BALL CHARACTERIZATIONS IN PLANES AND SPACES OF CONSTANT CURVATURE, I

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Abstract. High proved the following theorem. If the intersections of any two congruent copies of a plane convex body are centrally symmetric, then this body is a circle. In our paper we extend the theorem of High to the sphere and the hyperbolic plane.

Let us have in S^2, $\mathbb{R}^2$ or H^2 a pair of convex bodies (for S^2 different from S^2), such that the intersections of any congruent copies of them are centrally symmetric. Then our bodies are congruent circles. If the intersections of any congruent copies of them are axially symmetric, then our bodies are (incongruent) circles.

Let us have in S^2, $\mathbb{R}^2$ or H^2 proper closed convex subsets K, L with interior points, such that the numbers of the connected components of the boundaries of K and L are finite. If the intersections of any congruent copies of K and L are centrally symmetric, then K and L are congruent circles, or, for $\mathbb{R}^2$, parallel strips. For $\mathbb{R}^2$ we exactly describe all pairs of such subsets K, L, whose any congruent copies have an intersection with axial symmetry (there are five cases).

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1. Introduction

We write S^d, $\mathbb{R}^d$, H^d, with $d \geq 2$, for the d-dimensional spherical, Euclidean and hyperbolic spaces, resp. *Convexity of a set* $K \subset H^d$ is defined as for $K \subset \mathbb{R}^d$. *Convexity of* $K \subset S^d$, *with* $\operatorname{int} K \neq \emptyset$, is meant as follows: for any two non-antipodal points of K the shorter great circle arc connecting them belongs to K. Then for $\pm x \in K$, $y \in \operatorname{int} K$ and $y \neq \pm x$, the shorter arcs $\widehat{(\pm x)y}$ belong to K, hence some half large circle connects $\pm x$ in K. By a *convex body* in S^d, $\mathbb{R}^d$, H^d we mean a compact convex set, with nonempty interior. *In* S^d, when saying *ball, or sphere*, we always mean one *of radius at most* $\pi/2$ (thus a ball is convex). A proper closed convex subset of S^d, $\mathbb{R}^d$ or H^d, with nonempty interior, is *strictly convex*, if its boundary does not contain a non-trivial segment. A *convex surface* is the boundary of a proper closed convex subset of S^d, $\mathbb{R}^d$ or H^d with nonempty interior. For $d = 2$ a convex surface will be called a *convex curve.*

R. High proved the following theorem.

Theorem. ([7]) *Let* $K \subset \mathbb{R}^2$ *be a convex body. Then the following statements are equivalent:*

(1) *All intersections* $(\varphi K) \cap (\psi K)$*, having interior points, where* $\varphi, \psi : \mathbb{R}^2 \to \mathbb{R}^2$ *are congruences, are centrally symmetric.*
(2) *K is a circle.* ■

It seems, that his proof gives the analogous statement, when φ, ψ are only allowed to be orientation preserving congruences.

Problem 1. Describe the pairs of closed convex sets with interior points, in S^d, $\mathbb{R}^d$ and H^d, whose any congruent copies have a centrally symmetric intersection, provided this intersection has interior points. Evidently, two congruent balls (for S^d of radii at most $\pi/2$), or two parallel slabs in $\mathbb{R}^d$, have a centrally symmetric intersection, provided it has a nonempty interior.

It was proved in [8], Theorem 2, that in the C^2 case for S^d, and in the C^2_+ case for $\mathbb{R}^d$ and H^d, the only possibility is two congruent balls (for S^d of radii at most $\pi/2$).

The authors are indebted to L. Montejano (Mexico City) and G. Weiss (Dresden) for having turned their interest to characterizations of pairs of convex bodies with all translated/congruent copies having a centrally or axially symmetric intersection or convex hull of the union, resp., or with other symmetry properties, e.g., having some affine symmetry.

The aim of our paper is to give partial answers to this problem. To exclude trivialities, we always suppose that *our sets are different from the whole plane, or space*, and also we investigate only such cases, when the *intersection has interior points*. We prove the analogue of the theorem of High for S^2 and H^2. Namely, we characterize the pairs of proper closed convex subsets with interior points, in S^2, $\mathbb{R}^2$ and H^2, having centrally symmetric intersections of all congruent copies, provided these intersections have nonempty interiors. However, for H^2 we have to suppose that if the connected components of the boundaries of both subsets are straight lines, then there are altogether finitely many of them. Also we investigate a variant of this question, for S^2 and $\mathbb{R}^2$, when we prescribe not central but axial symmetry of all intersections, having nonempty interiors. We exactly describe all pairs of proper closed convex subsets with interior points, with the above property: for S^2 and $\mathbb{R}^2$ there are one and five cases, resp. (The case of H^2 is postponed to [9].)

Suppose that in S^2, $\mathbb{R}^2$ and H^2, all small intersections of congruent copies of two closed convex proper subsets with interior points, having a nonempty interior, admit some non-trivial congruence. Then all connected components of the boundaries of the two sets are cycles or straight lines.

We plan to publish additional results, related to [8] and to this paper, in [9] and [10], with subjects described in their titles.

Surveys about characterizations of central symmetry, for convex bodies in $\mathbb{R}^d$, cf. in [3], §14, pp. 124-127, and, more recently, in [6], §4.

2. New results, Theorems 1–4

We mean by a *non-trivial congruence* a congruence different from the identity. We write $\mathrm{conv}\,(\cdot)$, $\mathrm{diam}\,(\cdot)$, $\mathrm{int}\,(\cdot)$, $\mathrm{cl}\,(\cdot)$, $\mathrm{bd}\,(\cdot)$ and $\mathrm{perim}\,(\cdot)$ for the convex hull, diameter, interior, closure, boundary and perimeter of a set. A *paracircle* (also called a horocircle) is a closed convex set in H^2, bounded by a paracycle.

As a general hypothesis in our theorems, we have that

$$(1)\quad \begin{cases} X \text{ is } S^d,\ \mathbb{R}^d \text{ or } H^d,\ \text{with } d \geq 2,\ \text{and } K, L \subsetneqq X \text{ are closed convex sets} \\ \text{with interior points. Moreover, } \varphi, \psi : X \to X,\ \text{sometimes with indices,} \\ \text{are orientation preserving congruences, with } \mathrm{int}\,[(\varphi K) \cap (\psi L)] \neq \emptyset. \end{cases}$$

Sometimes we will say *direct/indirect congruence* for orientation preserving/reversing congruence.

The following Theorem 1 is the basis of our considerations.

Theorem 1. *Assume* (1) *with* $d = 2$. *Then we have* (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3), *where*

(1) *There exists some* $\varepsilon(K, L) > 0$, *such that for each* φ, ψ, *for which* diam $[(\varphi K) \cap (\psi L)] \leq \varepsilon(K, L)$, *we have that* $(\varphi K) \cap (\psi L)$ *is axially symmetric.*

(2) *There exists some* $\varepsilon(K, L) > 0$, *such that for each* φ, ψ, *for which* diam $[(\varphi K) \cap (\psi L)] \leq \varepsilon(K, L)$, *we have that* $(\varphi K) \cap (\psi L)$ *admits some non-trivial congruence.*

(3) *Each connected component of the boundaries of both* K *and* L *is a cycle (for* $X = S^2$ *a circle of radius at most* $\pi/2$*), or a straight line. If either for* K*, or for* L*, one connected component is a circle, or paracycle, then this is the unique component, and* K*, or* L *is a circle (disk), or a paracircle, resp.*

In particular, if the congruences in (2) *are central symmetries, then in* (3) *the connected components of the boundaries of both* K *and* L *are congruent.*

For $X = S^2$ *and* $X = \mathbb{R}^2$ *we have* (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3). *For* $X = H^2$, *if both for* K *and* L, *the infimum of the positive curvatures of its boundary components is positive, and at most one of its boundary components has* 0 *curvature, then* (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3).

Let $X = H^2$. *If for, e.g.,* K, *the infimum of the positive curvatures of its boundary components is* 0, *or two of its boundary components have* 0 *curvatures, then* (3) $\not\Longrightarrow$ (2). *Even, supposing* (3) *for* K, *we may prescribe in any way the curvatures of the connected hypercycle or straight line boundary components of* K *(with multiplicity), in case that the infimum of the positive curvatures of the boundary* *components of K is 0, or two boundary components of K have 0 curvatures. Then we can find a K with these prescribed curvatures of the connected hypercycle or straight line boundary components of K (with multiplicity), and an L, such that for them* (3) *holds, but* (2) *does not hold.*

As follows from Theorem 1, the compact case is particularly simple. This of course includes the case when $X = S^2$.

Theorem 2. *Assume* (1) *with $d = 2$. Let both K and L be compact. Alternatively, as a particular case of this, let $X = S^2$.*

Then we have (1) $\iff$ (2) $\iff$ (3), *where*

(1) *For each φ, ψ we have that $(\varphi K) \cap (\psi L)$ is centrally symmetric.*
(2) *There exists some $\varepsilon(K, L) > 0$, such that for each φ, ψ, for which* diam $[(\varphi K) \cap (\psi L)] \leq \varepsilon(K, L)$, *we have that $(\varphi K) \cap (\psi L)$ is centrally symmetric.*
(3) *K and L are congruent circles, for S^2 of radius at most $\pi/2$.*

Also we have (4) $\iff$ (5) $\iff$ (6) $\iff$ (7) $\iff$ (8), *where*

(4) *For each φ, ψ we have that $(\varphi K) \cap (\psi L)$ is axially symmetric.*
(5) *For each φ, ψ we have that $(\varphi K) \cap (\psi L)$ admits some non-trivial congruence.*
(6) *There exists some $\varepsilon(K, L) > 0$, such that for each φ, ψ, for which* diam $[(\varphi K) \cap (\psi L)] \leq \varepsilon(K, L)$, *we have that $(\varphi K) \cap (\psi L)$ is axially symmetric.*
(7) *There exists some $\varepsilon(K, L) > 0$, such that for each φ, ψ, for which* diam $[(\varphi K) \cap (\psi L)] \leq \varepsilon(K, L)$, *we have that $(\varphi K) \cap (\psi L)$ admits some non-trivial congruence.*
(8) *K and L are* (in general incongruent) *circles, for S^2 of radii at most $\pi/2$.*

Theorem 3. *Assume* (1) *with $d = 2$ and let $X = \mathbb{R}^2$. Then we have* (1) $\iff$ (2), *where*

(1) *For each φ, ψ we have that $(\varphi K) \cap (\psi L)$ admits some non-trivial congruence.*
(2) *K and L are* (in general incongruent) *circles, or one of them is a circle and the other one is a parallel strip or a half-plane, or they are two parallel strips, or they are two half-planes.*

In particular, writing in (1) *central symmetries (rather than non-trivial congruences) is equivalent to writing in* (2) *either two congruent circles or two parallel strips. Similarly, writing in* (1) *axial symmetries is equivalent to adding to* (2) *that for the case of two parallel strips, these strips are congruent.*

The following two theorems give two different characterizations for H^2, under different additional hypotheses. Of these, Theorem 4 deals with central symmetry, and Theorem 5 (in Part II, i.e., [9]) deals with axial symmetries, or non-trivial congruences. Recall Theorem 1, (2) $\Longrightarrow$ (3).

Theorem 4. *Assume* (1) *with $d = 2$ and let $X = H^2$. If all connected components of the boundaries of both of K and L are straight lines, let their total number be finite. Then we have* (1) $\iff$ (2), *where*

(1) *For each φ, ψ we have that $(\varphi K) \cap (\psi L)$ is centrally symmetric.*
(2) *K and L are congruent circles.*

Problem 2. Is the finiteness hypothesis in Theorem 4 necessary?

Remark 1. As will be seen from the proof of Theorem 4, namely in the proof of Lemma 4.1, rather than the finiteness hypothesis in Theorem 4, we may suppose only the following. There holds one of the following properties.
(1) One of K and L, e.g., K has a boundary component K_1, with the following property. Let us pass on $\mathrm{bd}\, K$, meant in B^2 containing the model circle, in the positive sense. Then there is a non-empty open arc A_K of S^1, which immediately follows K_1 on this boundary (thus K_1 and A_K have a common infinite point), and which does not contain any infinite point of any boundary component of K. Simultaneously, the other set L has a boundary component L_1, with the following property. Let us pass on $\mathrm{bd}\, L$, meant in B^2 containing the model circle, in the negative sense. Then there is a non-empty open arc A_L of S^1, which immediately follows L_1 on this boundary (thus L_1 and A_L have a common infinite point), and which does not contain any infinite point of any boundary component of L.
(2) The same as (1), only with "positive sense" and "negative sense" interchanged. As will follow from Part II (i.e., [9]), Remark 3, rather than the finiteness hypothesis in Theorem 4, we may suppose the following.
(3) Either K, or L has two boundary components with at least one common infinite point.

In the proofs of our Theorems we will use some ideas of [7].

3. Preliminaries

We write $\| \cdot \|$ for the norm of a vector, and B^d for the closed unit ball, in $\mathbb{R}^d$. For x, y in $S^d, \mathbb{R}^d$ or H^d, we write $d(x, y)$ for their distance in the respective space, and $[x, y]$ or (x, y) for the closed or open segment (shorter segment in S^d) with end-points x, y. (We will not apply this last notation for antipodal points on S^d.) For x a point, and A a subset of $S^d, \mathbb{R}^d$ or H^d, we write $\mathrm{dist}\,(x, A)$ for the distance of x to A. The line xy is the line spanned by x, y (this notation will not be applied for $x = y$, or for S^d and $x + y = 0$). Suppose for $d = 2$, that x_1, x_2 on the boundary of a closed convex set $K \subset X$ with interior points are "close" to each other. Then we write $\widehat{x_1 x_2}$ *for the (shorter, or unique) arc of* $\mathrm{bd}\, K$*, with these endpoints*, which set K will be clear from the context. Also we will specify, whether we mean a closed or an open arc.

We recall that if the boundary of a closed convex set $K \subset \mathbb{R}^d$ with interior points is differentiable, then it is C^1. By the collinear models of S^d and H^d, this takes over to S^d and H^d. We will say in this case that K *is* C^1.

We write X for S^d, $\mathbb{R}^d$, H^d for $d \geq 2$. (Except in Part II, i.e., [9], Theorems 6 and 7, we will be concerned with the case $d = 2$.) For $x \in X$ and $r > 0$, we write $B(x, r) \subset X$ for the *closed ball in* X*, of centre* x *and radius* r (for $X = S^d$ supposing $r \leq \pi/2$). For $Z \subset Y \subset X$ we write $\mathrm{relint}_Y Z$ for the relative interior of Z w.r.t. Y.

For hyperbolic plane geometry we refer to [2], [4], [11], [12]. For geometry of hyperbolic space we refer to [1], [5]. For elementary differential geometry we refer to [15].

We shortly recall some of the concepts to be used later.

Two straight lines in H^2 can have at least one common finite point, then they are *intersecting*. They can have at least one common infinite point, then they are *parallel*. (In these cases coincidence of the lines is allowed.) If none of these cases occurs, then they are *ultraparallel*.

