = b(\lambda) \ \ for\ each\ \ \lambda \in V.$$

Theorem 22. *Let G be an open subset of $\mathbb{R}$, containing 0. Let A and B be unital complex Banach algebras with continuous involutions, and let $\pi : U \to \mathcal{B}(A, B)$ be a real analytic map, whose values are unit preserving *-homomorphisms $A \to B$, such that $\pi(0)$ is surjective. Suppose that the spectrum of each element of $\mathrm{Ker}\,\pi(0)$ is $\{0\}$. Let $b(\cdot) : G \to S_p(B)$ $(\subset B)$ be a real analytic map, then there exist an open set $H \subset \mathbb{R}$, such that $0 \in H \subset G$, and a real analytic map $a(\cdot) : H \to S_p(A)$ $(\subset A)$ such that*

$$\pi(\lambda)a(\lambda) = b(\lambda) \ \ for\ each\ \ \lambda \in H.$$

Remark H. The following statements follow from the proofs in [AMMZ14], cf. in particular [AMMZ14], Remark 1. In Theorem 18 we may replace U by a Stein manifold. In Theorem 19 we may suppose $G \subset \mathbb{R}^n$ open, provided each connected component of U has a neighbourhood base in $\mathbb{C}^n$ consisting of domains of holomorphy, when $\mathbb{R}^n$ is embedded in $\mathbb{C}^n$ in the canonical way. In Theorem 20 we may suppose that $0 \in U \subset \mathbb{C}^n$ is open. In Theorem 21 we may suppose that $0 \in G \subset \mathbb{R}^n$ is open.

Remark I. We repeat a question posed in [...] that has relations to the spectral hypotheses in Theorems 19–22. Let H be a Hilbert space, and let $k(\lambda)$ be an analytic family of compact operators in $\mathcal{B}(H)$, for $\lambda \in \mathbb{C}$, with $|\lambda| < 1$. Suppose that for some sequence $\{\lambda_n\} \subset \mathbb{C}$ with $|\lambda_n| < 1$. Converging to 0, we have that the spectrum of $k(\lambda_n)$ is $\{0\}$. Is then the spectrum of $k(\lambda)$ equal to $\{0\}$ *for each λ with* $|\lambda| < 1$? [...] gave a positive answer if each $k(\lambda)$ has finite rank.

Remark J. In [AMMZ03] we asked whether there exists an infinite dimensional Banach space X, such that its Calkin algebra $\mathcal{C}(X)$ (i.e., $\mathcal{B}(X)/\mathcal{K}(X)$, where $\mathcal{B}(X)$ and $\mathcal{K}(X)$ are the Banach algebras of all bounded, or compact linear operators

on X, respectively) is commutative. Then, in particular, by [Ze], cited in the Introduction under number 4, each point of $E(\mathcal{C}(X))$, being central, is isolated in $E(\mathcal{C}(X))$.

It is known that there exists an infinite dimensional Banach space X, such that

$$\mathcal{B}(X) = \{\lambda I + K \mid \lambda \in \mathbb{C}, \ K \in \mathcal{K}(X)\},$$

cf. [...]. Then $\mathcal{C}(X) \cong \mathbb{C}$ is commutative, and thus $E(\mathcal{C}(X)) = \{0, 1\}$ consists of two isolated points.

However, it remains open, whether $E(\mathcal{C}(X))$ can consist of $n \in [3, \infty)$ isolated points.

3. Proofs

Proof of Proposition 1. **1.** We begin with the proof of the implication that $a \in E_p(A)$ implies $a = \sum\limits_{i=1}^{n} \lambda_i e_i$, with $\{e_1, \ldots, e_n\}$ as in the proposition.

Let $p_1, \ldots, p_n$ denote the Lagrange interpolation polynomials, i.e.,

$$p_i(\lambda) = \prod_{\substack{j=1 \\ j \neq i}}^{n} (\lambda - \lambda_j) \Big/ \prod_{\substack{j=1 \\ j \neq i}}^{n} (\lambda_i - \lambda_j).$$

Then

$$p : (\lambda_j) = \delta_{ij} \quad \text{for} \quad 1 \leq i, j \leq n,$$

and any polynomial f over F, of degree at most $n - 1$ can be written, in a unique way, as a linear combination of these polynomials, namely as

$$f(\lambda) = \sum_{i=1}^{n} f(\lambda_i) p_i(\lambda).$$

In particular, we have

$$1_F = \sum_{i=1}^{n} p_i(\lambda), \quad \text{and} \quad \lambda = \sum_{i=1}^{n} \lambda_i p_i(\lambda). \tag{1.1}$$

(Recall that, by hypothesis, $n \geq 2$.)

Let us define

$$e_i := p_i(a).$$

Then $1_F = \sum\limits_{i=1}^{n} p_i(\lambda)$ implies

$$1_A = \sum_{i=1}^{n} p_i(a) = \sum_{i=1}^{n} e_i, \tag{1.2}$$

and $\lambda = \sum\limits_{i=1}^{n} \lambda_i p_i(\lambda)$ implies

$$a = \sum_{i=1}^{n} \lambda_i p_i(a) = \sum_{i=1}^{n} \lambda_i e_i. \tag{1.2}$$

Let $i \neq j$. Then $p_i(\lambda)p_j(\lambda)$ is a multiple of $p(\lambda)$, and $p(a) = 0_A$, hence

$$0_A = p_i(a)p_j(a) = e_ie_j. \tag{1.4}$$

Now we calculate e_i^2. Let us divide $p_i(\lambda)$ by $\lambda - \lambda_i$ (with remainder), obtaining

$$p_i(\lambda) = (\lambda - \lambda_i)q_i(\lambda) + p_i(\lambda_i) = (\lambda - \lambda_i)q_i(\lambda) + 1.$$

Multiplying both sides with $p_i(\lambda)$, we obtain

$$p_i(\lambda)^2 = p_i(\lambda)(\lambda - \lambda_i)q_i(\lambda) + p_i(\lambda) = \frac{p(\lambda)}{\prod\limits_{\substack{i=1 \\ j \neq i}}^{n}(\lambda - \lambda_j)}q_i(\lambda) + p_i(\lambda).$$

Substitute here a for λ, and observe $p(a) = 0$. Then we obtain

$$e_i^2 = p_i(a)^2 = p_i(a) = e_i. \tag{1.5}$$

2. We turn to the proof of the converse implication. So, let $a := \sum\limits_{i=1}^{n} \lambda_i e_i$, with $\{e_1, \ldots, e_n\}$ as in the proposition. We calculate $p(a)$. We have, using $\sum\limits_{i=1}^{n} e_i = 1_A$, that

$$p(a) = p\left(\sum_{i=1}^{n}\lambda_i e_i\right) = \prod_{j=1}^{n}\left(\sum_{i=1}^{n}\lambda_i e_i - \lambda_j\right) = \prod_{j=1}^{n}\left(\sum_{i=1}^{n}\lambda_i e_i - \lambda_j \cdot 1_A\right)$$

$$= \prod_{j=1}^{n}\left(\sum_{i=1}^{n}\lambda_i e_i - \lambda_j \cdot \sum_{i=1}^{n}e_i\right) = \prod_{j=1}^{n}\left(\sum_{i=1}^{n}(\lambda_i - \lambda_j)e_i\right).$$

Here the last expression is an n-fold product of sums of n terms of the form $(\lambda_i - \lambda_j)e_i$. Rewrite this as a sum of n^n terms, each term being an n-fold product, and recall that the e_i's commute. Then each of these n-fold products, containing different e_i's, vanishes. There remains a sum of n terms, each term being an n-fold product, containing only a single e_i. Such an n-fold product equals

$$\prod_{j=1}^{n}(\lambda_i - \lambda_j) \cdot e_i^n = 0 \cdot e_i = 0_A.$$

Then $p(a)$ is a sum of n terms, each being equal to 0_A, so

$$p(a) = 0_A, \quad \text{i.e.,} \quad a \in E_p(A). \tag{1.6}$$

3. By **1** the function $a \mapsto e_i(a)$ is the polynomial function $\lambda \mapsto p_i(\lambda)$.

4. By the hypothesis of the last statement of the proposition we have, for each integer $N \geq 0$, that

$$a^N = \sum_{i=1}^{n}\lambda_i^N e_i = \sum_{j=1}^{m}\mu_j^N f_j, \quad \text{thus} \quad \sum_{i=1}^{n}\lambda_i^N e_i + \sum_{j=1}^{m}\mu_j^N(-f_j) = 0.$$

We use these equations for $0 \leq N \leq n + m - 1$. We have $n, m > 0$, and we may suppose $n \leq m$.

First suppose that $\lambda_i \neq \mu_j$ for each $1 \leq i \leq n$, $1 \leq j \leq m$. Then we have a system of linear equations on the vector space A, with determinant the Vandermonde determinant of $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_m$, that are all distinct. Then the Vandermonde determinant is not 0, and the unique solution of this system is $e_1 = \ldots = e_n = f_1 = \ldots = f_m = 0$, a contradiction. Hence, say, $\lambda_1 = \mu_1$.

Then our system of equations reduces to

$$\lambda_1{}^N (e_1 - f_1) + \sum_{i=2}^{n} \lambda_i{}^N e_i + \sum_{j=2}^{m} \mu_j{}^N (-f_j) = 0.$$

If $\lambda_i \neq \mu_j$ for each $2 \leq i \leq n$, $2 \leq j \leq m$, then we use these equations for $0 \leq N \leq n + m - 2$. Again the determinant is the Vandermonde determinant of λ_1 $(= \mu_1)$, $\lambda_2, \ldots, \lambda_n$, $\mu_2, \ldots, \mu_m$. Therefore

$$e_1 - f_1 = e_2 = \ldots = e_n = f_2 = \ldots = f_m = 0.$$

If $n = m = 1$, we have $\lambda_1 = \mu_1$, $e_2 = f_2$, and we are done. Else we have a contradiction to $e_i \neq 0$, $f_j \neq 0$. Therefore we may suppose, say, $\lambda_2 = \mu_2$. We have

$$\lambda_2{}^N (e_1 - f_1) + \lambda_2{}^N (e_2 - f_2) + \sum_{i=3}^{n} \lambda_i{}^N e_i + \sum_{j=3}^{m} \mu_j{}^N f_j = 0.$$

If $\lambda_i \neq \mu_j$ for each $3 \leq i \leq n$, $3 \leq j \leq m$, then we use our equations only for $0 \leq N \leq n + m - 3$. Like above, we obtain

$$e_1 - f_1 = e_2 - f_2 = e_3 = \ldots = e_n = f_3 = \ldots = f_m = 0.$$

If $n = m = 2$, we have $\lambda_1 = \mu_1$, $e_1 = e_2$, and also $\lambda_2 = \mu_2$, $e_2 = f_2$, and we are done. Else we have a contradiction to $e_i \neq 0$, $f_j \neq 0$. Therefore we may suppose, say, that $\lambda_3 = \mu_3$.

We continue analogously at each step, either the statement of the proposition becomes proved, or we have a contradiction to $e_i \neq 0$, $f_j \neq 0$.

If we have not obtained the statement of the proposition earlier, we get to

$$e_1 - f_1 = e_2 - f_2 = \ldots = e_n - f_n = f_{n+1} = \ldots = f_m = 0.$$

Then either $m = n$, and we have the statement of the proposition, or $m \geq n + 1$, and we have a contradiction. This proves the last statement of the proposition. ∎

Proof of Proposition 3. In **1** of the proof of Proposition 1 we showed that $a \in E_p(A)$ implied $a = \sum_{i=1}^{n} \lambda_i e_i$, with $\{e_1, \ldots, e_n\}$ as in Proposition 1. Now we have $a \in S_p(A) \subset E_p(A)$, so these considerations apply. There remains to prove $e_i{}^* = e_i$. We have

$$e_i = p_i(a) = \prod_{\substack{j=1 \\ j \neq i}}^{n} (a - \lambda_j) \Big/ \prod_{\substack{j=1 \\ j \neq i}}^{n} (\lambda_i - \lambda_j).$$

Therefore, $a = a^*$ and $\lambda_i^* = \lambda_i$ imply

$$e_i = p_i(a) = p_i(a^*) = p_i(a)^* = e_i^*. \tag{3.1}$$

In **2** of the proof of Proposition 1 we showed that $a = \sum_{i=1}^{n} \lambda_i e_i$ with $\{e_1, \ldots, e_n\}$ as in Proposition 1, implied $a \in E_p(A)$. Now $\lambda_i = \lambda_i^*$ and $e_i = e_i^*$ imply

$$a^* = \Big(\sum_{i=1}^{n} \lambda_i e_i\Big)^* = \sum_{i=1}^{n} \lambda_i^* e_i^* = \sum_{i=1}^{n} \lambda_i e_i = a.$$

Then $a \in E_p(A)$ and $a^* = a$ imply

$$a \in S_p(A). \tag{3.2}$$

By Proposition 1 (cf. **3** of its proof), we know that $a \mapsto e_i(a)$ is a polynomial, with coefficients in F. But this polynomial is $\lambda \mapsto p_i(\lambda)$, whose coefficients are rational functions of the self-adjoint elements λ_i, hence themselves are self-adjoint. ∎

Proof of Theorem 6. By Proposition 1 we have

$$a_0 = \sum_{i=1}^{n} \lambda_i e_{0i}, \quad a_1 = \sum_{i=1}^{n} \lambda_i e_{1i},$$

where $e_{0i}, e_{1i} \in E(A)$. By Corollary 5, we have

$$\|e_{1i} - e_{0i}\| < 1, \quad \text{for all } \ 1 \le i \le n,$$

provided $\|a_1 - a_0\|$ is sufficiently small. Therefore, by J. Zemánek [Ze], Lemma 3.1 and its proof, there exist $s_i \in A$ involutions. For all $1 \le i \le n$, such that

$$e_{1i} = s_i^{-1} e_{0i} s_i. \tag{6.1}$$

Moreover, for $\|a_1 - a_0\|$ sufficiently small we have $\|e_{1i} - e_{0i}\|$ sufficiently small, and then by [Ze], Lemma 3.1, Proof

$$\|s_i - (2e_{0i} - 1)\| \ \text{ is sufficiently small as well.} \tag{6.2}$$

Here $2e_{0i} - 1$ is an involution, thus $\sigma(2e_{0i} - 1) \subset \{-1, 1\}$ so for $\|s_i - (2e_{0i} - 1)\|$ sufficiently small, by upper semicontinuity of the spectrum,

$$s_i = e^{c_i}, \quad \text{hence} \ \ e_{1i} := e^{-c_i} e_{0i} e^{c_i}, \tag{6.3}$$

where $c_i \in A$. Moreover, we may choose $c_i', c_i'' \in A$ so that

$$e_{1i} = e^{-c_i''} e^{-c_i'} e_{0i} e^{c_i'} e^{c_i''} \ \text{ and } \ (c_i')^2 = (c_i'')^2 = 0, \tag{6.4}$$

cf. J. Esterle [Es], Theorem, Proof. Here even one of the factors containing c_i' can be deleted, without changing the value of the right-hand side expression of the

first equality in (6.4), leaving (6.4) valid, cf. [Es], Theorem, Proof. For $\|a_1 - a_0\|$ sufficiently small, we have for each $1 \leq i \leq n$ that $\|e_{1i} - e_{0i}\|$ is sufficiently small, and then for each $1 \leq i \leq n$ we have that

$$\text{both} \quad \|c_i'\| \quad \text{and} \quad \|c_i''\| \quad \text{are small,} \tag{6.5}$$

cf. [Es], Theorem, Proof.

Now we define

$$s := \sum_{i=1}^{n} e_{0i}s_i = \sum_{i=1}^{n} s_i e_{1i} \quad \text{and} \quad s' := \sum_{i=1}^{n} s_i^{-1} e_{0i} = \sum_{i=1}^{n} e_{1i}s_i^{-1}. \tag{6.6}$$

(The equalities (6.6) follow from (6.1).) Then

$$s's := \sum_{i=1}^{n} s_i^{-1} e_{0i}s_i = \sum_{i=1}^{n} e_{1i} = 1, \quad ss' := \sum_{i=1}^{n} s_i e_{1i} s_i^{-1} = \sum_{i=1}^{n} e_{0i} = 1, \tag{6.7}$$

hence

$$s' = s^{-1}.$$

Then we have

$$s^{-1}a_0 s = \sum_{i=1}^{n} s_i^{-1} e_{0i} \cdot \sum_{j=1}^{n} \lambda_j e_{0j} \cdot \sum_{k=1}^{n} e_{0k}s_k = \sum_{i=1}^{n} s_i^{-1} e_{0i}\lambda_i s_i = \sum_{i=1}^{n} \lambda_i e_{1i} = a_1. \tag{6.8}$$

Rewriting this, by (6.6) and (6.3) we have

$$a_1 = \sum_{i=1}^{n} s_i^{-1} e_{0i} \cdot a_0 \cdot \sum_{i=1}^{n} e_{0i}s_i = \sum_{i=1}^{n} e^{-c_i} e_{0i} \cdot a_0 \cdot \sum_{i=1}^{n} e_{0i}e^{c_i}. \tag{6.9}$$