In S^2, $\mathbb{R}^2$ and H^2 there are the following (maximal, connected, twice differentiable) curves of constant curvature (in S^2 meaning geodesic curvature). In S^2 these are the circles, of radii $r \in (0, \pi/2]$, with (geodesic) curvature $\cot r \in [0, \infty)$. In $\mathbb{R}^2$, these are circles of radii $r \in (0, \infty)$, with curvature $1/r \in (0, \infty)$; and straight lines, with curvature 0. In H^2, these are circles of radii $r \in (0, \infty)$, with curvature $\coth r \in (1, \infty)$; paracycles, with curvature 1; and hypercycles, i.e., distance lines, with distance $l \in (0, \infty)$ from their base lines (i.e., the straight lines that connect their points at infinity), with curvature $\tanh l \in (0, 1)$; and straight lines, with curvature 0. Paracycles are also called horocycles, and their unique infinite points are also called their *centres*. Either in S^2, $\mathbb{R}^2$, or in H^2, each sort of the above curves have different curvatures, and for one sort, with different r or l, they also have different curvatures. The common name of these curves is, except for straight lines in $\mathbb{R}^2$ and H^2, *cycles*. In S^2 also a great circle is called a *cycle*, but when speaking about straight lines, for S^2 this will mean great circles. An elementary method for the calculation of these curvatures in H^2 cf. in [16].

Sometimes we will include straight lines among the hypercycles (with $l = 0$). Then the base line of a straight line is meant to be itself. In this case cycles occur as orthogonal trajectories of all straight lines incident to a finite point p (all circles with centre p); of all straight lines incident to an infinite point q (all paracycles with infinite point q); of all straight lines orthogonal to a straight line l (all hypercycles with base line l).

The space H^d has two usual models, in $\operatorname{int} B^d$, namely the collinear (Beltrami-Caley-Klein) model (cf. [17]), and the conformal (Beltrami-Poincaré) model (cf. [19]). (Sometimes we will consider the closed unit ball, when we have to consider the infinite points as well. Then we say closure of the model in $\mathbb{R}^d$.) In analogy, we will speak about collinear and conformal models of S^d in $\mathbb{R}^d$. By this we mean the ones obtained by central projection (from the centre), or by stereographic projection (from the north pole), to the tangent hyperplane of S^d, at the south pole, in $\mathbb{R}^{d+1}$. These exist of course only on the open southern half-sphere, or on S^d minus the north pole, resp. Their images are $\mathbb{R}^d$. We call the *centre of the model* the south pole of S^d. The collinear and conformal models of $\mathbb{R}^d$ are meant as itself, with *centre* the origin.

Sometimes we will consider the (collinear or conformal) model circle of H^2 as the unit circle of the complex plane $\mathbb{C}$. Thus we will speak about its points 1, i, etc.

A *paraball* (also called a horoball) is a closed convex set in H^d, bounded by a parasphere.

The congruences of S^2, $\mathbb{R}^2$ and H^2 can be given as follows (cf. [1], p. 70, and Theorems 4.1 and 4.2, where H^2 denotes the Poincaré upper halfplane model, [1], p. 43). The orientation preserving ones are rotations in S^2, rotations and translations in $\mathbb{R}^2$, and rotations, "rotations about an infinite point", and translations along a straight line (preserving this line) in H^2. The orientation reversing ones are glide reflections in each of S^2, $\mathbb{R}^2$ and H^2. For H^2 each congruence can be uniquely extended by continuity to the closure of the (collinear or conformal) model circle in $\mathbb{R}^2$, to a homeomorphism of this closure.

For a topological space Y we say that some property of a point $y \in Y$ holds *generically*, if it holds outside a nowhere dense closed subset.

4. Proofs of Theorems 1–4

In the proofs of our theorems by the *boundary components of a set* we will mean the connected components of the boundary of that set.

The following (2), (3), (4) and (5) will serve to exclude in H^2 in many cases congruences of $(\varphi K)\cap(\psi L)$, which are non-trivial translations, or glide reflections which are not reflections, or non-trivial rotations about an infinite point, or rotations about a finite point which are not central symmetries.

If a closed convex set $M \subsetneqq H^2$ with nonempty interior admits a glide reflection which is not a reflection, as a congruence to itself, then it admits the square of this glide reflection as well, which is a non-trivial translation. (Therefore we will not need to exclude glide reflections which are not reflections, but exclusion of non-trivial translations will suffice.) If M admits a non-trivial translation, then it contains the closed convex hull of the orbit of some point, w.r.t. the subgroup generated by this translation. Thus, M contains a straight line. Consequently

(2)
$$\begin{cases} \text{Let a closed convex set } M \subsetneqq H^2 \text{ with nonempty interior admit a non-trivial} \\ \text{translation, or a glide reflection which is not a reflection. Then } M \text{ has two} \\ \text{different infinite points (in particular, it is not compact).} \end{cases}$$

In an analogous way we have

(3)
$$\begin{cases} \text{Let a closed convex set } M \subsetneqq H^2 \text{ with nonempty interior admit a non-trivial} \\ \text{rotation about an infinite point. Then } M \text{ contains a paracircle with this} \\ \text{infinite point (centre; in particular, } M \text{ is not compact).} \end{cases}$$

Now let $x_1x_2x_3x_4$ be a strictly convex quadrangle in H^2. Suppose that it admits a congruence χ which is a *combinatorial central symmetry* of our quadrangle (i.e., $\chi x_1 = x_3$, $\chi x_2 = x_4$, $\chi x_3 = x_1$ and $\chi x_4 = x_2$). Then χ is orientation preserving. Further, $\{x_1, x_3\}$ is invariant under χ, hence also $[x_1, x_3]$ is invariant, and also its midpoint o is invariant. Since χ is orientation preserving, it is a rotation about o, and from above the angle of rotation is π. Hence χ is a central symmetry of our quadrangle, w.r.t. a unique centre o. In fact, if there were two such centres, then the composition of the two central symmetries would be a non-trivial translation admitted by our quadrangle, contradicting (2). Rephrasing this,

$$(4)\quad \begin{cases} \text{Let } x_1x_2x_3x_4 \text{ be a strictly convex quadrangle in } H^2. \text{ Suppose that} \\ \text{it admits a congruence } \chi \text{ which is combinatorial central symmetry} \\ \text{(cf. above). Then } \chi \text{ is a central symmetry w.r.t. a unique centre } o. \end{cases}$$

Further, let a closed convex set $M \subsetneqq H^2$ with nonempty interior have a connected boundary component M' of its boundary, with the following property. It is invariant either under a non-trivial translation, or a glide reflection which is not a reflection, or a non-trivial rotation about an infinite point. Suppose that M' has a nonsmooth point. Then it has infinitely many nonsmooth points. Rephrasing this,

(5)
$$\begin{cases} \text{Let a closed convex set } M \subsetneqq H^2 \text{ with nonempty interior have a boundary} \\ \text{component } M', \text{ which has a positive finite number of nonsmooth points.} \\ \text{Then a non-trivial congruence admitted by } M' \text{ is either a rotation about} \\ \text{a finite point (e.g., a central symmetry), or is a reflection w.r.t. a straight} \\ \text{line (according to as the congruence is orientation preserving, or orientation} \\ \text{reversing).} \end{cases}$$

Let $X = S^2$, $\mathbb{R}^2$ or H^2. Let $K \subset X$ be a closed convex set with interior points. Let $x \in \operatorname{bd} K$, and let x' be a point of $\operatorname{bd} K$ very close to x, that follows x on $\operatorname{bd} K$ in the positive sense. We will often consider the *shorter counterclockwise arc* $\widehat{xx'}$ *of* $\operatorname{bd} K$. This makes sense if K is compact, i.e., $\operatorname{bd} K$ is homeomorphic to S^1. If the connected component of the boundary of K, containing x, is homeomorphic to $\mathbb{R}$, and thus connects two possibly coincident infinite points, then there is just one such arc. Later, when writing *shorter arc*, we mean the shorter one in the first case (the other arc has a length almost the perimeter of K) and the unique one in the second case.

The *distortion of the arc element in* $X = S^d$, $\mathbb{R}^d$ *or* H^d, *in the collinear, or conformal model, resp.*, is the quotient of the corresponding arc element in the collinear, or conformal model, resp., as a subset of $\mathbb{R}^d$, and of the arc element in X. For $x, y \in X$ we write x', y' for their images in the collinear or conformal model, and $d'(x', y')$ for the distance of x', y' in the collinear, or conformal model, resp., as a subset of $\mathbb{R}^d$. (We will always tell, which model do we mean.)

We recall ([17]) and ([19]) that in the collinear, or the conformal model of H^d, resp., the arc element at $x' \in \operatorname{int} B^d$ is given by

(6)
$$\begin{cases} ds^2 = \|dx'\|^2/(1-\|x'\|^2) + (\langle x', dx'\rangle)^2/(1-\|x'\|^2)^2 \in \big[\|dx'\|^2, \|dx'\|^2 \times \\ [1/(1-\|x'\|^2) + \|x'\|^2/(1-\|x'\|^2)^2]\big], \text{ or } ds^2 = 4\|dx'\|^2/(1-\|x'\|^2)^2, \text{ resp.} \\ \text{So, in compact sets } C \subset H^d, \text{ the distortion of the arc element in } H^d, \text{ in the} \\ \text{collinear, or conformal model, resp., is bounded below and above. Hence,} \\ \text{for distinct } x, y \in C, \text{ we have that } d'(x', y')/d(x, y) \text{ is bounded below and} \\ \text{above.} \end{cases}$$

The first statement of the following (7) is elementary.

$$(7) \quad \begin{cases} \text{In compact sets } C \text{ of the open southern hemisphere of } S^d, \text{ the} \\ \text{distortion of the arc element in } S^d, \text{ in the collinear, or conformal} \\ \text{model, resp., is bounded below and above. Hence, for distinct } x, y \\ \in C, \text{ we have that } d'(x', y')/d(x, y) \text{ is bounded below and above.} \end{cases}$$

Now we prove the last statements in (6) and in (7).

Observe that in the collinear model geodesic segments (shorter geodesic segments in the open southern hemisphere of S^d) are preserved, hence the distances, in X, or in $\mathbb{R}^d$, can be obtained by integrating the respective arc-elements along them. This proves the last statements for the collinear model.

For the conformal model we may suppose that $C \subset X$ is a ball, whose centre is mapped to 0 in the model, and for the case of S^d that its radius is smaller than $\pi/2$. Thus $C \subset X$ is convex, and also its image C' in the conformal model is convex (it is a ball of centre 0). Now let us consider for $x, y \in C$ the (shorter) geodesic segment $[x, y]$ in X, which lies in C. Its image in the model is a curve joining x' and y', having a length at most $\mathrm{const}_C \cdot d(x, y)$, hence $d'(x', y') \leq \mathrm{const}_C \cdot d(x, y)$. Changing the role of C and C', in the same way we gain $d(x, y) \leq \mathrm{const}_C \cdot d'(x', y')$.

Proof of Theorem 1. **1.** The implication (1) $\Longrightarrow$ (2) in Theorem 1 is evident.

We turn to the proof of the implication (2) $\Longrightarrow$ (3) in Theorem 1. This will be finished by Lemma 1.7.

The following lemma is surely known, but we could not locate a proof for it. Therefore we give its simple proof. It is some analogue of (6) and (7).

Lemma 1.1. *Let $X = S^2$ or $X = H^2$. Let $o \in X$ be fixed, and let its image in the collinear model be the centre of the model. Let $p \in X$, let $d(o, p) \leq r$, and let us consider an angle in X with apex p. Then the quotient of the measure of the image of our angle in the collinear model, as a subset of $\mathbb{R}^2$, and of the measure of this angle in X lies in $[\cos r, 1/\cos r]$ for $X = S^2$ and $r < \pi/2$, and in $[1/\cosh r, \cosh r]$ for $X = H^2$. In both cases the lower and upper bounds are sharp.*

Proof. **1.** We begin with the case of S^2. Its collinear model is obtained by central projection in $\mathbb{R}^3$ of the open southern hemisphere, from the origin, to the tangent plane of S^2 at the south pole. We may assume $d(o, p) = r$, and $p = (\sin r, 0, -\cos r)$. In the tangent plane of S^2 at p we consider an orthonormal coordinate system with basis vectors $e_1' := (\cos r, 0, \sin r)$ and $e_2' := (0, 1, 0)$. We consider a rotating unit vector in this coordinate system, given by $f(\Phi) := (e_1' \cos \Phi, e_2' \sin \Phi)$, for $\Phi \in [0, 2\pi]$. Let $\varepsilon \in (0, \pi/2 - r)$. Then consider p and

$$p + \tan\varepsilon \cdot f(\Phi) = (\sin r + \tan\varepsilon \cdot \cos r \cdot \cos\Phi, \tan\varepsilon \cdot \sin\Phi, -\cos r + \tan\varepsilon \cdot \sin r \cdot \cos\Phi) \tag{8}$$

(which lies in the open half-space $z < 0$ by $\varepsilon < \pi/2 - r$) and their images p' and $[p + \tan\varepsilon \cdot f(\Phi)]'$ by the map $(x, y, z) \mapsto (-x/z, -y/z, -1)$ from the open lower half-space to the plane $z = -1$. (This map, when restricted to the open southern hemisphere of S^2, gives the map to the collinear model. The image of $p + \tan\varepsilon \cdot f(\Phi)$ by this map runs over the image of the small circular line on S^2, of centre p, and radius ε.) Then $p' = (\tan r, 0, -1)$, and

$$[p + \tan\varepsilon \cdot f(\Phi)]' = \left(\frac{\sin r + \tan\varepsilon \cdot \cos r \cdot \cos\Phi}{\cos r - \tan\varepsilon \cdot \sin r \cdot \cos\Phi}, \frac{\tan\varepsilon \cdot \sin\Phi}{\cos r - \tan\varepsilon \cdot \sin r \cdot \cos\Phi}, -1 \right). \tag{9}$$

Substracting from the last expression p', after some simplification we get

$$\left(\frac{\tan\varepsilon \cdot \cos\Phi}{(\cos r - \tan\varepsilon \cdot \sin r \cdot \cos\Phi \cdot \cos r)\cos r}, \frac{\tan\varepsilon \cdot \sin\Phi}{\cos r - \tan\varepsilon \cdot \sin r \cdot \cos\Phi \cdot \cos r}, 0 \right). \tag{10}$$

Then the slope of the line with this direction vector in the plane is $\tan\Phi \cdot \cos r$. The angle of this line with the basic vector e_1' is $\arctan(\tan\Phi \cdot \cos r)$.

Obviously it is sufficient to prove the statement of the lemma for "infinitesimally small" angles. Then integration will prove the same inequality for "finite" angles.

Therefore, differentiating $\arctan(\tan\Phi \cdot \cos r)$ w.r.t. Φ, after some simplifications this derivative becomes

$$(\cos r)/(1 - \sin^2 r \cdot \sin^2 \Phi) \in [\cos r, 1/\cos r]. \tag{11}$$

Here both endpoints of the interval are attained, for $\Phi = 0, \pi$, and for $\Phi = \pi/2, 3\pi/2$, resp. (The tan function has a singularity at $\Phi = \pi/2, 3\pi/2$. However, then we can exchange the basic vectors e_1' and e_2', and the slope changes to its reciprocal. Then we can use the cot function of the angles in X and in the model, which is analytic there. Thus if we choose the branches of the arctan function so that $\arctan(\tan t \cdot \cos r)$ remains continuous at $(k + 1/2)\pi$ for k any integer, then $\arctan(\tan t \cdot \cos r)$ becomes a strictly increasing analytic function on $\mathbb{R}$.)

2. We turn to the case of H^2. We may suppose $d(o,p) = r$ and $p = (\tanh r, 0)$ in polar coordinates, by the definition of the collinear model of H^2. Consider a circle of centre p and of radius $\varepsilon > 0$. Again we take a rotating segment $[p,q]$, with $d(p,q) = \varepsilon$, from the centre p of this circle to its boundary point q, with $\angle opq = \pi - \Phi$, where $\Phi \in [0, 2\pi]$. We denote by $'$ the points, quantities in the collinear model corresponding to points, quantities in H^2.

We consider the positively oriented triangle opq. We denote $\Psi := \angle poq \in (0,\pi)$ and $s := d(o,q)$. Consider r, s, Ψ as given. Then we have by the cosine law $\cosh\varepsilon = \cosh r \cdot \cosh s - \sinh r \cdot \sinh s \cdot \cos\Psi$, and by the sine law $\sin\Phi = \sinh s \cdot (\sin\Psi)/\sinh\varepsilon = \sinh s \cdot (\sin\Psi)/(\cosh^2\varepsilon - 1)^{1/2}$, where we substitute the value of $\cosh\varepsilon$ from above. Last we calculate $|\tan\Phi| = \sin\Phi/(1 - \sin^2\Phi)^{1/2}$.

In the collinear model we have the image $0p'q'$ of the triangle opq. Here $\Psi' = \Psi$ and $r' = \tanh r$ and $s' = \tanh s$. Like above, we determine first the side ε' of our triangle, then $\sin\Phi'$, and last $|\tan\Phi'|$. We claim

$$\tan\Phi' = \tan\Phi \cdot \cosh r. \tag{12}$$

Since the signs of $\tan\Phi$ and $\tan\Phi'$ are the same, it suffices to show the squared equality. To show this, we perform the calculations indicated above, expressing everything with the variables r, s, Ψ. We cancel with $\sin\Psi = \sin\Psi'$, and clear all the denominators. Thus we obtain two equal quantities, quadratic polynomials of $\cos\Psi$, with coefficients depending on r and s, namely

$$\sinh^2 r \cdot \cosh^2 s - 2\cosh r \cdot \sinh r \cdot \cosh s \cdot \sinh s \cdot \cos\Psi + \cosh^2 r \cdot \sinh^2 s \cdot \cos^2\Psi. \tag{13}$$

Thus (12) is proved.

Again, we need to calculate the derivative $d\Psi'/d\Psi = (d/d\Psi)\arctan(\tan\Psi \cdot \cosh r)$, and to determine its minimum and maximum. This derivative is

$$(\cosh r)/(1 + \sinh^2 r \cdot \sin^2\Psi) \in [1/\cosh r, \cosh r]. \tag{14}$$

Here the left endpoint of the interval is attained for $\Psi = \pi/2$, and the right endpoint is asymptotically attained for $\Psi \to 0$ and for $\Psi \to \pi$. ■

Lemma 1.2. *Assume* (1) *with* $d = 2$. *Then* (2) *of Theorem* 1 *implies that both* K *and* L *are* C^1.

Proof. **1.** A closed convex set $K \subset S^2, \mathbb{R}^2, H^2$ with interior points is not necessarily differentiable. However, at each boundary point x it has two half-tangents, that is, the limit of the line xx', when $x' \to x$, so that x' follows x on $\mathrm{bd}\, K$ in the positive, or in the negative sense.

Suppose, e.g., that K is not C^1. Let $x \in \mathrm{bd}\, K$ be a point of non-smoothness. Let $\alpha \in (0, \pi)$ denote the angle of the positively oriented half-tangents of K at x. We will call this angle the *outer angle of* K *at* x. (This is π minus the inner angle of K at x.)

Let $x', x'' \in \mathrm{bd}\, K$ be points very close to x, such that the shorter, say, counter-clockwise open arc $\widehat{x'x''}$ contains x. Furthermore, we choose the points x', x'' so that, additionally, we have $d(x, x') = b\varepsilon$ and $d(x, x'') = c\varepsilon$. Here $\varepsilon > 0$ is small, and $b, c \in (0, 1]$ fixed satisfy, that a Euclidean triangle T with one angle $\pi - \alpha$ and adjacent sides b, c is not isosceles. Observe that, by continuity, all sufficiently small distances occur as $d(x, x')$ and $d(x, x'')$. Similarly, let $y, y' \in \mathrm{bd}\, L$ be such that y' is very close to y, and y' follows y on $\mathrm{bd}\, L$ in the positive sense. Like above, all sufficiently small distances occur as $d(y, y')$. Let $d(x', x'') = d(y, y')$.

Then there exist orientation preserving congruences φ and ψ, with the following properties. We have $\varphi(x') = \psi(y')$, and $\varphi(x'') = \psi(y)$, and $(\varphi K) \cap (\psi L)$ is bounded by the shorter arcs $\widehat{\varphi(x')\psi(x'')}$ of $\mathrm{bd}\,(\varphi K)$ and $\widehat{\psi(y)\psi(y')}$ of $\mathrm{bd}\,(\psi L)$. Thus this intersection is an arc-triangle A, with "vertices" $\varphi x, \varphi(x') = \psi(y')$ and $\varphi(x'') = \psi(y)$. Let T be the triangle with the same vertices.

We are going to prove that

(15) any congruence admitted by A preserves the set of its three "vertices".

2. First we deal with the case $X = \mathbb{R}^2$. By one-sided differentiablity of $\mathrm{bd}\, K$ at x, and of $\mathrm{bd}\, L$ at y, the total angular rotations (i.e., curvature measures) of all the three open arc-sides of A are $o(1)$, for $\varepsilon \to 0$. In particular, the half-tangents of A at its "vertices" enclose with the respective half-tangents of T at the same vertices, i.e., with the respective side lines of T, an angle $o(1)$. (I.e., the half-tangents enclose small angles with the secant lines, whose limits are the half-tangents.) Therefore the inner angles of A differ from the respective inner angles of T by $o(1)$. Thus the inner angles of T are $\pi - \alpha + o(1)$, and two other angles, whose sum is $\alpha + o(1)$. Therefore both of these last mentioned inner angles of T are at most $\alpha + o(1)$, and the respective outer angles are therefore at least $\pi - \alpha + o(1)$. Therefore each outer angle of A at its "vertices" is at least $\min\{\pi - \alpha + o(1), \alpha + o(1)\}$.

Now let us consider one of the open arc-sides of A. Consider all the points p in this open arc-side of A, such that the outer angle at p is positive. The sum of all the outer angles at such points p is at most the total angular rotation of the considered open arc-side, which is $o(1)$ for $\varepsilon \to 0$. Hence for each open arc-side of A, all the points p in this open arc-side of A, for which the outer angle is positive, satisfy that this outer angle is $o(1)$, for $\varepsilon \to 0$.

Therefore, for $\varepsilon > 0$ sufficiently small, the outer angles of A are either large, namely at least $\min\{\pi - \alpha + o(1), \alpha + o(1)\}$, or small, namely $o(1)$. Therefore any congruence admitted by A preserves the three largest outer angles of A, which occur at the "vertices" of A. This proves (15) for $X = \mathbb{R}^2$.

3. Second we deal with the case of $X = S^2, H^2$. Here total angular rotation makes no sense, therefore we have to go to $\mathbb{R}^2$ via the collinear model. Denote the images of A and T in the collinear model by A' and T', resp. *Let* φx *be mapped to the centre* 0 *of the collinear model.* Then, for $\varepsilon > 0$ sufficiently small, the total angular rotations in $\mathbb{R}^2$ of the open arc-sides of A' are arbitrarily small. Like in **2**, this implies that in $\mathbb{R}^2$, the half-tangents of A' at its "vertices" enclose with the respective half-tangents of T' at the same vertices, i.e., with the respective side lines of T', an angle $o(1)$. Therefore, like in **2**, in $\mathbb{R}^2$, the inner angles of A' differ from the respective inner angles of T' by $o(1)$. This implies, like in **2**, that in $\mathbb{R}^2$, each outer angle of A' at its "vertices" is at least $\min\{\pi - \alpha + o(1), \alpha + o(1)\}$.

However, the collinear model distorts the angles. By Lemma 1.1, we have lower and upper bounds of the form $1 + o(1)$ for the ratio of the angles in X and their images in $\mathbb{R}^2$, for $\varepsilon \to 0$. Therefore the outer angles of A at its "vertices" in X are at least $(\min\{\pi - \alpha + o(1), \alpha + o(1)\}) \cdot [1 + o(1)] = \min\{\pi - \alpha + o(1), \alpha + o(1)\}$.

Now let us consider one of the open arc-sides of A. Consider all the points p in this open arc-side of A, such that the outer angle at p is positive (this means the same for X and for the images in $\mathbb{R}^2$). The sum of all the outer angles, at the images in $\mathbb{R}^2$ of such points p, is at most the total angular rotation of the considered open arc-side in $\mathbb{R}^2$, which is $o(1)$ for $\varepsilon \to 0$. Hence for each open arc-side of A, all the points p in this open arc-side of A, for which the outer angle is positive, satisfy that the image of this outer angle in $\mathbb{R}^2$ is $o(1)$, for $\varepsilon \to 0$. Then the same angle in X is at most $o(1) \cdot [1 + o(1)] = o(1)$. Hence, like in **2**, any congruence admitted by A preserves the three largest outer angles of A, which occur at the "vertices" of A. This proves (15) for $X = S^2, H^2$.

4. We turn once again to $X = \mathbb{R}^2$. The inner angle of T at φx is $\pi - \alpha + o(1)$. If the side of T opposite to this angle is $a\varepsilon$, then $a^2 = b^2 + c^2 - 2bc\cos[\pi - \alpha + o(1)] = b^2 + c^2 - 2bc\cos(\pi - \alpha) + o(1)$.

Observe that $\operatorname{int} A \neq \emptyset$, since it has an inner angle in $(0, \pi)$. We claim that also $\operatorname{diam} A$ is small. Clearly $\operatorname{diam} A$ is attained for a pair of points on the arc-sides of A. For T $(\subset A)$ we have that $\operatorname{diam} T$ is at most $\max\{a, b, c\} \cdot \varepsilon$, which is small. Now it suffices to observe that an arc-side of A has a distance $o(\varepsilon)$ from the respective side of T. This follows from the fact that the angles of the arc-sides of A and the respective sides of T at both of their endpoints are $o(1)$, hence the arc-sides are contained in isosceles triangles with base the respective side of T, and height $o(1)$. Therefore $\operatorname{diam} A \leq [\max\{a, b, c\} + o(1)]\varepsilon$, thus is arbitrarily small. Since also $\operatorname{int} A \neq \emptyset$, therefore A admits a non-trivial congruence.

If for some sequence of ε's, tending to 0, the arc-triangle A admitted a non-trivial congruence, then it would preserve its "vertices". Hence it would be a non-trivial congruence admitted by T as well. That is, T would be an isosceles triangle. Then the limit triangle, satisfying $a^2 = b^2 + c^2 - 2bc\cos(\pi - \alpha)$, would be isosceles too, contradicting the choice of b and c.

Hence for all sufficiently small ε the triangle T is not isosceles, and thus A admits no non-trivial congruence, contradicting the hypothesis of Lemma 1.2 (i.e., (2) of Theorem 1).

5. We turn once again to $X = S^2, H^2$. Analogously as in **4**, now we have to write the spherical and hyperbolic cosine laws for the side of length $a\varepsilon$ of the triangle T in X. Like usual, we subtract 1 from both sides of this equation, and then divide both sides by ε^2. Thus we obtain $a^2 = b^2 + c^2 - 2bc\cos[\pi - \alpha + o(1)] + O(\varepsilon^2) = b^2 + c^2 - 2bc\cos(\pi - \alpha) + o(1)$. This is an analogous formula as in the beginning of **4**.

As in **4**, $\operatorname{int} A \neq \emptyset$. Now we have $\operatorname{diam} T \leq \max\{b, c\} \cdot 2\varepsilon$, since T can be included in a circle of centre φx and radius $\max\{b, c\} \cdot \varepsilon$. Also now the arc-sides of A have a distance at most $o(\varepsilon)$ from the respective sides of T. This follows by including the arc-side to an isosceles triangle like in **4**, and using spherical and hyperbolic trigonometric formulas. Then the diameter of A can occur between two points of T, or one point of T and one point in the "half-lens like domains" between the sides of T and the respective arc-sides of A, or between two points in such "half-lens like domains". In each case we have $\operatorname{diam} A \le O(\varepsilon) + o(\varepsilon) = o(1)$.

The limit argument is the same as for $\mathbb{R}^2$, hence we obtain a contradiction once more.

6. The conclusions of **4** and **5** contradict our indirect hypothesis about non-smoothness of K. This proves smoothness of K, and of L. ■

Lemma 1.3. *Assume* (1) *with* $d = 2$. *Let* K *be* C^1. *For* K *compact let* $K'' := K' := \operatorname{bd} K$ (this is homeomorphic to S^1). *For* K *non-compact let* K' *be a connected component of* $\operatorname{bd} K$ (this is homeomorphic to $\mathbb{R}$, and tends to infinity at both of its "ends", for $\mathbb{R}^2$ and H^2), *moreover, let* K'' *be a compact subarc of* K'. *Then there exists an* $\varepsilon(K, K'') > 0$ *such that for each* $x \in K''$ *and each* $\varepsilon \in (0, \varepsilon(K, K'')]$ *there hold the following statements.*
(1) *The intersection* $(\operatorname{bd} K) \cap B(x, \varepsilon)$ *is a closed subarc of* K'.
(2) *The intersection* $(\operatorname{bd} K) \cap \operatorname{bd}[B(x, \varepsilon)]$ *consists of exactly two points* x_ε^+ *and* x_ε^-, *the endpoints of the subarc in* (1). *The directed arcs of* $\operatorname{bd} K$ *from* x *to* x_ε^+ *and* x_ε^- *are positively and negatively directed, resp.*
(3) *The angles of* $\operatorname{bd} K$ *and* $\operatorname{bd}[B(x, \varepsilon)]$, *at the two points of intersection from* (2), *are* $\pi/2 + o(1)$, *for* $\varepsilon \to 0$, *uniformly for each* $x \in K''$.
(4) *The functions* $x \mapsto x_\varepsilon^+$ *and* $x \mapsto x_\varepsilon^-$ *are continuous, and are strictly monotonous in the following sense. A small motion of* x *along* K'' *strictly in positive or negative sense implies small motions of* x_ε^+ *and* x_ε^- *along* K'' *strictly in positive or negative sense, resp.*

Proof. **1.** We consider K and K'' as fixed. Hence dependence of quantities on K and K'' will not be explicitly written in the notations.

We begin with the case when $\operatorname{bd} K$ *is connected* (i.e., $\operatorname{bd} K = K'$).

Let $\delta > 0$ *be sufficiently small.* We define $K''(\delta) \subset K'$ as follows. *For* K *compact*, we let $K''(\delta) = \operatorname{bd} K$. *For* K *noncompact* first we extend K'' to the closed 2δ-neighbourhood $K''_{2\delta}$ of K'' in K', and then to the closed 4δ-neighbourhood $K''_{4\delta}$ of K'' in K', in the arc length metric of K'. Then we extend $K''_{4\delta}$ further as follows. Observe that now K' is homeomorphic to $\mathbb{R}$, and tends to infinity at both of its "ends". We take such a long compact subarc $K''(\delta)$ of K', containing $K''_{4\delta}$, that we have $\operatorname{dist}[K''_{4\delta}, K' \setminus K''(\delta)] \ge 1$, with distance meant in X. This will ensure for $x \in K''$ and $\delta < 1$ that

$$K' \cap B(x, \delta) = [[K''(\delta) \cup [K' \setminus K''(\delta)]] \cap B(x, \delta) = K''(\delta) \cap B(x, \delta). \tag{16}$$

Therefore it will suffice to show (1), (2) and (3) of the lemma, with K' replaced by $K''(\delta)$.

2. In this proof we will use the collinear model, with coordinates ξ, η. This clearly works for $\mathbb{R}^2$ and H^2, however for S^2 this exists only on the open southern hemisphere. Let $X = S^2$.