Analogously, we have by (6.4)

$$\begin{aligned} a_1 &= \sum_{i=1}^{n} \lambda_i e_{1i} = \sum_{i=1}^{n} e^{-c_i''} e^{-c_i'} e_{0i} \cdot a_0 \cdot \sum_{i=1}^{n} e_{0i}e^{c_i'}e^{c_i''} \\ &= \sum_{i=1}^{n} (1 - c_i'')(1 - c_i')e_{0i} \cdot a_0 \cdot \sum_{i=1}^{n} e_{0i} \cdot (1 + c_i')(1 + c_i''). \end{aligned} \tag{6.10}$$

This implies an analytic, or polynomial path

$$a(t) := \sum_{i=1}^{n} e^{-c_i t} e_{0i} \cdot a_0 \cdot \sum_{i=1}^{n} e_{0i}e^{c_i t}, \tag{6.11}$$

or

$$\begin{aligned} \tilde{a}(t) :&= \sum_{i=1}^{n} e^{-c_i'' t} e^{-c_i t} e_{0i} \cdot a_0 \cdot \sum_{i=1}^{n} e_{0i}e^{c_i' t}e^{c_i'' t} \\ &= \sum_{i=1}^{n} (1 - c_i'' t)(1 - c_i' t)e_{0i} \cdot a_0 \cdot \sum_{i=1}^{n} e_{0i}(1 + c_i' t)(1 + c_i'' t), \end{aligned} \tag{6.12}$$

between $a(0) = a_0$ and $a(1) = a_1$, with $t \in [0, 1]$. In (6.12), for all $1 \leq i \leq n$, one of the factors containing c'_i can be deleted, leaving the value of the last expression in (6.12) unchanged. This means that we have a cubic polynomial path between $\tilde{a}(0) = a_0$ and $\tilde{a}(1) = a_1$.

Since by (6.5) $\|c'_i\|$ and $\|c''_i\|$ are small, for each $1 \leq i \leq n$, the distance of this cubic polynomial path to a_0 for $t \in [0, 1]$ is small, proving local connectedness of $E_p(A)$ via cubic polynomial paths.

For $\|a_1 - a_0\|$ sufficiently small, for all $1 \leq i \leq n$ $\|s_i - (2e_{0i} - 1)\|$ is small (cf. (6.2)), and therefore

$$\left\| \sum_{i=1}^{n} e_{0i} \big(s_i - (2e_{0i} - 1)\big) \right\| = \left\| \sum_{i=1}^{n} e_{0i} s_i - \sum_{i=1}^{n} e_{0i} \big(2e_{0i} - 1\big) \right\|$$

$$= \left\| \sum_{i=1}^{n} e_{0i} s_i - \sum_{i=1}^{n} e_{0i} \right\| = \|s - 1\|$$

is small as well. As soon as $\|s - 1\| < 1$, we have

$$s = e^c, \tag{6.13}$$

where $\|c\|$ is small as well, for $\|s - 1\|$ small. Then, by (6.8) and (6.13)

$$a_1 = s^{-1} a_0 s = e^{-c} a_0 e^c,$$

that implies an analytic path

$$\hat{a}(t) = e^{-ct} a_0 e^{ct} \tag{6.14}$$

between $\hat{a}(0) = a_0$ and $\hat{a}(1) = a_1$, for $t \in [0, 1]$.

For $\|a_1 - a_0\|$ sufficiently small, $\|c\|$ is sufficiently small, and then the distance of this whole path from a_0 is small. This proves local connectedness of $E_p(A)$, via similarities with exponential functions. ∎

Remark K. Even though the inidividual s_i's can be chosen as involutions, however most probably

$$s = \sum_{i=1}^{n} e_{0i} s_i$$

will not be an involution in general. Thus in this respect $E_p(A)$ behaves differently from $E(A)$. We do not know if s could be chosen as an involution, in the proof of Theorem 6.

The same remark concerns also the proof of Theorem 10, for the analogous question for C^*-algebras.

Proof of Theorem 7. By Theorem 6, $E_p(A) \subset A$ is locally (pathwise) connected. This implies that all connected components of $E_p(A)$ are relatively open subsets of $E_p(A)$.

Now let us fix $a_0 \in C$. We let

$$C(a_0) := \big\{ a \in E_p(A) \mid \exists m \geq 1 \text{ integer}, \exists c_1, ..., c_m \in A, \ a = e^{-c_m} ... e^{-c_1} a e^{c_1} ... e^{c_m} \big\}. \tag{7.1}$$

Since $a_0 = e^{-0} a_0 e^0$, we have $a_0 \in C$. Then for $a \in C(a_0)$ there exists an analytic path

$$a(t) = e^{-c_m t} \ldots e^{-c_1 t} a_0 e^{c_1 t} \ldots e^{c_m t}, \tag{7.2}$$

hence $a \in C$. That is,

$$C(a_0) \subset C. \tag{7.3}$$

By Theorem 6, $C(a_0)$ is a relatively open subset of $E_p(A)$. In fact, for $a \in C(a_0)$, any $a' \in E_p(A)$ sufficiently close to a is of the form

$$a' = e^{-c} a e^c = e^{-c} e^{-c_m} \ldots e^{-c_1} a_0 e^{c_1} \ldots e^{c_m} e^c,$$

hence $a' \in C(a_0)$ as well (with integer $m + 1$ and elements $c_1, \ldots, c_m, c \in A$).

Now suppose that $C(a_0) \subsetneqq C$. Since C is connected, it cannot have a non-trivial open-and-closed subset. Since $a_0 \in C(a_0) \subsetneqq C$, and $C(a_0)$ is open, therefore $C(a_0)$ is not closed. That is, there is a point

$$b \in \overline{C(a_0)} \cap \left(C \setminus C(a_0) \right) \tag{7.4}$$

($\overline{}$ meaning closure in $E_p(A)$, that is closure in A, by closedness of $E_p(A)$.) By $b \in \overline{C(a_0)}$ any neighbourhood of b, relative to $E_p(A)$, intersects $c(a_0)$. By Theorem 6, some neighbourhood of b consists of elements of the form $e^{-c} b e^c$, where $c \in A$. That is,

$$e^{-c} b e^c = e^{-c_m} \ldots e^{-c_1} a_0 e^{c_1} \ldots e^{c_m}, \tag{7.5}$$

which implies

$$b = e^c e^{-c_m} \ldots e^{-c_1} a_0 e^{c_1} \ldots e^{c_m} e^{-c}. \tag{7.6}$$

Therefore $b \in C(a_0)$ (with integer $m + 1$, and elements $c_1, \ldots, c_m, -c \in A$). However, by (7.4) $b \notin C(a_0)$, a contradiction. This proves

$$C(a_0) = C. \tag{7.7}$$

Then for $a_1 \in C = C(a_0)$ (7.1) implies an analytic path $a(t) \in E_p(A)$, thus $a(t) \in C$, from a_0 to a_1. Namely,

$$a(t) = e^{-c_m t} \ldots e^{-c_1 t} a_0 e^{c_1 t} \ldots e^{c_m t}, \tag{7.8}$$

for $t \in [0, 1]$.

As we have seen in the proof of Theorem 6, (6.4), (6.12), we may additionally suppose $c_1^2 = \ldots = c_m^2 = 0$, which implies a polynomial path

$$\tilde{a}(t) = e^{-c_m t} \ldots e^{-c_1 t} a_0 e^{c_1 t} \ldots e^{c_m t} = (1 - c_m t) \ldots (1 - c_1 t) a_0 (1 + c_1 t) \ldots (1 + c_m t), \tag{7.9}$$

from a_0 to a_1. For $t \in [0, 1]$, and here one of the factors containing c_1 can be deleted, without changing the value of the right-hand side of (7.9), by J. Esterle [Es], Theorem, Proof (cf. the proof of Theorem 6). ∎

Proof of Corollary 8. Let $a_0 \in E_p(A)$, and let C be the connected component of a_0 in $E_p(A)$.

Suppose that a_0 is in the centre of A, then for any $a_1 \in C$, by Theorem 7, for some invertible s we have

$$a_1 = s^{-1}a_0 s = a_0 s^{-1}s = a_0.$$

Hence $C = \{a_0\}$.

Now suppose that a_0 is not in the centre of A. Let $x \in A$, $a_0 x \neq x a_0$. Then, for $t \to 0$,

$$C \ni e^{-tx}a_0 e^{tx} = (1 - tx)a_0(1 + tx) + O(t^2) = a_0 + t(-xa_0 + a_0 x) + O(t^2) \neq a_0.$$

Then C is not the singleton $\{a_0\}$. ∎

Proof of Theorem 9. Let $a \in E_p(A)$ be an arbitrary element of $E_p(A)$. That is,

$$a = \sum_{i=1}^{n} \lambda_i e_i,$$

where $\{e_1, e_2, e_3, \ldots, e_n\}$ is an arbitrary partition of unity to mutually orthogonal idempotents in A. We want to perturb e_1 and e_2 to e_1' and e_2'. Leaving $e_3, \ldots, e_n$ unchanged, so that we obtain a new partition of unity to mutually orthogonal idempotents in A.

Let $x \in A$ be arbitrary, and let

$$e_1' := e_1 + e_1 x e_2, \quad e_2' := e_2 - e_1 x e_2, \quad \text{and} \quad e_i' = e_i \text{ for } i \geq 3.$$

Then

$$e_1' + e_2' + e_3' + \ldots + e_n' = 1, \quad (e_1')^2 = e_1', \quad (e_2')^2 = e_2', \quad e_1'e_2' = e_2'e_1' = 0,$$

and for $i, j \geq 3$, $i \neq j$

$$e_1'e_i' = e_i'e_1' = e_2'e_i' = e_i'e_2' = 0 \quad \text{and} \quad e_i'e_j' = 0.$$

Thus $\{e_1', e_2', e_3', \ldots, e_n'\}$ is a new partition of unity to mutually orthogonal idempotents in A.

We consider the element

$$a' := \lambda_1 e_1' + \lambda_2 e_2' + \lambda_3 e_3' + \ldots + \lambda_n e_n' \in E_p(A).$$

Multiplying it from left by e_1', e_2' we obtain

$$e_1'a' = \lambda_1 e_1' = \lambda_1(e_1 + e_1 x e_2) \quad \text{and} \quad e_2'a' = \lambda_2 e_2' = \lambda_2(e_2 - e_1 x e_2).$$

Also we have either $\lambda_1 \neq 0$ or $\lambda_2 \neq 0$. In the first case we use the equation for $e_1'a'$, in the second case we use the equation for $e_2'a'$. Thus unless the continuous linear map $A \ni x \mapsto \lambda_1 \cdot e_1 x e_2$ (or $\lambda_2 \cdot e_1 x e_2$) is identically 0, its image contains a (complex) line. In the second one of these cases also the image of the continuous linear map

$$A \ni x \mapsto a' = \lambda_1 e_1' + \lambda_2 e_2' + \lambda_3 e_3' + \ldots + \lambda_n e_n'$$
$$= \lambda_1(e_1 + e_1 x e_2) + \lambda_2(e_2 - e_1 x e_2) + \lambda_3 e_3 + \ldots + \lambda_n e_n$$

contains a (complex) line. For $x = 0$ we have $a' = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \ldots + \lambda_n e_n$, so the above line contans the arbitrarily chosen element a of $E_p(A)$. Now $a' \in E_p(A)$, and a' is in the same connected component of $E_p(A)$ as a.

We can replace e_1, e_2 by any other e_i, e_j $(i \neq j)$ in the above considerations. Therefore either

1) for each $x \in A$ and each $i \neq j$ we have $e_i x e_j = 0$, or

2) for some $i \neq j$ the image of the map $A \ni x \mapsto \sum_{i=1}^{n} \lambda_i e_i'$ contains a complex line.

In Case 2) the theorem is proved.

In Case 1)

$$e_i x = e_i x \cdot 1 = \sum_{i \neq j}^{n} e_i x e_j + e_i x e_i = e_i x e_i \quad \text{and}$$

$$x e_i = 1 \cdot x e_i = \sum_{i \neq j}^{n} e_j x e_i + e_i x e_i = e_i x e_i.$$

Therefore

$$e_i x = e_i x e_i = x e_i.$$

Here $x \in A$ is arbitrary, hence $e_i \in Z(A)$, and

$$a = \sum_{i=1}^{n} \lambda_i a_i \in Z(A).$$

Then by Corollary 8 the connected component of $E_p(A)$ containing a is the singleton $\{a\}$. ∎

Proof of Corollary 10. By Corollary 8 and Theorem 9, the central points of $E_p(A)$ are isolated in $E_p(A)$, and the non-central points of $E_p(A)$ lie on complex lines contained in $E_p(A)$. ∎

Proof of Theorem 11. The proof is analogous to that of Theorem 6.

We have that s_i, that was an involution in the proof of Theorem 6, is now even a self-adjoint involution, by S. Maeda [Ma], Lemma 2. Moreover, s, defined in (6.6) in the proof of Theorem 6, now satisfies, also using Proposition 3, (6.7) from the proof of Theorem 6, and $s_i^2 = 1$, that

$$s^* = \sum_{i=1}^{n} s_i^* e_{0i}^* = \sum_{i=1}^{n} s_i e_{0i} = \sum_{i=1}^{n} s_i^{-1} e_{0i} = s' = s^{-1}, \tag{11.1}$$

so s is unitary as well. Moreover, from the proof of Theorem 6, for $\|a_1 - a_0\|$ sufficiently small, $\|s - 1\|$ is small as well. As soon as $\|s - 1\| < 1$, we have

$$s = e^{ic}, \tag{11.2}$$

where $\|c\|$ is small for $\|s - 1\|$ small. In fact, letting log the branch of the logarithm function that takes the value 0 at 1, we define

$$ic := \log s. \tag{11.3}$$

By (10.1)

$$c = c^*. \tag{11.4}$$

Then

$$a(t) = a(t)^* = e^{-ict} a_0 e^{ict} \tag{11.5}$$

is, for $t \in [0, 1]$, a self-adjoint analytic path between a_0 and a_1.

Like in Theorem 6, for $\|a_1 - a_0\|$ sufficiently small, the distance of this whole path from a_0 is small, showing local connectedness of $S_p(A)$, by similarities via unitary valued exponential functions. ∎

Proof of Theorem 12. Like in the deduction of the global Theorem 7 from the local Theorem 6, we have from the local Theorem 9 that

$$a_1 = e^{-ic_m} \dots e^{-ic_1} a_0 e^{ic_1} \dots e^{ic_m},$$

where $c_i = c_i^* \in A$ for $2 \leq i \leq m$.

The statement about the self-adjoint analytic path follows immediately. ∎

Proof of Corollary 13. Let $a_0 \in S_p(A)$ and let C be the connected component of a_0 in $S_p(A)$.

Suppose that a_0 is in the centre of A. Then, like in the proof of Corollary 8 we obtain $C = \{a_0\}$. (By the way, this actually follows from Corollary 8.)

Now suppose that a_0 is not in the centre of A. Let $x \in A$, $a_0 x \neq x a_0$. Let $x = y + i_2$, $y = y^*$, $z = z^*$. Then a_0 does not commute either with y, or with z. That is, we can assume $x = x^*$. Then, like in the proof of Corollary 8 for $t \to 0$ we have

$$C \ni e^{-itx} a_0 e^{itx} = a_0 + it(-x a_0 + a_0 x) + O(t^2) \neq a_0.$$

Then C is not the singleton $\{a_0\}$. ∎

Proof of Theorem 14. **1.** We have by Proposition 1 and Corollary 5

$$a_0 = \sum_{i=1}^{n} \lambda_i e_{0i}, \quad a_1 = \sum_{i=1}^{n} \lambda_i e_{1i}, \text{ with } \|e_{1i} - e_{0i}\| \text{ small for each } 1 \leq i \leq n. \tag{14.1}$$

By Corollary 2, we have

$$X = \bigoplus_{i=1}^{n} X_{0i} = \bigoplus_{i=1}^{n} X_{1i}, \tag{14.2}$$

with coordinate projections

$$e_{0i} : X \to X_{0i}, \quad e_{i1} = X \to X_{1i}.$$

We will consider $n + 1$ decompositions of X as direct sums, namely

$$X = \bigoplus_{i=1}^{n} X_{0i}, \quad X = X_{11} \oplus \left(\bigoplus_{i=2}^{n} X_{0i} \right),$$

$$X = X_{11} \oplus X_{12} \oplus \left(\bigoplus_{i=3}^{n} X_{0i} \right), \dots, \tag{14.3}$$

$$X = \left(\bigoplus_{i=1}^{n-1} X_{1i} \right) \oplus X_{0n}, \quad X = \bigoplus_{i=1}^{n} X_{1i}.$$

We will show that these are in fact direct decompositions of X, and that all their respective coordinate projections are close to $e_{01}, \dots, e_{0n}$. Moreover, we will show that for any two neighbouring decompositions in (13.3), say, the jth and $(j+1)^{\text{st}}$ one, where $j \in \{1, \dots, n\}$, a simultaneous linear interpolation between their first, $\dots, n^{\text{th}}$ coordinate projections, for a parameter $t \in [0,1]$, gives a linear path of n pairwise orthogonal idempotents, summing to $I \in B(X)$, for each $t \in [0,1]$.