First suppose $\operatorname{diam} K = \pi$. Then either K is a halfsphere, or a digon with angle in $(0, \pi)$. In the first of these cases the statement of the lemma is evident, for any $\varepsilon \in (0, \pi)$. In the second of these cases K is not C^1.

Second suppose $\operatorname{diam} K < \pi$. Then by the first four sentences of the second paragraph of the proof of Lemma 1.4 in [8], applied for $d = 2$, we get that K lies in an open hemisphere. Then we may suppose that this is the open southern hemisphere, hence the collinear model exists on some neighbourhood of K.

3. By (6), (7) and Lemma 1.1, for S^d and H^d, we have the following. Both the arc elements and the angles have quotients bounded below and above on $K''(\delta)$ (with vertex of the angle on $K''(\delta)$), when considered in X, and in the collinear model, as a subspace of $\mathbb{R}^2$. For $\mathbb{R}^2$ these are obvious.

We claim that

(17)
$$\begin{cases} \text{for a sufficiently short subarc of } K''(\delta), \text{ the chord length and the arc length} \\ \text{in } X \text{ have a quotient } 1+o(1), \text{ uniformly, if the arc length tends to 0.} \end{cases}$$

For sufficiently short arcs of $K''(\delta)$ the tangent lines in the collinear model, as a subspace of $\mathbb{R}^2$, change very little, uniformly. We will identify $\operatorname{bd} K$ and $K''(\delta)$ by their images in the collinear model, resp. We cover $K''(\delta)$ by four subsets, according to as the the tangent direction in the positive sense belongs to the open angular intervals $(-\pi/3, \pi/3)$, or $(\pi/6, 5\pi/6)$, or $(2\pi/3, 4\pi/3)$, or $(7\pi/6, 11\pi/6)$. Thus we obtain four open subsets $I_1, \ldots, I_4$ of $K''(\delta)$, covering $K''(\delta)$. On I_1 or I_3 we have that $\operatorname{bd} K$ can be given by an equation $\eta = f(\xi)$, with f convex, or concave, resp. On I_2 or I_4 we have that $\operatorname{bd} K$ can be given by an equation $\xi = g(\eta)$, with g concave, or convex, resp. We have $|f'(\xi)|, |g'(\eta)| < \sqrt{3}$. Moreover, the domains of definition of these functions in all four cases are the respective projections of the open subsets I_i of $K''(\delta)$ to the ξ- or η-axis, resp.

Let us consider a sufficiently short arc of $K''(\delta)$. Its first endpoint, in the positive sense, can have a positively directed tangent direction lying in $[-\pi/4, \pi/4]$, $[\pi/4, 3\pi/4]$, $[3\pi/4, 5\pi/4]$ or $[5\pi/4, 7\pi/4]$. Suppose the first case. (The other three cases are settled analogously. If we can find an $\varepsilon(K, K'')$ for the first case, then analogously we can find $\varepsilon(K, K'')$ for the other three cases as well. Then the minimum of these four values will satisfy the statement of the lemma.) Then our sufficiently short arc of $K''(\delta)$ lies in I_1. Moreover, on it we have $\eta = f(\xi)$, with f convex, and $|f'(\xi)| < \sqrt{3}$. Let the endpoints of our sufficiently short arc (and chord) be (ξ_1, η_1) and (ξ_2, η_2). Then our arc is given as $\{(\xi, f(\xi)) \mid \xi \in [\xi_1, \xi_2]\}$.

The length of this arc is

$$\int_{\xi_1}^{\xi_2} \left[g_{11}(\xi, f(\xi)) + 2g_{12}(\xi, f(\xi)) f'(\xi) + g_{22}(\xi, f(\xi)] [f'(\xi)]^2\right]^{1/2} d\xi. \tag{18}$$

Here g_{ij} is the metric tensor, in the (ξ, η) coordinate system. Moreover, the chord length is the same expression, with $f(\xi)$ replaced by $\overline{f}(\xi) := \eta_1 + (\eta_2 - \eta_1) \cdot (\xi - \xi_1)/(\xi_2 - \xi_1)$, and hence with $(\overline{f})'(\xi) = (\eta_2 - \eta_1)/(\xi_2 - \xi_1)$. (Since chords in X and in the collinear model, as a subset of $\mathbb{R}^2$, coincide.) Here, by the mean value theorem, also using that f is C^1, on the interval $[\xi_1, \xi_2]$ we have $f'(\xi) - (\overline{f})'(\xi) = o(1)$, uniformly, if the arc length tends to 0 (hence also $\xi_2 - \xi_1$ tends to 0). Hence on this interval also $f(\xi) - \overline{f}(\xi) = o(\xi_2 - \xi_1) = o(1)$, uniformly, if the arc length tends to 0.

This implies that the difference of (18), and the analogous expression, which is obtained from (18) by replacing $f(\xi)$ in it by $\overline{f}(\xi)$, is $o(\xi_2 - \xi_1)$. E.g., we show

$g_{22}\left(\xi, f(\xi)\right)[f'(\xi)]^2 - g_{22}\left(\xi, \overline{f}(\xi)\right)[(\overline{f})'(\xi)]^2 = o(1)$, uniformly for $\xi \in [\xi_1, \xi_2]$. (The analogous estimates for the first and second summands of the square of the integrand in (18) are obtained analogously, but even simpler.) We have

$$
\text{(19)} \qquad \begin{cases} g_{22}\left(\xi, f(\xi)\right)[f'(\xi)]^2 - g_{22}\left(\xi, \overline{f}(\xi)\right)[(\overline{f})'(\xi)]^2 = [g_{22}\left(\xi, f(\xi)\right) \\ -g_{22}\left(\xi, \overline{f}(\xi)\right)] \cdot [f'(\xi)]^2 + g_{22}\left(\xi, \overline{f}(\xi)\right) \cdot [(f'(\xi) + (\overline{f})'(\xi))] \times \\ [(f'(\xi) - (\overline{f})'(\xi))] = o(1) \cdot O(1) + O(1) \cdot O(1) \cdot o(1) = o(1). \end{cases}
$$

It remains to observe that the function $t \mapsto t^{1/2}$ is Lipschitz on any closed subinterval of $(0, \infty)$. This we apply to the square of the integrand in (18), which has a positive lower (and upper) bound. In fact, the metric tensor can be estimated from below by some $\text{const}_{K''} \cdot (d\xi^2 + d\eta^2) \geq \text{const}_{K''} \cdot d\xi^2$. Hence the square of the integrand in (18) is at least $\text{const}_{K''}$.

The last lower estimate yields a lower bound for (18), namely $\text{const}_{K''}^{1/2} \cdot (\xi_2 - \xi_1)$. From above the difference of (18) and of the analogous expression, with $f(\xi)$ replaced by $\overline{f}(\xi)$, is $o(\xi_2 - \xi_1)$. *These imply our claim* (17).

4. The C^1 curve $K''(\delta)$ has an inherited Riemannian submanifold (with boundary) structure, with the intrinsic metric. This is isometric either to some circular line in $\mathbb{R}^2$, with the metric the length of the not longer connecting arc on the circular line, or to some closed interval of $\mathbb{R}$, with the metric inherited from $\mathbb{R}$ (according to whether K is compact, or not). We denote this metric by d_{arc}. On the other hand, we have the metric d_X in X, which for $K''(\delta)$ is the chord length in X. Then $d_{\text{arc}} \geq d_X$, hence the identical map of the compact metric space $(K''(\delta), d_{\text{arc}})$ to the compact metric space $(K''(\delta), d_X)$ is continuous. Hence this bijective map is a homeomorphism. Hence its inverse map $(K''(\delta), d_X) \to (K''(\delta), d_{\text{arc}})$ also is continuous, with compact metric domain, hence it is even uniformly continuous. Let $U_{\text{arc}}(\delta) := \{(k_1, k_2) \in K''(\delta) \times K''(\delta) \mid d_{\text{arc}}(k_1, k_2) \leq \delta\}$ and $U_X(\delta) := \{(k_1, k_2) \in K''(\delta) \times K''(\delta) \mid d_X(k_1, k_2) \leq \delta\}$. Therefore, for each $\delta > 0$ there exists a $\gamma(\delta) > 0$, such that $U_X[\gamma(\delta)] \subset U_{\text{arc}}(\delta)$. Therefore,

$$
\text{(20)} \qquad \begin{cases} \text{for } x \in K'' \text{ and } x' \in K''(\delta) \text{ and} \\ d_X(x, x') \leq \gamma(\delta) \text{ we have } d_{\text{arc}}(x, x') \leq \delta. \end{cases}
$$

$$
\text{(21)} \qquad \text{If } K \text{ is compact, we still assume } \delta \leq (\operatorname{perim} K)/8.
$$

In this case, for $d_{\text{arc}}(x, x') \leq 4\delta \leq (\operatorname{perim} K)/2$ we have that $d_{\text{arc}}(x, x')$ equals the integral of ds on the positively or negatively oriented arc $\widehat{xx'}$.

We will see (in (26)) that $\varepsilon(K, K'')$ can be chosen as $\min\{\delta, \gamma(\delta)\}$, for some suitably small $\delta > 0$.

5. Now consider a sufficiently short arc $\widehat{xx'}$ of $K''(\delta)$, with $x \in K''$, and with x' following x on $K''(\delta)$ in the positive sense, with arc length δ, say. (The case when x' follows x on $K''(\delta)$ in the negative sense, can be settled analogously.) We investigate the direction, in the collinear model, as a subset of $\mathbb{R}^2$, of the positively oriented segment (chord) S' with endpoints x, x', with orientation inherited from the positively oriented $K''(\delta)$. Further, we investigate the direction of the positively oriented tangent line $(l(x'))'$ of $K''(\delta)$ at x', in the collinear model, as a subset of $\mathbb{R}^2$.

By the mean value theorem, and the C^1 property of $K''(\delta)$, we have the following. The angle, in the collinear model, as a subset of $\mathbb{R}^2$, of the direction of the oriented segment S', and the direction of the oriented tangent line $(l(x'))'$ is $o(1)$, uniformly, if the arc length δ tends to 0. Observe that our oriented segment and oriented tangent line in the model, as a subspace of $\mathbb{R}^2$, have x' as a common point.

Therefore *the direction of the corresponding oriented segment $S := [x, x']$ and oriented tangent line $l(x')$ of $K''(\delta)$ at x', both taken in X, have in X an angle, at x', which is also $o(1)$, uniformly*, by Lemma 1.1. Now observe that S is a radius of the circle $B\,(x, d_X(x', x))$ in X, hence is perpendicular to $\mathrm{bd}\,[B(x, d_X(x', x)]$ in X. Therefore

$$(22)\qquad \begin{cases} l(x') \text{ encloses in } X \text{ an angle } \pi/2 + o(1) \text{ with } \mathrm{bd}\,[B(x,\delta)] \\ \text{at } x', \text{ with } o(1) \text{ uniform, for } x \in K'' \text{ and } \delta \to 0. \end{cases}$$

Also taking into consideration (16), *this would prove* (3) *of the lemma, provided we already knew* (2) *of the lemma.*

Let $x \in K''$. We write r_x for the distance from x in X, and s_x for the arc length distance of a point $x' \in K''$ from x (i.e., the length of the not longer connecting arc $\widehat{xx'} \subset K''$). (We have $s_x(x) = r_x(x) = 0$.) For K compact we have $s_x \leq (\mathrm{perim}\,K)/2$; therefore in this case we will consider only subarcs of $\mathrm{bd}\,K$ not longer than $(\mathrm{perim}\,K)/2$. This ensures that *the integral of ds on such arcs equals s_x* (cf. the sentence following (21)). The analogue of this italicized statement for the noncompact case is obvious (there is only one such arc). Then by the trigonometry of Euclidean, spherical and hyperbolic triangles ([18]), we have the following. Formula (22) implies

$$(23)\qquad \begin{cases} \text{uniformly for all } r \text{ in an interval of the form } [0, r_0], \text{ where } r_0 \text{ is} \\ \text{small, that } 1 \leq (ds_x/dr_x)(r) = 1 + o(1), \text{ with } o(1) \text{ uniform. Hence} \\ r_x \leq s_x = r_x \cdot [1 + o(1)], \text{ with } o(1) \text{ uniform, for } x \in K'' \text{ and} \\ \delta \to 0; \text{ hence } s_x \leq 2r_x, \text{ for } x \in K'' \text{ and } \delta \text{ sufficiently small.} \end{cases}$$

Then

$$(24)\qquad \begin{cases} 1 \geq dr_x/ds_x = 1 + o(1) \geq 1/2, \text{ with } o(1) \text{ uniform, for } \delta \text{ sufficiently small.} \\ \text{Therefore passing with } x' \text{ away from } x \in K'', \text{ along an arc of } K''(\delta), \\ \text{in the positive sense, of arc length at most } 4\delta \text{ from } x, \text{ the distance} \\ d_X(x, x') \text{ strictly increases, for a } \textit{uniform sufficiently small } \delta > 0. \end{cases}$$

Now we fix this value of δ.

Here we can attain d_X-distance any $\varepsilon \in (0, \delta]$, for some $x' \in K''_{2\delta} \subset K''(\delta)$. In fact, for $x \in K''$ and $x' \in K''_{2\delta} \subset K''(\delta)$ with $d_{\mathrm{arc}}(x, x') = 2\delta$ – such an x' exists, by the definition of $K''_{2\delta}$ – we have by (23) $d_X(x, x') = 2\delta[1 + o(1)] \geq \delta \geq \varepsilon$. By the strictly increasing property of $d_X(x, x')$, on the arc of $K''_{2\delta}$, consisting of the (x')'s, following x on $K''_{2\delta}$ in the positive sense and satisfying $d_{\mathrm{arc}}(x, x') \leq 2\delta$, we have the following. The distance $d_X(x, x')$ can be equal to ε only for one point x'; which point x' in fact exists, as pointed out above. By (16), (2) of the lemma would be proved, if we proved it with $K''(\delta)$ rather than K'. This in turn would be proved, provided we knew already (1) of the lemma, also with $K''(\delta)$ rather than K'. As observed in (22), this would prove also (3) of the lemma.

6. First we will investigate the case of non-compact K. Then K' tends to infinity at both of its ends (for $\mathbb{R}^2$ and H^2).

From (23), on a positively oriented arc of $K''(\delta)$ with starting point $x \in K''$, of arc length 2δ, both of r_x and s_x are strictly monotonically increasing C^1 functions of each other: $r_x = r_x^+(s_x)$, and $s_x = s_x^+(r_x)$. Therefore,

(25)
$$\begin{cases} \text{for } x \in K'' \text{ and for } \varepsilon \in (0, \delta], \text{ we have } A_\varepsilon^+ := \{x' \in K''(\delta) \mid x' \text{ follows } x \\ \text{on } K''(\delta) \text{ in the positive sense}\} \cap B(x, \varepsilon) = \{x' \in K''(\delta) \mid x' \text{ follows } x \text{ on} \\ K''(\delta) \text{ in the positive sense, and } d_{\text{arc}}(x, x') = s_x^+\left(d_X(x, x')\right) \leq s_x^+(\varepsilon) = \varepsilon \times \\ [1 + o(1)] \leq 2\varepsilon \leq 2\delta\}, \text{ by (23), with } o(\cdot) \text{ uniform, for } x \in K'' \text{ and } \delta \to 0. \end{cases}$$

Then $A_\varepsilon^+ \subset K''(\delta)$ is a positively oriented closed arc of $K''(\delta)$, with starting point x and of arc length $\varepsilon \cdot [1 + o(1)]$, with $o(\cdot)$ uniform. If on A_ε^+ a variable point x'' moves from x in the positive orientation, then $d_X(x, x'')$ is a strictly increasing continuous function of the position of x''. Hence it assumes each value till its maximum value ε just once. We let

$$0 < \varepsilon \leq \varepsilon(\delta) := \min\{\delta, \gamma(\delta)\}. \tag{26}$$

By (16), (1) of the lemma would be proved, if we proved it with $x' \in K''(\delta)$ rather than $x' \in K'$.

By (20),

$$\begin{cases} \text{for } x \in K'' \text{ and } x' \in K''(\delta) \text{ and } d_{\text{arc}}(x, x') > \delta \\ \text{we have } x' \notin B(x, \gamma(\delta)) \supset B(x, \varepsilon(\delta)) \supset B(x, \varepsilon). \end{cases} \tag{27}$$

There remains the case when $x \in K''$ and $x' \in K''(\delta)$, and $d_{\text{arc}}(x, x') \leq \delta$. By (25) and (26) we have

$$\begin{cases} \{x' \in K''(\delta) \mid x' \text{ follows } x \text{ on } K''(\delta) \text{ in the positive sense,} \\ \text{and } d_{\text{arc}}(x, x') \leq \delta\} \cap B(x, \varepsilon) = \big\{x' \in K''(\delta) \mid x' \text{ follows } x \text{ on} \\ K''(\delta) \text{ in the positive sense, and } d_{\text{arc}}(x, x') \leq \min\{\delta, s_x^+(\varepsilon)\}\big\}. \end{cases} \tag{28}$$

Hence, by (26), (27) and (28) we get

(29)
$$\begin{cases} \{x' \in K''(\delta) \mid x' \text{ follows } x \text{ on } K''(\delta) \text{ in the positive sense}\} \cap B(x, \varepsilon) \\ = [\{x' \in K''(\delta) \mid x' \text{ follows } x \text{ on } K''(\delta) \text{ in the positive sense, and} \\ d_{\text{arc}}(x, x') \leq \delta\} \cap B(x, \varepsilon)] \cup [\{x' \in K''(\delta) \mid x' \text{ follows } x \text{ on } K''(\delta) \text{ in} \\ \text{the positive sense, and } d_{\text{arc}}(x, x') > \delta\} \cap B(x, \varepsilon)] = \big\{x' \in K''(\delta) \mid x' \\ \text{follows } x \text{ on } K''(\delta) \text{ in the positive sense, and } d_{\text{arc}}(x, x') \leq \min\{\delta, s_x^+(\varepsilon)\}\big\}. \end{cases}$$

Let us turn to points x' following x on $K''(\delta)$ in the negative sense. Then analogously to (25) we obtain a negatively oriented closed arc A_ε^- of $K''(\delta)$, with starting point x, of arc length $\varepsilon \cdot [1 + o(1)]$, with $o(\cdot)$ uniform. Further, analogously to the functions $r_x^+(\cdot)$ and $s_x^+(\cdot)$ we obtain functions $r_x^-(\cdot)$ and $s_x^-(\cdot)$. Moreover, there holds the analogue of (29):

$$\begin{cases} \{x' \in K''(\delta) \mid x' \text{ follows } x \text{ on } K''(\delta) \text{ in the negative} \\ \text{sense}\} \cap B(x, \varepsilon) = \big\{x' \in K''(\delta) \mid x' \text{ follows } x \text{ on } K''(\delta) \\ \text{in the negative sense, and } d_{\text{arc}}(x, x') \leq \min\{\delta, s_x^-(\varepsilon)\}\big\}. \end{cases} \tag{30}$$

Now (29) and (30) imply

$$(31)\quad \begin{cases} K''(\delta)\cap B(x,\varepsilon)=\{x'\in K''(\delta)\mid x' \text{ follows } x \text{ on } K''(\delta) \text{ in the positive} \\ \text{sense, and } d_{\rm arc}(x,x')\le \min\{\delta, s_x^+(\varepsilon)\}\}\cup\{x'\in K''(\delta)\mid x' \text{ follows } x \\ \text{on } K''(\delta) \text{ in the negative sense, and } d_{\rm arc}(x,x')\le \min\{\delta, s_x^-(\varepsilon)\}\}. \end{cases}$$

Applying this to $\varepsilon=\varepsilon(\delta)$ (cf. (26)) we get (1) of Lemma 1.3, for $\operatorname{bd} K$ connected (cf. the beginning of **1**) and K non-compact.

In (24) we have seen that if on $A_\varepsilon^+\subset B(x,\varepsilon)\subset B(x,4\delta)$ a variable point x'' moves from x in the positive orientation, then $d_X(x,x'')$ is a strictly increasing continuous function of the position of x''. Hence it assumes each value till its maximum value ε just once. Analogously we have that if on A_ε^- a variable point x'' moves from x in the negative orientation, then $d_X(x,x'')$ is a strictly increasing continuous function of the position of x''. Hence it assumes each value till its maximum value ε just once. Applying these for $\varepsilon:=\varepsilon(\delta)$, we get (2) of Lemma 1.3, for $\operatorname{bd} K$ connected (cf. the beginning of **1**) and K non-compact. As shown in **5**, (2) of Lemma 1.3 implies (3) of Lemma 1.3, for $\operatorname{bd} K$ connected and K non-compact. Thus, for $\operatorname{bd} K$ connected and K non-compact, statements (1), (2), (3) of the lemma are proved.

7. Now we turn to the case of compact K (then $\operatorname{bd} K$ is connected).

Now it makes no sense to say that x' follows x on $K''(\delta)=\operatorname{bd} K$ in the positive, or negative sense. Therefore we make the following changes in the above arguments. Choose $\tilde{x}\in \operatorname{bd} K$ such that it together with x divides $\operatorname{bd} K$ to two arcs of equal lengths. Recall that in (21) we have assumed for the compact case that $\delta\le(\operatorname{perim} K)/8$. Then instead of saying that x' follows x on $K''(\delta)=\operatorname{bd} K$ in the positive, or negative sense, we will say that x' lies in the positively, or negatively oriented subarc $\widehat{x\tilde{x}}$ of $\operatorname{bd} K$. (Then $x,\tilde{x}$ lie in both of these arcs, but this causes no problem.) In the last paragraph of **5** we have shown that the distance $d(x,x')$ assumes the value δ for some point of an arc of $\operatorname{bd} K$, beginning at x and having arc length 2δ. Observe that by the hypothesis on δ this arc still lies in one of the subarcs $\widehat{x\tilde{x}}$ of $\operatorname{bd} K$. Then we can repeat the considerations from **6**. In (25) we will have $d_{\rm arc}(x,x')\le s_x^+(\varepsilon)=\varepsilon\cdot(1+o(1))\le 2\varepsilon\le 2\delta\le(\operatorname{perim} K)/4$. We also have the analogous statement with $s_x^-(\varepsilon)$.

Then with these notational changes all arguments of **6** carry over to the case of compact K, and this proves all statements of this lemma for compact K. Then **6** and the previous parts of **7** prove (1), (2), (3) of the lemma for the case when $\operatorname{bd} K$ is connected.

8. Second suppose that $\operatorname{bd} K$ has several connected components. Then $\operatorname{bd} K$ has at most countably infinitely many connected components K'_n, which are relatively closed and open subsets of the closed set $\operatorname{bd} K$. Let the component K' from the statement of this lemma be K'_1. Then for $n\ge 2$ we have that $\cup_{n\ge 2}K'_n$ is open and closed in $\operatorname{bd} K$, hence is closed in X. Now recall that in a metric space the distance of a compact set, and a closed set disjoint to it, is positive. This we apply to $K''(\delta)$ and $\cup_{n\ge 2}K'_n$. Then we can choose an $\varepsilon_1>0$, only depending on $K''(\delta)$ and K, such that the closed ε_1-neighbourhood of $K''(\delta)$ is disjoint to $\cup_{n\ge 2}K'_n$. Thus, in particular,

$$(32)\qquad \text{for any } x\in K'' \text{ we have that } B(x,\varepsilon_1) \text{ is disjoint to } \cup_{n\ge 2}K'_n.$$

Now let $\varepsilon_2>0$ be the value $\varepsilon:=\varepsilon(\delta)$ which we have obtained just below (31), but with K replaced by the closed convex set $K^*\supset K$, with boundary K'. Then for $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$ statements (1), (2) and (3) of Lemma 1.3 are valid by **6** and **7** of this proof and (32).

9. There remains to prove the "strict monotonicity" properties of the maps $x \mapsto x_\varepsilon^+$ and $x \mapsto x_\varepsilon^+$, asserted in (4) of the lemma. We will show this for the map $x \mapsto x_\varepsilon^+$ (for the other map the proof is analogous).

By (24), for $x \in K''$ we have $x_\delta^+ \in K''_{2\delta} \subset K''_{4\delta} \subset K''(\delta)$. This implies

$$(x_\delta^+)_\delta^+ \in K''_{4\delta} \subset K''(\delta). \tag{33}$$

By applying (24) to $K''(\delta)$, rather than K'' (this requires a smaller δ, say, $\delta(0) \in (0,\delta)$), on any arc A of $K''(\delta)$, of arclength at most $2\delta(0)$, we have the following. For a proper subarc A' of A, with one endpoint common with one endpoint of A, the corresponding chord length is strictly smaller than that for A. This implies the same statement for any proper subarc A' of A.

Now let x^* begin to move along $K''(\delta)$, in the positive sense, from $x \in K''$ till $x_{\delta(0)}^+ \in K''_{2\delta}$. Then for $x^* \neq x$ we cannot have that $(x^*)_{\delta(0)}^+$ belongs to the shorter arc $\widehat{x(x_{\delta(0)}^+)}$, as follows from what precedes. Hence $(x^*)_{\delta(0)}^+$, which can be reached from x^* by moving along $K''(\delta)$ in the positive sense, lies strictly "beyond" $x_{\delta(0)}^+$.

We still have to show that at the beginning of this motion $(x^*)_{\delta(0)}^+$ cannot move "too far beyond" $x_{\delta(0)}^+$. Let $\beta \in (0, \delta(0)]$ be small. Suppose that, in the collinear model, x and $x_{\delta(0)}^+$ have images on the negative and positive ξ-axis, resp., and these images are symmetrical w.r.t. 0. Let the length of the shorter arc $\widehat{x(x^*)}$ of $K''(\delta)$ be β. Then we assert that $d_X(x_{\delta(0)}^+, (x^*)_{\delta(0)}^+) \leq \beta\,(1+o(1))$, for $\beta \to 0$.

Let π denote projection on the ξ-axis in X.

For $X = \mathbb{R}^2, H^2$ we have that π is a contraction in X. The points $x, x^*, x_{\delta(0)}^+$, $(x^*)_{\delta(0)}^+$ (and $(x_{\delta(0)}^+)_{\delta(0)}^+$, by (33)) follow each other on $K''(\delta)$ in the positive order. Therefore, by the C^1-property of K, we have that $x, \pi(x^*), x_{\delta(0)}^+, \pi\left((x^*)_{\delta(0)}^+\right)$ follow each other on the ξ-axis in the positive order. Then $d_X\left(x, \pi(x^*)\right) \leq \beta$ by the contraction property. This implies $d_X\left(\pi(x^*), x_{\delta(0)}^+\right) \geq \delta(0) - \beta$. Then once more by the contraction property, we have $d_X\left(\pi(x^*), \pi\left((x^*)_{\delta(0)}^+\right)\right) \leq d_X\left(x^*, (x^*)_{\delta(0)}^+\right) = \delta(0)$. From these two inequalities we have $d_X\left(x_{\delta(0)}^+, \pi\left((x^*)_{\delta(0)}^+\right)\right) \leq \beta$. Once more by the C^1-property of K, we have $d_X\left(x_{\delta(0)}^+, \left((x^*)_{\delta(0)}^+\right)\right) \leq \beta\,(1+o(1))$, for $\beta \to 0$, as asserted.

Next let $X = S^2$. Since the length of the shorter arc $\widehat{x(x^*)}$ of $K''(\delta)$ is β, therefore $d_X\left(x, \pi(x^*)\right) \leq d_X(x, x^*) \leq \beta$ (like we had above). Now let $\tilde{x}$ lie on the line $x(x_{\delta(0)}^+)$, on the other side of $x_{\delta(0)}^+$, as x, with $d_X(x_{\delta(0)}^+, \tilde{x}) = \beta$. Then the line passing through $\tilde{x}$ and orthogonal to the line $x(x_{\delta(0)}^+)$ is a great circle D, whose distance from x is $d_X(x, x_{\delta(0)}^+) + d_X(x_{\delta(0)}^+, \tilde{x}) = \delta + \beta$. From above $d_X(x, x^*) \leq \beta$, and then the distance of x^* from D is at least $(\delta(0)+\beta) - \beta = \delta(0)$. In particular, by $d_X\left(x^*, (x^*)_{\delta(0)}^+\right) = \delta(0)$, we have that $(x^*)_{\delta(0)}^+ \in B(x^*, \delta(0))$ cannot lie "beyond" D. In other words, $\pi\left((x^*)_{\delta(0)}^+\right)$ cannot lie "beyond" D. Once more by the C^1-

property of K, we have $d_X\left(x^+_{\delta(0)}, (x^*)^+_{\delta(0)}\right) \le \beta\,(1+o(1))$, for $\beta \to 0$, as asserted. ■

Lemma 1.4. *Assume* (1) *with* $d = 2$. *Let* K *and* L *be* C^1. *Let* K' *and* K'' *be as in Lemma* 1.3. *Let* L' *and* L'' *be defined analogously for* L, *as* K' *and* K'' *were defined for* K *in Lemma* 1.3. *Then there exists an* $\varepsilon(K, L, K'', L'') > 0$ *such that for each* $\varepsilon \in (0, \varepsilon(K, L, K'', L'')]$ *the following holds.*

Let $[x_1, x_2]$ *and* $[y_1, y_2]$ *be chords of* K' *and* L', *resp., of length* ε, *where* x_2 *follows* x_1 *on* K' *in the positive sense, and* y_2 *follows* y_1 *on* L' *in the negative sense. Let at least one of* x_1, x_2 *belong to* $\mathrm{relint}_{K'} K''$, *and at least one of* y_1, y_2 *belong to* $\mathrm{relint}_{L'} L''$. *Let us choose* φ *and* ψ *so, that* $\varphi(x_i) = \psi(y_i)$ $(i = 1, 2)$. *Let us consider the shorter arcs* $\widehat{(\varphi x_1)(\varphi x_2)}$ *and* $\widehat{(\varphi y_2)(\varphi y_1)}$. *If they are both segments, then this segment is a subset of* $(\varphi K) \cap (\psi L)$. *Else* $(\varphi K) \cap (\psi L)$ *is the compact convex set with boundary* $\widehat{(\varphi x_1)(\varphi x_2)} \cap \widehat{(\varphi y_2)(\varphi y_1)}$.

Proof. **1.** We begin with the case when $\mathrm{bd}\, K$ and $\mathrm{bd}\, L$ are connected.

We choose $\varepsilon \in (0, \min\{\varepsilon(K, K''), \varepsilon(L, L'')\}]$ sufficiently small (with $\varepsilon(L, L'')$ defined analogously to $\varepsilon(K, K'')$, cf. Lemma 1.3).