Hence, multiplying these n orthogonal idempotents by $\lambda_1, \dots, \lambda_n$, respectively, and summing these, we obtain a linear path in $E_p(A)$, joining its values for $t = 0$ and $t = 1$. These values are obtained from the first, $\dots, n^{\text{th}}$ projections of the jth and $(j+1)^{\text{st}}$ decompositions (for some $1 \le j \le n$), by multiplying them by $\lambda_1, \dots, \lambda_n$, respectively, and summing them.

We will consider the case $j = 1$, i.e. the construction of the second direct decomposition of X from the first one in (14.3). The further cases are analogous.

2. Let $e_0, e_1 \in E(A) = E(B(X))$, where e_0 is fixed, and $\|e_1 - e_0\|$ is small. Then the Kovarik element

$$g = g(e_0, e_1) \tag{14.4}$$

is defined in the following way, cf. Z. V. Kovarik [Ko], p. 343, (7), Definition of S_1, C_1 and p. 345, (b), and p. 347, Definition of F, and J. Esterle [Es], p. 253, Theorem, Proof, first paragraph. Suppose $\|e_1 - e_0\| < 1$. Then $e_0 + e_1 - 1$ is invertible ([Ko], p. 345, (b) and [Es], above cited), and we let

$$g = g(e_0, e_1) := e_1(e_0 + e_1 - 1)^{-2} e_0 = e_1 \big[1 - (e_1 - e_0)^2\big]^{-1} e_0, \tag{14.5}$$

then

$$
\begin{aligned}
&g = g^2 \in E(A), \\
&e_1 g = g, \quad \text{thus} \quad R(g) \subset R(e_1), \\
&g e_1 = e_1, \quad \text{thus} \quad R(e_1) \subset R(g), \quad \text{hence} \quad R(g) = R(e_1), \\
&e_0 g = e_0, \quad \text{thus} \quad N(g) \subset N(e_0), \\
&g e_0 = g, \quad \text{thus} \quad N(e_0) \subset N(g), \quad \text{hence} \quad N(g) = N(e_0)
\end{aligned}
\tag{14.6}
$$

(here $R(\cdot)$ is range, $N(\cdot)$ is kernel), cf. [Ko], p. 347, [Es], p. 253). Clearly, a projection g is uniquely determined by $R(g) = R(e_1)$ and $N(g) = N(e_0)$ for each $t \in [0,1]$ (even for each $t \in \mathbb{C}$) we have

$$(1-t)e_0 + tg, \; (1-t)e_1 + tg \in E(A) \tag{14.7}$$

cf. [Ko], p. 347. (By the way, all these calculations are rather straightforward.)

3. As told in **1**, we want to change the direct sum decomposition $X = \bigoplus_{i=1}^{n} X_{0i}$ to the direct sum decomposition $X = X_{10} \oplus \left(\bigoplus_{i=2}^{n} X_{0i} \right)$, in such a way that the simultaneous linear interpolation (for $t \in [0,1]$) of the first, $\dots, n^{\text{th}}$ projections gives for all $t \in [0,1]$ a system of orthogonal projections summing to 1.

The projections corresponding to the direct sum decomposition $X = \bigoplus_{i=1}^{n} X_i$ are $e_{01}, \dots, e_{0n}$. The projection to X_{11}, corresponding to the direct sum decomposition

$$X = X_{11} \oplus \left(\bigoplus_{i=2}^{n} X_{0i} \right), \tag{14.8}$$

has range $X_{11} = R(e_{11})$, and has kernel $\bigoplus\limits_{i=2}^{n} X_{0i} = N(e_{01})$, hence supposing $\|e_{11} - e_{01}\| < 1$, it exists and equals

$$g(e_{01}, e_{11}) = e_{11}(e_{01} + e_{11} - 1)^{-2} e_{01}. \tag{14.9}$$

Since later we will use only $g(e_{01}, e_{11})$, we most often will write for it just g. Thus

$$R(g) = X_{11} \quad \text{and} \quad N(g) = \bigoplus_{i=2}^{n} X_{0i}. \tag{14.10}$$

Observe that for $\|a_1 - a_0\|$ small we have $\|e_{1i} - e_{0i}\|$ small for each $1 \leq i \leq n$, and $\|g(e_{01}, e_{11}) - e_{01}\|$ small.

For the complementary projection $1 - g$ we have

$$R(1 - g) = \bigoplus_{i=1}^{n} X_{0i} \quad \text{and} \quad N(1 - g) = X_{11}. \tag{14.11}$$

Now we consider

$$e_{0i}(1 - g) \quad \text{for} \quad 2 \leq i \leq n. \tag{14.12}$$

This is obtained by first applying the projection $1 - g$, having range $\bigoplus\limits_{i=2}^{n} X_{0i}$, and then applying the restriction to $\bigoplus\limits_{i=2}^{n} X_{0i}$ of the $i^{\text{rm th}}$ projection $X = \bigoplus\limits_{i=1}^{n} X_{0i} \to X_{0i}$. This shows that $e_{0i}(1 - g)$ is a projection of X to X_{0i}, so

$$R\big(e_{0i}(1 - g)\big) = X_{0i}. \tag{14.13}$$

Similarly one sees that

$X_{11} = N(1 - g) \subset N\big(e_{0i}(1 - g)\big)$ and for $2 \leq j \leq n$, $j \neq i$ we have $X_{0j} \subset N\big(e_{0i}(1 - g)\big)$.

Hence

$$N\big(e_{0i}(1 - g)\big) = X_{11} \oplus \left(\bigoplus_{\substack{j=2 \\ j \neq i}}^{n} X_{0j} \right). \tag{14.14}$$

Now we turn the opposite way. We define $g(e_{01}, e_{11})$ by (14.10), which is possible as soon as $\|e_{11} - e_{01}\| < 1$. Then we have

$$R(g) = R(e_{11}) \quad \text{and} \quad N(g) = (e_{01}). \tag{14.15}$$

We will show that

$$g \quad \text{and} \quad e_{0i}(1 - g) \quad \text{for} \quad 2 \leq i \leq n \tag{14.16}$$

form an orthogonal system of idempotents summing to 1. Their sum is

$$g + \sum_{i=2}^{n} e_{0i}(1 - g) = g + (1 - e_{01})(1 - g) = 1 + e_{01}g - e_{01} = 1 \tag{14.17}$$

by (13.6).

4. It remained to show that the elements in (13.16) form an orthogonal system of idempotents. We will show more: a simultaneous linear interpolation between e_{01} and g and between e_{0i} and $e_{0i}(1-g)$, for $t \in [0,1]$, gives an orthogonal system of idempotents, summing to 1.

That is

$$(1-t)e_{01} + tg \text{ and } (1-t)e_{0i} + te_{0i}(1-g) \text{ for } 2 \le i \le n \qquad (14.18)$$

are orthogonal systems of idempotents summing to 1.

Since $e_{01} + \sum_{i=2}^{n} e_{0i} = 1$ and $g + \sum_{i=2}^{n} e_{0i}(1-g) = 1$ (cf. (14.17)), the sum of all elements in (14.16) is 1 for all $t \in [0,1]$.

The inclusion

$$(1-t)e_{01} + tg(e_{01}, e_{11}) \in E(A) \qquad (14.19)$$

is contained in Z. V. Kovarik [Ko], p. 347.

For $2 \le i \le n$ we have

$$(1-t)e_{0i} + te_{0i}(1-g) = e_{0i} - te_{0i}g. \qquad (14.20)$$

Its square is

$$e_{0i} - te_{0i}g - te_{0i}ge_{0i} + t^2 e_{0i}ge_{0i}g.$$

Here the third and fourth summands contain

$$g(e_{01}, e_{11})e_{0i} = e_{11}(e_{01} + e_{11} - 1)^{-2}e_{01}e_{02} = 0.$$

Therefore the element (14.20) belongs to $E(A)$.

There remains to show that the elements in (14.18) are pairwise orthogonal.

Let $2 \le i \le n$. Then, using the shorter form in (14.19),

$$[(1-t)e_{01}+tg]\cdot[e_{0i}-te_{0i}g] = (1-t)e_{01}e_{0i}-(1-t)te_{01}e_{0i}g+tge_{02}-t^2ge_{02}g. \qquad (14.21)$$

Here the first and second summand contain $e_{01}e_{0i} = 0$, and the third and fourth summands contain $ge_{02} = 0$. Hence (14.21) is 0.