In the second paragraph of **5** of the proof of Lemma 1.3 we have seen the following. The direction of the oriented segment $S := [x, x']$ and the oriented tangent line $l(x')$ of $K''(\delta)$ at x', both taken in X, have in X an angle, at x', which is $o(1)$, uniformly. We apply this to $x := x_1,\ x' := x_2$, and to $x := x_2,\ x' := x_1$, resp. Analogously, we apply this to $L''(\delta)$ (defined analogously as $K''(\delta)$, in **1** of the proof of Lemma 1.3), to y_1, y_2 or y_2, y_1 in place of x, x', resp. Therefore *the compact convex sets ("half-lens like domains"), bounded by* $[\varphi(x_1), \varphi(x_2)] = [\psi(y_1), \psi(y_2)]$, *and by the shorter arcs* $\widehat{(\varphi x_1)(\varphi x_2)}$ *of* $\mathrm{bd}\,(\varphi K)$ *or* $\widehat{(\psi(y_2))(\psi(y_1))}$ *of* $\mathrm{bd}\,(\psi L)$, *resp., have at* $\varphi(x_1) = \psi(y_1)$ *and* $\varphi(x_1) = \psi(y_1)$ *angles which are* $o(1)$, *uniformly.* Hence

(34) the union M of these two compact convex sets ("half-lens like domains")

is also a compact set, which has inner angles at $\varphi(x_1) = \psi(y_1)$ and at $\varphi(x_1) = \psi(y_1)$, which are $o(1)$, uniformly, for $\varepsilon \to 0$. Hence M is compact convex. (In a suitable orthogonal coordinate system, in the collinear model, it can be given as $\{(\xi, \eta) \mid \xi \in [\xi_1, \xi_2],\ \eta \in [f(\xi), g(\xi)]$, with f convex and g concave on $[\xi_1, \xi_2]$, and $f(\xi_i) = g(\xi_i) = 0$.) However, M may have an empty interior.

We are going to prove $(\varphi K) \cap (\psi L) = M$, except if both $\widehat{(\varphi x_1)(\varphi x_2)}$ and $\widehat{(\psi(y_2))(\psi(y_1))}$ are segments.

2. First we prove $M \subset (\varphi K) \cap (\psi L)$. It is sufficient to show $M \subset \varphi K$.

(35) $\begin{cases} \text{For this it suffices to show that the compact convex set } M_1, \\ \text{bounded by } [\varphi x_1, \varphi x_2] \cup \widehat{(\psi y_2)(\psi y_1)}, \text{ is a subset of } \varphi K. \end{cases}$

By Lemma 1.3, $\widehat{(\psi y_2)(\psi y_1)} \subset B(\psi y_1, \varepsilon) = B(\varphi x_1, \varepsilon)$. Also by Lemma 1.3, $\varphi A := \mathrm{bd}\,(\varphi K) \cap B(\varphi x_1, \varepsilon)$ is an arc of $\mathrm{bd}\,(\varphi K)$, with endpoints at a distance ε from φx_1 (one of them being φx_2, the other one is denoted by φx_0). Thus

(36) $\begin{cases} \text{no point of } [\mathrm{bd}\,(\varphi K)] \setminus (\varphi A) = [\mathrm{bd}\,(\varphi K)] \setminus \\ B(\psi x_1, \varepsilon) \text{ lies in } \widehat{(\psi y_2)(\psi y_1)} \subset B(\psi y_1, \varepsilon). \end{cases}$

Also,

$$\begin{cases} \text{no point of the shorter arc } \widehat{(\varphi x_0)(\varphi x_1)} \\ \text{lies in } \widehat{(\psi y_2)(\psi y_1)}, \text{ except } \varphi x_1 = \psi y_1. \end{cases} \tag{37}$$

In fact, in the collinear model, as a subset of $\mathbb{R}^2$, we have that $\widehat{(\psi y_2)'(\psi y_1)'}$ lies close to $[(\psi y_1)', (\psi y_2)'] = [(\varphi x_1)', (\varphi x_2)']$. (Here $(\cdot)'$ denotes image in the collinear model.) More exactly, it lies in an angular domain of vertex $(\varphi x_1)'$, with one leg passing through $(\varphi x_2)'$, and angular measure $o(1)$. Similarly, in the collinear model, as a subset of $\mathbb{R}^2$, we have that $\widehat{(\varphi x_0)'(\varphi x_1)'}$ lies close to $[(\varphi x_0)', (\varphi x_1)']$. Hence it lies also close to $[(\varphi x_0^*)', (\varphi x_1)']$, where φx_0^* is the mirror image of φx_2 w.r.t. φx_1. More exactly, it lies in an angular domain of vertex $(\varphi x_1)'$, with one leg passing through $(\varphi x_0^*)'$, and angular measure $o(1) + o(1) = o(1)$. Now $\widehat{(\psi y_2)'(\psi y_1)'} \cap \widehat{(\varphi x_0)'(\varphi x_1)'}$ lies in the intersection of these two angular domains, which is, for H^2 and $\mathbb{R}^2$, $\{(\varphi x_1)'\} = \{(\psi y_1)'\}$. Hence, for H^2 and $\mathbb{R}^2$, we have $\widehat{(\psi y_2)(\psi y_1)} \cap \widehat{(\varphi x_0)(\varphi x_1)} = \{\varphi x_1\}$, which proves (37) in these cases. For S^2 we still have to take into account that both $\widehat{(\varphi x_0)(\varphi x_1)}$ and $\widehat{(\psi y_2)(\psi y_1)}$ are subsets of $B(\varphi x_1, \varepsilon)$. Now the intersection of the above two angular domains and of $B(\varphi x_1, \varepsilon)$ is $\{\varphi x_1\}$, which proves (37) in this case as well.

If both arcs $\widehat{(\varphi x_1)(\varphi x_2)}$ and $\widehat{(\psi y_2)(\psi y_1)}$ are equal to the common chord $[\varphi x_1, \varphi x_2] = [\psi y_1, \psi y_2]$, then clearly this common chord is a subset of $(\varphi K) \cap (\psi L)$, as asserted. (But this inclusion may be proper.) This is valid also for $\operatorname{bd} K$ or $\operatorname{bd} L$ disconnected. *Henceforward we exclude the case that both* $\widehat{(\varphi x_1)(\varphi x_2)}$ *and* $\widehat{(\psi y_2)(\psi y_1)}$ *are segments (also for the case when the boundary of* K *or* L *is disconnected).*

If both these arcs are different from this common chord, then this common chord strictly separates these two arcs in M, except for their endpoints. Now let, e.g., $\widehat{(\varphi x_1)(\varphi x_2)} = [\varphi x_1, \varphi x_2] \neq \widehat{(\psi y_1)(\psi y_2)}$. Then, in a suitable rectangular coordinate system in the collinear model, $\widehat{(\psi y_2)'(\psi y_1)'} = \{(\xi, \eta) \mid \xi \in [\xi_1, \xi_2],\ \eta = h(\xi)\}$. Here h is concave on $[\xi_1, \xi_2]$ and positive on (ξ_1, ξ_2), and $h(\xi_i) = 0$. Hence $\{(\xi, \eta) \mid \xi \in [\xi_1, \xi_2],\ \eta = h(\xi)/2\}$ strictly separates these two arcs in M, except for their endpoints. So, in both of these cases,

$$\begin{cases} \text{no point of the shorter arc } \widehat{(\varphi x_1)(\varphi x_2)} \\ \text{lies in } \widehat{(\psi y_2)(\psi y_1)}, \text{ except } \varphi x_1 \text{ and } \varphi x_2. \end{cases} \tag{38}$$

Then by (36), (37) and (38),

$$\widehat{(\psi y_1)(\psi y_2)} \text{ intersects } \operatorname{bd}(\varphi K) \text{ only at } \varphi x_1 \text{ and } \varphi x_2. \tag{39}$$

3. Now we prove $M \subset (\varphi K) \cap (\psi L)$. By the angular conditions at φx_1 and φx_2 we see that in some of their neighbourhoods, except these points themselves, the arc $\widehat{(\psi y_2)(\psi y_1)}$ lies in $\operatorname{int}(\varphi K)$. Hence by (39) the relative interior of this arc lies in $\operatorname{int}(\varphi K)$. Therefore the closed convex curve $\operatorname{bd} M_1 = [(\varphi x_1), (\varphi x_2)] \cup \widehat{(\psi y_2)(\psi y_1)}$ (cf. (35)) is a subset of φK. Therefore $M_1 = \operatorname{conv}(\operatorname{bd} M_1) \subset \varphi K$. By (35) this proves $M \subset \varphi K$, thus $M \subset (\varphi K) \cap (\psi L)$.

4. Since in **2** we have excluded the case that both $\widehat{(\varphi x_1)(\varphi x_2)}$ and $\widehat{(\psi y_2)(\psi y_1)}$ are segments (also for $\mathrm{bd}\, K$ or $\mathrm{bd}\, L$ disconnected), therefore M has an interior point o. Now we show $M \supset (\varphi K) \cap (\psi L)$. Suppose $z \in [(\varphi K) \cap (\psi L)] \setminus M$. Then by **3**, $M \subset (\varphi K) \cap (\psi L)$, hence $o \in \mathrm{int}\, M \subset \mathrm{int}\, [(\varphi K) \cap (\psi L)]$. Then the open segment (o, z) satisfies $(o, z) \subset \mathrm{int}\, [(\varphi K) \cap (\psi L)] = [\mathrm{int}\, (\varphi K)] \cap [\mathrm{int}\, (\psi L)]$. However, it also intersects $\mathrm{bd}\, M$. Suppose, e.g., that $(o, z) \cap \widehat{(\varphi x_1)(\varphi x_2)} \neq \emptyset$. Then

$$\emptyset \neq (o, z) \cap [\widehat{(\varphi x_1)(\varphi x_2)}] \subset [\mathrm{int}\, (\varphi K)] \cap \mathrm{bd}\, (\varphi K) = \emptyset, \tag{40}$$

a contradiction, showing $M \supset (\varphi K) \cap (\psi L)$. Hence, by **3**, $M = (\varphi K) \cap (\psi L)$.

5. Second suppose that $\mathrm{bd}\, K$ or $\mathrm{bd}\, L$ has several connected components. This implies that X is $\mathbb{R}^2$ or H^2. Recall that in **2** we have excluded the case when both $\widehat{(\varphi x_1)(\varphi x_2)}$ and $\widehat{(\psi y_2)(\psi y_1)}$ are equal to the common chord $[\varphi x_1, \varphi x_2] = [\psi y_1, \psi y_2]$, also for disconnected $\mathrm{bd}\, K$ or $\mathrm{bd}\, L$. The sets $\mathrm{bd}\, K$ and $\mathrm{bd}\, L$ have at most countably infinitely many connected components K'_n and L'_m, resp. Let $K' = K'_1$ and $L' = L'_1$. Then by **8** of the proof of Lemma 1.3

$$\begin{cases} \text{we can choose an } \varepsilon_1, \text{ only depending on } K''(\delta), \ K, \\ L''(\delta) \text{ and } L, \text{ such that the closed } \varepsilon_1\text{-neighbourhood} \\ \text{of } K''(\delta) \text{ is disjoint to } \cup_{n \geq 2} K'_n, \text{ and the closed} \\ \varepsilon_1\text{-neighbourhood of } L''(\delta) \text{ is disjoint to } \cup_{m \geq 2} L'_m. \end{cases} \tag{41}$$

Let $K_n \supset K$ be the closed convex set, whose boundary is K'_n. Similarly we define L_m. Then

$$\begin{cases} (\varphi K) \cap (\psi L) = [\cap_n (\varphi K_n)] \cap [\cap_m (\psi L_m)] = P \cap Q, \text{ where} \\ P := (\varphi K_1) \cap (\psi L_1) \text{ and } Q := (\cap_{n \geq 2} K_n) \cap (\cap_{m \geq 2} L_m). \end{cases} \tag{42}$$

From the proof of Lemma 1.3 and the previous parts of the proof of this lemma, for $\varepsilon \in (0, \min\{\varepsilon(K, K''), \varepsilon(L, L''), \varepsilon_1\}]$ sufficiently small, we have $P \subset B(\varphi x_1, \varepsilon) \subset Q$.

The *radial function* of a nonempty closed convex set $C \subset X$, w.r.t. a point $c \in C$, is defined on the unit circle of the tangent space of X at c, with values in $[0, \infty]$, as follows. Its value at some u is the length of the maximal geodesic segment, starting from c, in direction u, contained in C. (Recall that now X is $\mathbb{R}^2$ or H^2!) The radial function of $P \cap Q$ w.r.t. φx_1 is the minimum of the radial functions of P and Q w.r.t. φx_1. The radial function of P w.r.t. φx_1 is at most ε, while that of Q is at least ε. Hence the radial function of $P \cap Q$, w.r.t. φx_1, equals the radial function of P, w.r.t. φx_1. Hence $(\varphi K) \cap (\psi L) = P \cap Q = P$. Thus, using (42), the case when $\mathrm{bd}\, K$ or $\mathrm{bd}\, L$ has several connected boundary components, is reduced to the case of connected boundaries (K'_1 and L'_1), which has been settled in **4.** ■

Proof of Theorem 1, **continuation. 2.** Now we continue the proof of the fact, that (2) $\Longrightarrow$ (3) in Theorem 1.

We will use the notations and hypotheses of Lemma 1.4. In particular, $[x_1, x_2]$ and $[y_1, y_2]$ have length ε.

First suppose that not both shorter arcs $\widehat{x_1 x_2}$ and $\widehat{y_1 y_2}$ are equal to the corresponding chords. Then $(\varphi K) \cap (\psi L)$ is bounded by the shorter arcs $\varphi\widehat{(x_1)\varphi(}x_2)$ and

$\widehat{\psi(y_1)\psi(y_2)}$, cf. Lemma 1.4. These shorter arcs have lengths at most $\varepsilon\,(1+o(1))$, cf. (17). Hence

$$\operatorname{diam}[(\varphi K)\cap(\psi L)] \le \operatorname{perim}[(\varphi K)\cap(\psi L)]/2 \le \varepsilon\,(1+o(1))\,. \tag{43}$$

Hence $(\varphi K)\cap(\psi L)$ has an arbitrarily small diameter, for $\varepsilon\to 0$. Moreover, by the hypothesis about the arcs, this intersection has a nonempty interior. Hence, by hypothesis, it admits a non-trivial congruence. Observe that this intersection has just two points of non-smoothness, namely $\varphi(x_1)=\psi(y_1)$ and $\varphi(x_2)=\psi(y_2)$. Thus, any non-trivial congruence admitted by $(\varphi K)\cap(\psi L)$ is a central symmetry, with centre the midpoint o of the segment (shorter segment for S^2) joining these two non-smooth points; or it is an axial symmetry, either with axis passing through these two non-smooth points, or with axis the perpendicular bisector of the segment with endpoints these two non-smooth points. (For S^2 also $-o$ is a centre of symmetry, but the symmetries w.r.t. the centres $\pm o$ coincide, so we only use o.)

Now consider the case that both above arcs $\widehat{x_1x_2}$ and $\widehat{y_1y_2}$ are equal to the corresponding chords, which have length ε. Then $(\varphi K)\cap(\psi L)$ may strictly contain the common chord $[\varphi x_1,\varphi x_2]=[\psi y_1,\psi y_2]$, thus, in particular, its diameter may be not small. In this case, therefore, we will consider, rather than this intersection, this common chord, as a degenerate closed convex set (i.e., with empty interior). Observe that this common chord (in general not equal to $(\varphi K)\cap(\psi L)$) has an arbitrarily small diameter, and admits all three non-trivial congruences from the last paragraph.

In both cases, the intersection (in the first case above), *or* the above common chord (in the second case above), has an arbitrarily small diameter, and admits (at least) one of the three above mentioned non-trivial congruences.