On the other hand,

$$\begin{aligned}
&[e_{0i} - te_{0i}g] \cdot [(1-t)e_{01} + tg] \\
&= (1-t)e_{0i}e_{01} + te_{0i}g - t(1-t)e_{0i}ge_{01} - t^2 e_{0i}gg \\
&= 0 + te_{0i}g - t(1-t)e_{0i}g - t^2 e_{0i}g = e_{0i}g \cdot (t - t(1-t) - t^2) = 0.
\end{aligned} \qquad (14.22)$$

Now let $2 \le i, j \le n$, $i \ne j$. Then

$$\begin{aligned}
&[e_{0i} - te_{0i}g] \cdot [e_{0j} - te_{0j}g] \\
&= e_{0i}e_{0j} - te_{0i}e_{0j}g - te_{0i}ge_{0j} + t^2 e_{0i}ge_{0j}g.
\end{aligned} \qquad (14.23)$$

Here the first and second summands contain $e_{0i}e_{0j} = 0$, and the third and fourth summands contain

$$g(e_{01}, e_{11})e_{0j} = e_{11}(e_{01} + e_{11} - 1)^2 e_{01}e_{0j} = 0.$$

5. Then multiplying the simultaneous interpolating elements in (12.15)?? (14.15)??, which form for each $t \in [0,1]$ a partition of unity, by the respective λ_i's, and summing them, we obtain a segment in $E_p(A)$, beginning from a_0. We have to show that its other endpoint is $\sum\limits_{i=1}^{n} \lambda_i f_i$, where the f_i's are the projections associated to the direct sum decomposition $X = X_{11} \oplus \left(\bigoplus\limits_{i=2}^{n} X_{0i} \right)$. For $i = 1$ we have seen this in **3**, (14.10), while for $2 \le i \le n$ we have seen this in **3**, (14.13) and (14.14). This means that our procedure changed the direct sum representation $X = \bigoplus\limits_{i=1}^{n} X_{0i}$ to $X = X_{11} \oplus \left(\bigoplus\limits_{i=2}^{n} X_{0i} \right)$, as we wanted to show. Moreover, the respective projections changed just a bit. Then, for $\|a_1 - a_0\|$ sufficiently small, we can change in the second analogous step X_{02} to $X_{12}, \dots$, in the nth analogous step X_{0n} to X_{1n}, the last direct sum decomposition will be thus $X = \bigoplus\limits_{i=1}^{n} X_{1i}$. Even in this last decomposition the respective projections remain close to the original projections e_{0i}, showing local connectedness of $E_p(A)$ by polygons consisting of n segments. Therefore, we obtained in $E_p(A)$ a polygon of n segments, joining $a_0 = \sum\limits_{i=1}^{n} \lambda_i e_0$ to $a_1 = \sum\limits_{i=1}^{n} \lambda_i f_i$, where $\{e_1, \dots, e_n\}, \{f_1, \dots, f_n\} \subset E(A)$ are partitions of unity.

6. As mentioned in **5**, by the first exchange all the respective projections changed just a bit, so they remained close to $e_{01}, \dots, e_{0n}$. The same holds for the second, $\dots$, n^{th} exchange. Then linearly interpolating between the successive respective first, $\dots$, n^{th} projections, we obtain polygonal paths close to the original projections $e_{01}, \dots, e_{0n}$. ∎

Proof of Theorem 15. **1.** We use the same formulas as in the proof of Theorem 13. That is, we define from $a_0 = \sum\limits_{i=1}^{n} \lambda_i e_{0i}$ and $a_1 = \sum\limits_{i=1}^{n} \lambda_i e_{1i}$, where $\|a_1 - a_0\|$ is small, hence by Corollary 5 also $\|e_{1i} - e_{0i}\|$ for each $1 \le i \le n$ is small, the elements

$$g = g(e_{01}, e_{11}), \quad 1 - g, \quad e_{0i}(1-g), \quad (1-t)e_{01} + tg, \quad (1-t)e_{0i} + te_{0i}(1-g)$$
$$\text{for each } 2 \le i \le n, \text{ and each } t \in [0,1].$$
$$(15.1)$$

Recall that in the proof of Theorem 12 we exchanged the direct summands X_{0i} with the direct summands X_{1i}, one by one, in the first step for $i = 1, \dots$, for the nth step for $i = n$. However, we did not directly use these subspaces, but only the respective projections, and the whole proof was done by algebraic manipulations with these projections. At the end of the proof it was established that after the last step of these n exchanges $X_{01}, \dots, X_{0n}$ was exchanged to $X_{11}, \dots, X_{1n}$. Equivalently, the first, $\dots$, n^{th} coordinate projection for the direct sum decomposition $X = \bigoplus\limits_{i=1}^{n} X_{0i}$ were exchanged, after the last of the n steps, by the first, $\dots$, n^{th} coordinate projection for the direct sum decomposition $X = \bigoplus\limits_{i=1}^{n} X_{1i}$.

We have to show that performing these exchanges, the n-tuple $e_{01}, \dots, e_{0n}$ will change just to $e_{11}, \dots, e_{1n}$, also for the case of a Banach algebra, and not only for the case of the algebra of bounded operators on a Banach space.

We recapitulate the formulas describing these exchanges in a form suitable for us. We consider an $(n+1) \times n$ matrix, with entries in $E(A)$, namely

$$
\begin{pmatrix}
f_{01} & \cdots & f_{0n} \\
f_{11} & \cdots & f_{1n} \\
\vdots & & \vdots \\
f_{n1} & \cdots & f_{nn}
\end{pmatrix}. \tag{15.2}
$$

We set $f_{01} = e_{01}, \ldots, f_{0n} = e_{0n}$. For $1 \leq i \leq n$ we set

$$
\begin{aligned}
f_{ii} &= g(f_{i-1,i}, e_{1i}) \quad \text{for} \quad 1 \leq i \leq n, \quad \text{and} \\
f_{ij} &= f_{i-1,j}(1 - g(f_{i-1,i}, e_{1i})) \quad \text{for} \quad 1 \leq i, j \leq n, \quad i \neq j.
\end{aligned} \tag{15.3}
$$

Observe that for $e_{01}, \ldots, e_{0n}$ fixed and $\|e_{1j} - e_{0j}\|$ small we have for each $1 \leq i, j \leq n$ that $\|f_{ij} - e_{0j}\|$ is small. This follows by induction for i. For $j = i$ this follows by continuity of the function $(e, f) \mapsto g(e, f)$ (where $e, f \in E(A)$ are close). For $j \neq i$, by the same continuity f_{ij} is close to $e_{0j}(1 - g(e_{0i}, e_{0i})) = e_{0j}(1 - e_{0i}) = e_{0j}$.

Using the recursive formulas (14.3), we see that each f_{ij} $(0 \leq i \leq n, 1 \leq j \leq n)$ is a "rational" function of $e_{01}, \ldots, e_{0n}, e_{11}, \ldots, e_{1n}$. More exactly, each f_{ij} can be expressed by $e_{01}, \ldots, e_{0n}, e_{11}, \ldots, e_{1n}$ by using the following operations: linear combinations, multiplication, and applying the function $x \mapsto (1-x)^{-1} = \sum_{m=0}^{\infty} x^m$, where $\|x\|$ is small ($\|x\| < 1$ is necessary). Therefore in particular f_{nj}, for each $1 \leq i \leq n$, belongs to the closed subalgebra of A, generated by $e_{01}, \ldots, e_{0n}, e_{11}, \ldots, e_{1n}$. We have to show $f_{nj} = e_{1j}$ for each $2 \leq j \leq n$. (Theoretically there would be a possibility to express $f_{21}, \ldots, f_{2n}$, then further $f_{31}, \ldots, f_{3n}$, etc. by these "rational" functions, superposed to each other, but these formulas soon become very complicated, and seem not to be treatable.)

2. Let us consider the regular representation

$$
\varphi : \ A \to B(A), \tag{15.4}
$$

that maps any $a \in A$ to the (bounded) operator $x \mapsto ax$ (i.e., left multiplication by a). This is a Banach algebra homomorphism, and even is an isometric embedding $A \to B(A)$. In particular, φ is injective.

Then φ preserves linear combinations, multiplication and the function $x \mapsto (1 - x)^{-1}$ (for $\|x\|$ small). Therefore considering $\varphi e_{01}, \ldots, \varphi e_{0n}, \varphi e_{11}, \ldots, \varphi e_{1n} \in E(B(A)) \subset B(A)$, and also $\varphi f_{ij} \in E(B(A)) \subset B(A)$, for each $0 \leq i \leq n_n$ and $1 \leq j \leq n$, we have the following. Each φf_{ij}, $0 \leq i \leq n$, $1 \leq j \leq n$, thus in particular each φf_{nj}, $1 \leq j \leq n$, is expressed by the same formula via $\varphi e_{01}, \ldots, \varphi e_{0n}, \varphi e_{11}, \ldots, \varphi e_{1n}$, as f_{ij}, in particular f_{nj}, is expressed via $e_{01}, \ldots, e_{0n}, e_{11}, \ldots, e_{1n}$. Observe that $\varphi e_{01}, \ldots, \varphi e_{0n}, \varphi e_{11}, \ldots, \varphi e_{1n}$ and $\varphi f_{n1}, \ldots, \varphi f_{nn}$ can be considered as to play the role of $e_{01}, \ldots, e_{0n}, e_{11}, \ldots, e_{1n}$ and $f_{n1}, \ldots, f_{nn}$ for the case of the operator algebra $B(A)$, rather than for A.