Lemma 1.5. *Suppose all hypotheses of Lemma* 1.4*. Further let $z_1,z_2\in X$, such that $d(z_1,z_2)=\varepsilon\in(0,\varepsilon(K,L,K'',L''))$. We define φ,ψ as the unique orientation preserving congruences of X satisfying $\varphi x_i=\psi y_i=z_i$ for $i=1,2$. Then in case* (A)*: $\widehat{(\varphi x_1)(\varphi x_2)}=\widehat{(\psi y_1)(\psi y_2)}=[z_1,z_2]$, we let $M(x_1,x_2,y_1,y_2)$ being this segment. Else* (B)*: let $M(x_1,x_2,y_1,y_2):=(\varphi K)\cap(\psi L)$. Then the map $(x_1,x_2,y_1,y_2)\mapsto M(x_1,x_2,y_1,y_2)$ is continuous from the set of (x_1,x_2,y_1,y_2)'s described in Lemma* 1.4 *(depending on K,K',K'' and L,L',L''), to the set of nonempty compact convex sets in X, with the Hausdorff-metric. Equivalently, the map which maps (x_1,x_2,y_1,y_2) to the image of $M(x_1,x_2,y_1,y_2)$ in the collinear model, is continuous in the Hausdorff metric of $\mathbb{R}^2$ (for S^2 and $\mathbb{R}^2$), and of the unit circle as a subset of $\mathbb{R}^2$ (for H^2).*

Proof. **1.**

(44) We denote by $(\cdot)'$ images in the collinear model, as a subset of $\mathbb{R}^2$

(for S^2, $\mathbb{R}^2$ and H^2, resp.).

For $X=H^2$ or $X=\mathbb{R}^2$ let $C\subset X$ be compact. For $X=S^2$ let C be a compact set of the open southern hemisphere. Then by (6) and (7) we have for distinct nonempty compact subsets A,B of C the following. The quotient of the Hausdorff distances of A and B in X, and of A' and B' in the collinear model, as a subset of $\mathbb{R}^2$, is bounded below and above. Hence for nonempty compact subsets A_n and B of C, convergence of A_n to B, in the Hausdorff metric of X, is equivalent to the convergence of A'_n to B', in the Hausdorff metric of the collinear model, as a subset of $\mathbb{R}^2$. This shows the equivalence in the last sentence of this lemma.

2. We suppose that the midpoint o of $[z_1, z_2]$ is mapped to 0 in the collinear model, and that z'_1, or z'_2 lies on the negative, or positive ξ-axis, resp. Then the straight line $z'_1 z'_2 = (\varphi x_1)'(\varphi x_2)' = (\psi y_1)'(\psi y_1)'$ is the *horizontal axis*, and the perpendicular bisector straight line of $[z'_1, z'_2]$ in $\mathbb{R}^2$ is the *vertical axis in* $\mathbb{R}^2$. We write $z'_i = (\zeta_i, 0)$.

In both cases (A) and (B) we cut $M(x_1, x_2, y_1, y_2)$ by $[z_1, z_2]$ to two nonempty compact convex parts, namely

$$\begin{cases} M_K, \text{ bounded by } \widehat{(\varphi x_1)(\varphi x_2)} \text{ and } [z_1, z_2], \\ \text{and } M_L, \text{ bounded by } \widehat{(\psi y_1)(\psi y_2)} \text{ and } [z_1, z_2]. \end{cases} \tag{45}$$

We will deal with the case of M_K (the case of M_L being analogous).

By the proof of Lemma 1.3, **5**, we have the following. The angle of the chord $[z_1, z_2]$, and of the tangents of the arc $\widehat{(\varphi x_1)(\varphi x_2)}$, at both endpoints z_1 and z_2 are $o(1)$ in X. The same statement holds also for their respective images in the collinear model, as subsets of $\mathbb{R}^2$ (by Lemma 1.1). Hence in the collinear model, as a subset of $\mathbb{R}^2$, we have that

$$M'_K = \{(\xi, \eta) \mid \zeta_1 \le \xi \le \zeta_2,\ f_K(\xi) \le \eta \le 0\}. \tag{46}$$

Here f_K is a C^1 convex function on $[\zeta_1, \zeta_2]$, with $f_K(\zeta_1) = f_K(\zeta_2) = 0$, and $f'_K(\zeta_1) = -o(1)$ and $f'_K(\zeta_2) = o(1)$ for $\varepsilon \to 0$ (with $o(1)$ nonnegative). Then on $[\zeta_1, \zeta_2]$ we have that f is non-positive, and $-o(1) \le f' \le o(1)$ for $\varepsilon \to 0$. These imply that for $\xi \in [\zeta_1, \zeta_2]$ we have $f(\xi) \ge \max\{-o(1) \cdot (\xi - \zeta_1), o(1) \cdot (\xi - \zeta_2)\}$. This implies that M'_K lies in the lower semicircle S' of the Thales circle of $[z_1, z_2]$, meant in $\mathbb{R}^2$ (for ε sufficiently small). Moreover,

$$M = M_K \cup M_L = \operatorname{conv}(M_K \cup M_L), \text{ hence } M' = \operatorname{conv}(M'_K \cup M'_L). \tag{47}$$

Hence for the *support functions* $h(M', u)$ etc. of M', M'_K and M'_L (defined for $u \in S^1$) we have

$$h(M', u) = \max\{h(M'_K, u), h(M'_L, u)\}. \tag{48}$$

We write $C(S^1) = \{\text{continuous functions } S^1 \to \mathbb{R}\}$, with the maximum norm, and hence with the topology of uniform convergence.

Recall that the Hausdorff distance of two convex sets in $\mathbb{R}^2$ equals the distance, in the maximum norm, of their support functions (defined on S^1). Therefore, in $\mathbb{R}^2$, convergence, in the Hausdorff metric, of nonempty compact convex sets to a limit nonempty compact convex set, is equivalent to the uniform convergence in $C(S^1)$ of their respective support functions. Now recall that the maximum operation of two functions in $C(S^1)$ preserves uniform limits. (I.e., uniform convergence of f_n to f and of g_n to g imply uniform convergence of $\max\{f_n, g_n\}$ to $\max\{f, g\}$.) Therefore, by (48), it suffices to prove continuous dependence of the support functions of M'_K and M'_L on x_1, x_2, y_1, y_2 in $C(S^1)$. Observe that M'_K only depends on x_1, x_2, and M'_L only depends on y_1 and y_2. (Since x_1 and x_2 are C^1 functions of each other, we could even say dependence only on x_1, or only on x_2, whichever lies in $\mathrm{relint}_{K'} K''$; and similarly for the y_i's.)

3. Therefore we are going to show continuous dependence of M'_K on x_1 and x_2. (That of M'_L on y_1, y_2 is shown analogously.)

We distinguish two cases.
(1): $M'_K \neq [z_1, z_2]$, and
(2): $M'_K = [z_1, z_2]$.

In case (1) we rewrite convergence of a sequence A_n of nonempty compact convex sets to a convex body B in the Hausdorff metric in $\mathbb{R}^2$. Choose some $b \in \mathrm{int}\, B$. Then a neighbourhood base of B is
(49)

$$\begin{cases} \{B^* \mid \emptyset \neq B^* \subset \mathbb{R}^2 \text{ is a compact convex set, and } h(B^*, u) \in [h(B,u) - \delta, \\ h(B,u) + \delta]\} = \{B^* \mid \emptyset \neq B^* \subset \mathbb{R}^2 \text{ is a compact convex set, and } h(B^* - b, u) \\ = h(B^*, u) - \langle b, u \rangle \in [h(B - b, u) - \delta, h(B - b, u) + \delta]\}, \text{ for } \delta \in (0,1). \end{cases}$$

Since $h(B^* - b, u)$ is bounded below and above, we may rewrite this as

$$(50)\quad \begin{cases} \{B^* \mid B^* \subset \mathbb{R}^2 \text{ is a nonempty compact convex set, and } h(B^* - b, u) \in \\ [(1-\delta)h(B - b, u), (1+\delta)h(B - b, u)]\} = \{B^* \mid B^* \subset \mathbb{R}^2 \text{ is a convex} \\ \text{body, and } (1-\delta)(B - b) \subset (B^* - b) \subset (1+\delta)(B - b)\}, \text{ for } \delta \in (0,1). \end{cases}$$

We denote by $\varrho(B - b, u)$ the radial function of the convex body $B - b \subset \mathbb{R}^2$ w.r.t. 0, for $0 \in \mathrm{int}\,(B - b)$. (Thus $\mathrm{bd}\,(B - b)$ has equation $r = \varrho(B - b, u)$, for $u \in S^1$, in polar coordinates u, r.) Hence we can rewrite the second half of (50) as

$$(51)\quad \begin{cases} \{B^* \mid B^* \subset \mathbb{R}^2 \text{ is a convex body, and } \varrho(B^* - b, u) \in \\ [(1-\delta)\varrho(B - b, u), (1+\delta)\varrho(B - b, u)]\}, \text{ for } \delta \in (0,1). \end{cases}$$

Since $\varrho(B - b, u)$ is bounded below and above, we can further rewrite (51) as

$$(52)\quad \begin{cases} \{B^* \mid B^* \subset \mathbb{R}^2 \text{ is a convex body, and } \varrho(B^* - b, u) \\ \in [\varrho(B - b, u) - \delta, \varrho(B - b, u) + \delta]\}, \text{ for } \delta \in (0,1) \end{cases}$$

or as

$$(53)\quad \begin{cases} \{B^* \mid B^* \subset \mathbb{R}^2 \text{ is a convex body, and the distance of } \varrho(B - b, u) \\ \text{ and } \varrho(B^* - b, u) \text{ in } C(S^1) \text{ is at most } \delta\}, \text{ for } \delta \in (0,1). \end{cases}$$

4. Let x_1, x_2 be as in this lemma, and let x_1^*, x_2^*, with the same properties, be close to x_1, x_2.

$$(54)\qquad \text{Let } b' \in \mathrm{int}\, M'_K.$$

Then for x_i^* sufficiently close to x_i, we have that the shorter arc $\widehat{(\varphi x_1^*)'(\varphi x_2^*)'}$ of $\mathrm{bd}\,(\varphi K)'$ is not equal to $[(\varphi x_1^*)', (\varphi x_2^*)']$. Moreover, b' lies on the same open side of the line $(\varphi x_1^*)'(\varphi x_2^*)'$ as this shorter arc.

Let H'_- denote the closed lower halfplane $\eta \leq 0$. Let $(H'_-)^*$ be the closed halfplane bounded by the line $(\varphi x_1^*)'(\varphi x_2^*)'$, containing the shorter arc $\widehat{(\varphi x_1^*)'(\varphi x_2^*)'}$

of $\mathrm{bd}\,(\varphi K)'$. Let $(M_K')^* := (\varphi K)' \cap (H_-')^*$ (a segment of $(\varphi K)'$). Then, for x_i^* sufficiently close to x_i, we have $b' \in \mathrm{int}\,(M_K')^*$. Let $(S')^*$ be the intersection of the Thales circle of $[(\varphi x_1^*)', (\varphi x_2^*)']$ (meant in $\mathbb{R}^2$) with $(H_-')^*$. By **2** of this proof we have $(M_K')^* \subset (S')^* \subset (H_-')^*$. Hence

$$\begin{cases} (M_K')^* = (M_K')^* \cap (S')^* = [(\varphi K)' \cap (H_-')^*] \cap (S')^* \\ = (\varphi K)' \cap [(H_-')^* \cap (S')^*] = (\varphi K)' \cap (S')^*. \end{cases} \tag{55}$$

Then for the radial functions we have

$$\varrho\left((M_K')^* - b', u\right) = \min\{\varrho\left((\varphi K)' - b', u\right), \varrho\left((S')^* - b', u\right)\}. \tag{56}$$

Now observe that, by Lemma 1.3, (4), and elementary geometry, for x_i^* sufficiently close to x_i, we have in the Hausdorff distance of $\mathbb{R}^2$, that $(S')^*$ is sufficiently close to S'. Then, by the equivalence of (49) and (53), we have that $\varrho\left((S')^* - b', u\right)$ is sufficiently close to $\varrho(S' - b', u)$. Now recall that the minimum operation of two functions in $C(S^1)$ preserves uniform limits (in the sense as the maximum operation of two functions). Then by (56), for x_i^* sufficiently close to x_i, we have that in the $C(S^1)$-norm, $\varrho\left((M_K')^* - b', u\right)$ is sufficiently close to $\varrho(M_K' - b', u)$. Therefore, once more using the equivalence of (49) and (53), for x_i^* sufficiently close to x_i, we have in the Hausdorff distance of $\mathbb{R}^2$, that $(M_K')^*$ is sufficiently close to M_K'.

Then, by **1** of this proof, for x_i^* sufficiently close to x_i, we have in the Hausdorff distance of X that M_K^* is sufficiently close to M_K (where $M_K^* \subset X$ is the inverse image of the subset $(M_K')^*$ of the model). This proves the statement of the lemma in case (1) in **3**.

5. There remains to prove the statement of the lemma in case (2) in **3**. Suppose that x_i^* is sufficiently close to x_i (for given $\varepsilon = d_X(x_1, x_2) = d_X(x_1^*, x_2^*)$), and that x_1^* follows x_1 on $\mathrm{bd}\,(\varphi K)$, e.g., in the positive sense. Then by Lemma 1.3, (4), the order of our points on $\mathrm{bd}\,(\varphi K)$, in the positive sense, is $\varphi x_1, \varphi x_1^*, \varphi x_2, \varphi x_2^*$. Then we have to show that the Hausdorff distance, in $\mathbb{R}^2$, of $M_K' = [z_1', z_2']$ and $(M_K')^*$ is small.

On the one hand, M_K lies in the $d_X(\varphi x_1, \varphi x_1^*)$-neighbourhood of $(M_K)^*$, where $d_X(\varphi x_1, \varphi x_1^*)$ is small.

So, by **1** of this proof, there remains to show that also $(M_K')^*$ lies in a small metric neighbourhood of M_K', meant in $\mathbb{R}^2$, containing the collinear model ($\mathbb{R}^2$ or the open unit circle). We are going to prove that a small metric neighbourhood of the segment $[(\varphi x_1^*)', (\varphi x_2)']$, meant in $\mathbb{R}^2$, contains $(M_K')^*$, which implies the above statement. Actually, since the small metric neighbourhood of the segment $[(\varphi x_1^*)', (\varphi x_2)']$, meant in $\mathbb{R}^2$, is convex in $\mathbb{R}^2$, it suffices to show that it contains $\mathrm{bd}\,((M_K')^*)$ (whose convex hull is $(M_K')^*$).

The boundary $\mathrm{bd}\,((M_K')^*)$ consists of three parts: the segments $[(\varphi x_1^*)', (\varphi x_2)']$ and $[(\varphi x_1^*)', (\varphi x_2^*)']$, and the shorter arc $\widehat{(\varphi x_2)'(\varphi x_2^*)'}$. The convex hull of $[(\varphi x_1^*)', (\varphi x_2)'] \cup \widehat{(\varphi x_2)'(\varphi x_2^*)'}$ contains $[(\varphi x_1^*)', (\varphi x_2^*)']$. Therefore it is sufficient to show that the small metric neighbourhood of the segment $[(\varphi x_1^*)', (\varphi x_2)']$ contains $[(\varphi x_1^*)', (\varphi x_2)']$ and $\widehat{(\varphi x_2)'(\varphi x_2^*)'}$. The first containment is obvious, so we only need to show that this small metric neighbourhood contains $\widehat{(\varphi x_2)'(\varphi x_2^*)'}$. In turn, this will be shown if we will show that a small neighbourhood of $(\varphi x_2)'$ contains $\widehat{(\varphi x_2)'(\varphi x_2^*)'}$. However, by Lemma 1.3, (4), $\widehat{(\varphi x_2)'(\varphi x_2^*)'}$ has a small Euclidean length. Since the chord length is always at most the corresponding arc length, therefore the Euclidean distance of $(\varphi x_2)'$ and any point of $\widehat{(\varphi x_2)'(\varphi x_2^*)'}$ is small. This proves the statement of the lemma in case (2) in **3**. ■

We will call a convex surface in $\mathbb{R}^d$ (i.e., the boundary of a proper closed convex subset of $\mathbb{R}^d$, with non-empty interior) at some of its points *twice differentiable* if the following holds. Locally it is the graph, in a suitable rectangular coordinate system, of a function having a Taylor series expansion of second degree at this point, with an error term $o(\| \cdot \|^2\|)$. By [13], pp. 31-32 (in both editions), convex surfaces in $\mathbb{R}^d$ are almost everywhere twice differentiable. This extends to S^d and H^d by using their collinear models.