However, by Theorem 14 we already know the equalities $f_{nj} = e_{1j}$ for $1 \leq j \leq n$, for the case of $A = B(X)$, where X is a Banach space. In particular, this holds for $B(A)$, i.e., we have

$$
\varphi f_{nj} = \varphi e_{1j} \quad \text{for} \quad 1 \leq j \leq n. \tag{15.5}
$$

Since φ is injective, this implies

$$f_{nj} = e_{1j} \quad \text{for} \quad 1 \leq j \leq n, \tag{15.6}$$

that was to be shown.

3. It remained to prove that the linear interpolations between f_{0i}, f_{1i}, and between $f_{1i}, f_{2i}, \ldots$, and between $f_{n-1,i}, f_{ni}$ together give a polygonal path which?? is close to the original projection $f_{0i} = e_{0i}$, for each $1 \leq i \leq n$. However, this was proved in Theorem 13 for $B(X)$ for any Banach space X, thus in particular for $B(A)$. Now recall that the regular representation $\varphi : A \to B(A)$ is an isometry into. From Theorem 13 we know that the image by φ of this polygonal path in $B(A)$ is close to φe_{0i}, for each $1 \leq i \leq n$. By the isometric property of φ this polygonal path in B is close to E_{0i}, for each $1 \leq i \leq n$. ∎

Proof of Theorem 16. The proof follows from Theorem 14 in an analogous way as the proof of Theorem 7 followed from Theorem 6 (and as the proof of Theorem 11 followed from Theorem 10). ∎

Proof of Corollary 17. Let C be a connected component of $E_p(A)$, or of $S_p(A)$, and let $a_0 \in C$. Then for any $a_1 \in C$, a_1 is similar to a_0 via some invertible, or unitary element of $A = B(H)$, cf. Theorems 7 and 11. This similarity implies the equality of the Hilbert space dimensions of the eigensubspaces of a_0 and a_1 corresponding to the eigenvalue λ_i, for each $1 \leq i \leq n$.

Conversely, let for $a_0, a_1 \in E_p(A)$, or $a_0, a_1 \in S_p(A)$ (in the second case with $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$) the Hilbert space dimensions of the eigensubspaces corresponding to each eigenvalue λ_i, $1 \leq i \leq n$, be equal. Then H is the direct sum of these eigensubspaces, both for a_0 and a_1. In the C^*-algebra case these eigensubspaces are even orthogonal, then we have

$$a_1 = s^{-1} a_0 s,$$

where $s \in B(H)$ is invertible, and in the C^*-algebra case s is unitary. Now recall that $Ge(H)$, as well as the unitary group $U(H)$ of H, is (pathwise) connected. Therefore we have a path in $Ge(H)$, or $U(H)$, from Id to s. Its image by the continuous map $t \mapsto t^{-1} a_0 t$ is a path from a_0 to a_1, in $E_p(A)$, or $S_p(A)$. For the C^*-algebra case (in the second case with $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$). ∎

Proof of Theorem 18. We proceed analogously to Remark C, 2), only will make use of the C^*-algebra property. Also we recall the proof of Theorem 6, and use the notations there.

So we have, also using Propositions 1 and 3 that

$$a_0 = \sum_{i=1}^n \lambda_i e_{0i} \sum_{i=1}^n \lambda_i e_i(a_0), \quad a_1 = \sum_{i=1}^n \lambda_i e_{1i} = \sum_{i=1}^n \lambda_i e_i(a_1). \tag{18.1}$$

As soon as

$$\|e_i(a_1) - e_i(a_0)\| < 1, \quad \text{for each} \quad 1 \leq i \leq n, \tag{18.2}$$

we have that $e_i(a_0), e_i(a_1)$ belong to the same connected component of $S(A)$, and their linear combinations

$$a_0 = \sum_{i=1}^n \lambda_i e_i(a_0), \quad a_1 = \sum_{i=1}^n \lambda_i e_i(a_1)$$

belong to the same connected component of $S_p(A)$. (Observe that then $(e_1(a_0), \ldots, e_n(a_0))$ and $(e_1(a_1), \ldots, e_1(a_n))$ belong to a connected component of $S(A)^n$, as the product of connected sets is connected. Then its image by the linear, thus continuous function $(e_1, \ldots, e_n) \mapsto \sum_{i=1}^{n} \lambda_i e_i$ is also connected.)

We will write

$$a_1 = a_0 + \min_{\substack{1 \le \alpha, \beta \le n \\ \alpha \ne \beta}} |\lambda_\alpha - \lambda_\beta| \cdot \varepsilon, \tag{18.3}$$

where by the problem before Theorem 17 we may suppose $\|\varepsilon\| < 1$. We look for upper estimates of $\|\varepsilon\|$ which assure (18.2). ?? Recall that

$$e_i(\lambda) = \prod_{\substack{j=1 \\ j \ne i}}^{n} (\lambda - \lambda_j) \Big/ \prod_{\substack{j=1 \\ j \ne i}}^{n} (\lambda_i - \lambda_j). \tag{18.4}$$

Hence for $a \in A$, $a = \sum_{k=1}^{n} \lambda_k e_k$ we have by $\sum_{k=1}^{n} e_k = 1$ that the numerator of $e_i(a)$ is

$$\prod_{\substack{j=1 \\ j \ne i}}^{n} (\lambda - \lambda_j) = \prod_{\substack{j=1 \\ j \ne i}}^{n} \left(\sum_{k=1}^{n} \lambda_k e_k - \lambda_j \right) = \prod_{\substack{j=1 \\ j \ne i}}^{n} \left(\sum_{k=1}^{n} (\lambda_k - \lambda_j) e_k \right). \tag{18.5}$$

Hence the numerator of $\|e_i(a_0 + \varepsilon) - e_i(a_0)\|$ is

$$\begin{aligned}
& \left\| \prod_{\substack{j=1 \\ j \ne i}}^{n} \left(a - \lambda_j + \min_{\substack{1 \le \alpha, \beta \le n \\ \alpha \ne \beta}} |\lambda_\alpha - \lambda_\beta| \cdot \varepsilon \right) - \prod_{\substack{j=1 \\ j \ne i}}^{n} (\alpha - \lambda_j) \right\| \\
& = \prod_{\substack{j=1 \\ j \ne i}}^{n} \left(\sum_{k=1}^{n} (\lambda_k - \lambda_j) e_k + \min_{\substack{1 \le \alpha, \beta \le n \\ \alpha \ne \beta}} |\lambda_\alpha - \lambda_\beta| \cdot \varepsilon \right) - \prod_{\substack{j=1 \\ j \ne i}}^{n} \sum_{k=1}^{n} (\lambda_k - \lambda_j) e_k \right\|.
\end{aligned} \tag{18.6}$$

(18.6) is a non-commutative polynomial of ε, but $\sum_{k=1}^{n} (\lambda_k - \lambda_j) e_k$ are self-adjoint commuting elements. Clearly in (18.6) ?? the coefficient of ε° is 0. All other terms contain ε at least once. Performing the multiplications in

$$\prod_{\substack{j=1 \\ j \ne i}}^{n} \left(\sum_{k=1}^{n} (\lambda_k - \lambda_j) e_k + \min_{\substack{1 \le \alpha, \beta \le n \\ \alpha \ne \beta}} |\lambda_\alpha - \lambda_\beta| \varepsilon \right), \tag{18.7}$$

we obtain 2^{n-1} additive terms, each of degree $n - 1$. Each of these additive terms is a product or $n - 1$ factors, each factor being of the form

$$\sum_{k=1}^{n} (\lambda_k - \lambda_j) e_k, \quad \text{or} \quad \min_{\substack{1 \le \alpha, \beta \le n \\ \alpha \ne \beta}} |\lambda_\alpha - \lambda_\beta| \varepsilon. \tag{18.8}$$

The norm of a single such product is at most the product of the norms of the factors. This last product contains some power of $\left\| \min_{\substack{1 \le \alpha, \beta \le n \\ \alpha \ne \beta}} |\lambda_\alpha - \lambda_\beta| \varepsilon \right\|$, say,

$$\left\| \min_{\substack{1 \le \alpha, \beta \le n \\ \alpha \ne \beta}} |\lambda_\alpha - \lambda_\beta| \varepsilon \right\|^k,$$

which we estimate from above by

$$\left(\min_{\substack{1\leq\alpha,\beta\leq n\\ \alpha\neq\beta}}|\lambda_\alpha-\lambda_\beta|\right)^k-\|\varepsilon\| \tag{18.9}$$

(observe that now $k\geq 1$, and $\|\varepsilon\|<1$, so $\|\varepsilon\|^k\leq\|\varepsilon\|$). The upper estimate of the last mentioned single product still contains $n-1-k$ factors of the form

$$\left\|\sum_{k=1}^n(\lambda_k-\lambda_j)e_k\right\|=\max_{1\leq k\leq n}|\lambda_k-\lambda_j|, \tag{18.10}$$

(i.e., here j can take $n-1-k$ values), the equality holding by the C^*-algebra property and $e_k^*=e_k$ (cf. Proposition 3). (We note yet that if at least two factors of the form $\sum_{k=1}^n(\lambda_k-\lambda_j)e_k$ follow each other in a summand after performing the multiplications in (18.7), then further simplifications become possible: since $e_k^2=e_k$ and $e_{k_1}e_{k_2}=0$ for $k_1\neq k_2$, so such a product will be a linear combination of the e_k's only. However, such a better result would give great complications in the estimate of Theorem 18.)

Then (18.6) can be estimated above

$$\left\|\prod_{\substack{j=1\\ j\neq i}}^n\left(\sum_{k=1}^n(\lambda_k-\lambda_j)e_k+\min_{\substack{1\leq\alpha,\beta\leq n\\ \alpha\neq\beta}}|\lambda_\alpha-\lambda_\beta|\right)-\prod_{\substack{j=1\\ j\neq i}}^n\sum_{k=1}^n(\lambda_k-\lambda_j)e_k\right\|\cdot\|\varepsilon\|$$

$$\leq\left[\prod_{\substack{j=1\\ j\neq i}}^n\left(\left\|\sum_{k=1}^n(\lambda_k-\lambda_j)e_k\right\|+\min_{\substack{1\leq\alpha,\beta\leq n\\ \alpha\neq\beta}}|\lambda_\alpha-\lambda_\beta|\right)-\prod_{\substack{j=1\\ j\neq i}}^n\left\|\sum_{k=1}^n(\lambda_k-\lambda_j)e_k\right\|\right]\cdot\|\varepsilon\|. \tag{18.11}$$

(Observe that here writing the first product as the sum of 2^{n-1} summands, one summand cancels with the second product, which corresponds to the fact that in (18.7) the coefficient of ε° is 0.) Then (18.11) with (18.10) imply

$$\left[\prod_{\substack{j=1\\ j\neq i}}^n\left(\max_{\substack{1\leq\alpha,\beta\leq n\\ \alpha\neq\beta}}|\lambda_k-\lambda_j|+\min_{\substack{1\leq\alpha,\beta\leq n\\ \alpha\neq\beta}}|\lambda_\alpha-\lambda_\beta|\right)-\prod_{\substack{j=1\\ j\neq i}}^n\max_{1\leq k\leq n}|\lambda_k-\lambda_j|\right]\cdot\|\varepsilon\|\ \Big/\ \prod_{\substack{j=1\\ j\neq i}}^n|\lambda_i-\lambda_j|. \tag{18.12}$$

Now recall (18.12) and the considerations following it. Then, as soon as

$$\|e_i(a_1)-e_i(a_0)\|=\|e_i(a_0+\varepsilon)-e_i(a_0)\|<1\ \ \text{for each}\ \ 1\leq i\leq n, \tag{18.13}$$

a_0 and a_1 belong to the same connected component of $S_p(A)$. That is, for

$$\max_{1\leq i\leq n}\|e_i(a_1)-e_i(a_0)\|<1$$

a_0 and a_1 belong to the same connected component of $S_p(A)$. Said otherwise, if a_0 and a_1 belong to different connected components of $S_p(A)$, then

$$\max_{1\leq i\leq n}\|e_i(a_i)-e_i(a_0)\|\geq 1.$$

Then (18.13) gives for $\min\limits_{\substack{1\le \alpha, \beta\le n \\ \alpha\ne\beta}} |\lambda_\alpha - \lambda_\beta|\ \ \|\varepsilon\| = \|a_1 - a_0\|$ the lower estimate given in the theorem. ∎

We give the proof of Theorem 19, and then will indicate the necessary changes to prove Theorems 20, 21 and 22.

Proof of Theorem 19. Let $\lambda \in U$, thus $b(\lambda) \in E_p(B)$. By Proposition 1, "only if" part

$$b(\lambda) = \sum_{i=1}^{n} \lambda_i f_i(\lambda) = \sum_{i=1}^{n} \lambda_i e_i(b(\lambda)),$$

where $\{f_1(\lambda), \ldots, f_n(\lambda)\} \subset E(B)$ is a partition of unity,

with $e_i(\cdot)$ the polynomials from Proposition 1. That is, $f_1(\lambda), \ldots, f_n(\lambda)$ are mutually orthogonal, and $f_1(\lambda) + \ldots + f_n(\lambda) = 1$. Observe that $U \ni \lambda \mapsto e_i(b(\lambda)) \in E(B) \subset B$, as a composition of two analytic functions, is analytic. By [AMMZ14], Theorem 3, using the hypotheses of this theorem, this analytic family of mutually orthogonal idempotents admits a lifting $\tilde{f}_i(\cdot) : U \to E(A) \subset A$, which is also an analytic family of mutually orthogonal idempotents, thus satisfies

$$\pi(\lambda)\tilde{f}_i(\lambda) = f_i(\lambda) \ \ \text{for each} \ \ \lambda \in U.$$

Of course we cannot guarantee $\tilde{f}_1(\lambda) + \ldots + \tilde{f}_n(\lambda) = 1$. Therefore we replace $\tilde{f}_n(\cdot)$ by $1 - \tilde{f}_1(\cdot) - \ldots - \tilde{f}_{n-1}(\cdot)$, which is also an analytic family of idempotents, and which forms with $\tilde{f}_1(\cdot), \ldots, \tilde{f}_{n-1}(\cdot)$ a mutually orthogonal system of idempotents, summing to 1, i.e., we have a partition of unity in $E(A)$.

Once more we apply Proposition 1, now the "if" part. Thus

$$\lambda_1 \tilde{f}_1(\cdot) + \ldots + \lambda_{n-1} \tilde{f}_{n-1}(\cdot) + \lambda_n(1 - \tilde{f}_1(\cdot) - \ldots - \tilde{f}_{n-1}(cdot)) \in E_p(A),$$

and thus we have in the last display an analytic family of elements of $E_p(A)$. Still we have to prove that this analytic family lifts $b(\cdot)$. In fact,

$$\pi(\lambda)\big(\lambda_1\tilde{f}_1(\lambda) + \ldots + \lambda_{n-1}\tilde{f}_{n-1}(\lambda)\big) + \lambda_n(1 - \tilde{f}_1(\lambda) - \ldots - \tilde{f}_{n=1}(\lambda))$$
$$= \lambda_1 f_1(\lambda) + \ldots + \lambda_{n-1}f_{n-1}(\lambda) + \lambda_n\big(1 - f_1(\lambda) - \ldots - f_{n-1}(\lambda)\big)$$
$$= \lambda_1 f_1(\lambda) + \ldots + \lambda_{n-1}f_{n-1}(\lambda) + \lambda_n f_n(\lambda) = b(\lambda).$$

∎

Proof of Theorem 20. We replace in the proof of Theorem 19 Proposition 1 by Proposition 3, and [AMMZ14] Theorem 3 by [AMMZ14], Theorem 4. ∎

Proof of Theorem 21. We use Proposition 1 and [AMMZ14], Theorem 5. Thus we obtain liftings $\tilde{f}_1(\lambda), \ldots, \tilde{f}_{n-1}(\lambda), 1 - \tilde{f}_1(\lambda) - \ldots - \tilde{f}_{n-1}(\lambda)$, not for each $\lambda \in U$, but only for $\lambda \in V_1, \ldots, \lambda \in V_{n-1}, \lambda \in V_1 \cap \ldots \cap V_{n-1}$, respectively, for some open subsets $V_1, \ldots, V_{n-1}$ of U, each containing 0. Then Theorem 21 holds for $V := V_1 \cap \ldots \cap V_{n-1}$. ∎

Proof of Theorem 22. We use Proposition 3, and [AMMZ14], Theorem 6, and finish like in the proof of Theorem 21. ∎

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