Lemma 1.6. *Assume* (1) *with* $d = 2$. *Let* K *and* L *be* C^1. *Suppose* (2) *of Theorem* 1, *and all hypotheses of Lemma* 1.4. *Suppose that there exists a sequence* $\varepsilon_n \to 0$, *where each* ε_n *is sufficiently small, such that we have the following. With the notations of Lemmas* 1.3 *and* 1.4, *either* K', *or* L' *has a chord* $[x_1, x_2]$, *or* $[y_1, y_2]$, *with* x_2 *following* x_1 *in the positive sense, or* y_2 *following* y_1 *in the negative sense, resp., and with at least one endpoint in* $\text{relint}_{K'} K''$, *or* $\text{relint}_{L'} L''$, *resp., such that the following holds. The chord* $[x_1, x_2]$, *or* $[y_1, y_2]$, *is of length* ε_n. *Moreover, the shorter arc determined by this chord, either on* K', *or on* L', *is not symmetrical w.r.t. the orthogonal bisector of the chord (*in particular, the respective shorter arc is different from the chord*). Then this leads to a contradiction.*

Proof. **1.** Let $[x_1, x_2]$, or $[y_1, y_2]$ be a chord of K', or of L', with at least one endpoint in $\text{relint}_{K'} K''$, or $\text{relint}_{L'} L''$, and of length ε_n (which replaces ε from Lemmas 1.4 and 1.5), with x_2 following x_1 on $\text{bd}\, K$ in the positive sense, or y_2 following y_1 on $\text{bd}\, L$ in the negative sense, resp. *Let* φ *and* ψ *be chosen so, that* $\varphi(x_i) = \psi(y_i) =: z_i$ (for $i = 1, 2$), where $d_X(z_1, z_2) = \varepsilon_n$. Then, by Lemma 1.4, $(\varphi K) \cap (\psi L)$ is bounded by the shorter arcs $\widehat{\varphi(x_1)\varphi(x_2)}$ and $\widehat{\psi(y_1)\psi(y_2)}$. (Observe that at least one of the arcs $\widehat{x_1 x_2}$ and $\widehat{y_1 y_2}$ is different from the respective chord. Therefore the case that $(\varphi K) \cap (\psi L)$ strictly contains this chord, and thus is degenerate, cannot occur, by Lemma 1.4.) Then the intersection $(\varphi K) \cap (\psi L)$ has a nonempty interior, and by (43) it has a diameter at most $\varepsilon_n (1 + o(1))$, which is arbitrarily small for n sufficiently large. Hence it admits some non-trivial congruence. By the hypothesis of the lemma this cannot be a symmetry w.r.t. the perpendicular bisector of $[z_1, z_2]$, which we call the *vertical axis*. That is, this congruence is a central symmetry w.r.t. the *midpoint* o *of* $[z_1, z_2]$, or is an axial symmetry w.r.t. the straight line $z_1 z_2$, which we call the *horizontal axis* (cf. the proof of Theorem 1, **2**).

Observe that both central symmetry w.r.t. the midpoint of $[z_1, z_2]$, and axial symmetry w.r.t. the horizontal axis, cannot occur. Namely, then we would have also an axial symmetry w.r.t. the vertical axis, which is excluded by the hypothesis of this lemma.

In the case of central symmetry w.r.t. o, the two (shorter) arcs $\widehat{x_1 x_2}$ of $\text{bd}\, K$ and $\widehat{y_1 y_2}$ of $\text{bd}\, L$, resp., are congruent, with x_1 corresponding to y_2, and x_2 corresponding to y_1. In case of axial symmetry w.r.t. the horizontal axis, once more the above arcs are congruent, but now with x_1 corresponding to y_1, and x_2 corresponding to y_2.

We will consider the one-sided curvatures, provided they exist, of K'' at x_i, in the sense towards x_{2-i} (i.e., of the shorter arc $\widehat{x_1 x_2}$), and similarly, of L'' at y_j, in the sense towards y_{2-j}. Here $x_i \in \text{relint}_{K'} K''$, and $y_j \in \text{relint}_{L'} L''$. For both considered symmetries (central, and axial w.r.t. the horizontal axis), the above considered two one-sided curvatures exist and are equal at the corresponding points, or they both do not exist at the corresponding points.

Now recall from Lemma 1.4, that any of x_1, x_2, or of y_1, y_2 could be any point of $\mathrm{relint}_{K'}K''$, or of $\mathrm{relint}_{L'}L''$, resp.

2. First suppose the case that, for all choices of x_1, x_2, y_1, y_2, we have central symmetry w.r.t. the midpoint of $[z_1, z_2]$. Then $\varphi(x_1)$ corresponds by this symmetry to $\psi(y_2)$. Here x_1, y_2 could be any points of $\mathrm{relint}_{K'}K''$ and $\mathrm{relint}_{L'}L''$. Therefore, for all points of $\mathrm{relint}_{K'}K''$ and $\mathrm{relint}_{L'}L''$, the considered one-sided curvatures exist and are equal, or they do not exist for any points. However, convex curves – and surfaces – are almost everywhere twice differentiable, in the sense as stated before this lemma. This rules out the second case. Now, replacing x_1, y_2 by x_2, y_1, we obtain the same for one-sided curvatures, but now in the opposite sense. Therefore, at all points of $\mathrm{relint}_{K'}K''$ and $\mathrm{relint}_{L'}L''$, the above considered two one-sided curvatures exist and are equal. Since the curvatures of K'' and L'' exist almost everywhere, the common values of the two one-sided curvatures are also equal.

3. Second suppose the case that, for all choices of x_1, x_2, y_1, y_2, we have axial symmetry, w.r.t. the horizontal axis. Then $\varphi(x_1)$ corresponds by this symmetry to $\psi(y_1)$. Now, x_1, y_1, and also x_2, y_2, could be any points of $\mathrm{relint}_{K'}K''$, and $\mathrm{relint}_{L'}L''$. Then, with this notational change, we repeat the arguments of the preceding paragraph. Thus we gain that, at all points of $\mathrm{relint}_{K'}K''$ and $\mathrm{relint}_{L'}L''$, the curvatures exist and are equal.

4. As third case, there remains the case that, for some choice of x_1, x_2, y_1, y_2 we have central symmetry w.r.t. o, and for some other choice of these points we have axial symmetry w.r.t. the horizontal axis.

$$(57)\quad \begin{cases} \text{We claim that the configurations of the points } x_1, x_2, y_1, y_2 \text{ in } K' \times K' \\ \times L' \times L', \text{ with } x_2 \text{ following } x_1 \text{ in the positive sense, and } y_2 \text{ following} \\ y_1 \text{ in the negative sense, where still we suppose, that one of } x_1, x_2 \\ \text{belongs to } \mathrm{relint}_{K'}K'', \text{ and one of } y_1, y_2 \text{ belongs to } \mathrm{relint}_{L'}L'', \text{ and} \\ \text{that } d(x_1, x_2) = d(y_1, y_2) = \varepsilon_n, \text{ is a connected topological space.} \end{cases}$$

Then the configuration space of the points x_1, x_2, y_1, y_2 is the product of the configuration spaces of the points x_1, x_2 and of the points y_1, y_2, and the product of connected spaces is connected. Therefore

$$(58)\quad \text{it suffices to show connectedness of the configuration space of the points } x_1, x_2$$

(then connectedness of the configuration space of the points y_1, y_2 follows similarly).

Suppose that x_1 belongs to $\mathrm{relint}_{K'}K''$ (which is by Lemma 1.3 homeomorphic either to an open segment, or to S^1, so it is connected). Then by Lemma 1.3, (4), x_2 depends continuously on x_1, hence also the ordered pair (x_1, x_2) depends continuously on x_1. Since a continuous image of a connected space is connected, therefore these ordered pairs form a connected space. We have the analogous statement if $x_2 \in \mathrm{relint}_{K'}K''$. Moreover, these two connected spaces intersect, provided some $x_1, x_2 \in \mathrm{relint}_{K'}K''$ have a distance ε_n. This happens if the arclength of K'' in X is greater than $2\varepsilon_n$ (cf. (24)), which can be supposed. Now it suffices to recall that the union of two intersecting connected spaces is itself connected. This ends the proof of (58), hence of (57).

Further, we claim that

(59)
$$\begin{cases} \text{the set of configurations of the points } x_1, x_2, y_1, y_2, \text{ for which one of the} \\ \text{considered symmetry properties holds, is a closed subset of } K' \times K' \times L' \times L'. \end{cases}$$

In fact, by Lemma 1.5, the map $(x_1, x_2, y_1, y_2) \mapsto M(x_1, x_2, y_1, y_2)'$ (this is the image, in the collinear model, of $M(x_1, x_2, y_1, y_2)$) is continuous in the Hausdorff metric of $\mathbb{R}^2$ (for $X = S^2, \mathbb{R}^2$), or of the unit circle, as a subset of $\mathbb{R}^2$ (for $X = H^2$). We suppose that z_1' and z_2' lie in the negative and positive ξ-axis, resp., and $(z_1' + z_2')/2 = 0$. Then $o' = 0$, and the above horizontal and vertical axes (cf. **1** of this proof) are mapped in the collinear model into the ξ- and η-axes in $\mathbb{R}^2$, resp.

Then central symmetry of $M(x_1, x_2, y_1, y_2)$ w.r.t. o, i.e., central symmetry of $M(x_1, x_2, y_1, y_2)'$ w.r.t. 0, can be expressed via the support function of $M(x_1, x_2, y_1, y_2)'$ as $h\,(M(x_1, x_2, y_1, y_2)', u) = h\,(M(x_1, x_2, y_1, y_2)', -u)$. Analogously, symmetry of $M(x_1, x_2, y_1, y_2)$ w.r.t. the horizontal axis, i.e., symmetry of $M(x_1, x_2, y_1, y_2)'$ w.r.t. the ξ-axis, can be expressed as $h\,(M(x_1, x_2, y_1, y_2)', (u_1, u_2)) = h\,(M(x_1, x_2, y_1, y_2)', (u_1, -u_2))$. Clearly both of these properties are preserved by (uniform) convergence of the functions $(x_1, x_2, y_1, y_2) \mapsto h\,(M(x_1, x_2, y_1, y_2)', u)$ to a limit function. This proves (59).

Further, the union of these two nonempty closed subsets is the entire space of all above configurations of the points x_1, x_2, y_1, y_2. By (57), these two closed subsets must intersect. That is, we must have a configuration, that simultaneously has both the central symmetry w.r.t. o, and the axial symmetry w.r.t. the horizontal axis. This, however, contradicts the second paragraph of **1** of the proof of this lemma.

5. So the third case (investigated in **4**) cannot occur. Therefore we must have either the first, or the second case (investigated in **2** and **3**, resp.). Both had the conclusion that, at all points of $\mathrm{relint}_{K'} K''$ and $\mathrm{relint}_{L'} L''$, the curvatures exist and are equal. In other words, both K'' and L'' have equal constant curvatures, i.e., both are arcs of congruent cycles (including entire compact cycles, i.e., circles), or are segments.

Recall that K'', or L'' were arbitrary compact subarcs of K', or L', if K', or L' were homeomorphic to $\mathbb{R}$, and they were equal to $K' = \mathrm{bd}\, K$, or $L' = \mathrm{bd}\, L$, if K', or L' was homeomorphic to S^1. Thus in both cases, K' and L' are congruent cycles, or are straight lines. However, this contradicts the assumptions about "not axial symmetry of some shorter arc w.r.t. the orthogonal bisector line of the corresponding chord" of this lemma. ■

Lemma 1.7. *Assume* (1) *with* $d = 2$. *Let* K *be* C^1. *Let* K' *and* K'' *be as in Lemma* 1.3. *Suppose that for each sufficiently small* $\varepsilon > 0$ *we have the following. For any chords* $[x_1, x_2]$ *of* K', *with* x_2 *following* x_1 *on* K' *in the positive sense, and with at least one endpoint in* $\mathrm{relint}_{K'} K''$, *and with length of these chords being* ε, *the following holds. The shorter arcs determined by these chords, on* K', *are symmetrical w.r.t. the perpendicular halving straight line of the chord (*in particular, the respective shorter arcs may coincide with these chords*). Then* K' *is a cycle, or a straight line. Moreover, if* K' *is a circle or paracycle, then* K *is a circle (disk) or a paracircle, resp. For* L' *and* L *there holds the analogous statements, with* y_2 *following* y_1 *on* L' *in the negative sense.*

Proof. Observe that the perpendicular halving straight line of such a chord also halves the shorter respective arc, and is perpendicular to it at its midpoint (it cannot touch this arc).

By the statement before Lemma 1.6 convex curves are almost everywhere twice differentiable, in the sense given there, hence have curvatures almost everywhere. Now let $x' \in \text{relint}_{K'} K''$, such that K' is twice differentiable at x'. Further, let $x'' \in \text{relint}_{K'} K''$ be arbitrary. Then there exist $x' = x_1, \dots, x_n = x'' \in \text{relint}_{K'} K''$, following each other in the same sense, and such, that the distance of x_i and x_{i+1} (in X) is less than ε, for $i = 1, \dots, n-1$. Then x_i and x_{i+1} are symmetrical w.r.t. the perpendicular bisector of the chord $[x_i, x_{i+1}]$. Then x_i and x_{i+1} are symmetrical also w.r.t. the perpendicular bisector of some other chord. For this the corresponding shorter arc I' contains the closed shorter arc $I = \widehat{x_i x_{i+1}}$ in its relative interior w.r.t. K', with I' being only slightly longer than I. Moreover, these two arcs have the same midpoint, and the chord corresponding to I' is still shorter than ε. Further, also I' is symmdtrical w.r.t. the perpendicular bisector of $[x_i, x_{i+1}]$.

Then by induction one sees that K'' is twice differentiable at each x_i, and the curvatures of K'' at each x_i are equal to the same number $\kappa \geq 0$. In particular, the curvature of K' at any $x'' \in \text{relint}_{K'} K''$ equals the constant κ. Recall from Lemma 1.3 that K'' was equal to $K' = \text{bd}\, K$, if K was compact and then K' was homeomorphic to S^1. Moreover, K'' was any compact arc of K' if K was noncompact and then K' was homeomorphic to $\mathbb{R}$. Hence the curvature of K' at any of its points equals the constant κ.

For K noncompact (thus $X = \mathbb{R}^2, H^2$) we have that K' joins two infinite points (possibly coinciding). Both for K compact and noncompact, and for any X, we have that K' is a maximal C^2 curve of constant curvature, i.e., a cycle, or a straight line, as asserted. If K' is a circle or a paracycle, then $\text{conv}\, K' \subset K$. If here we have an equality, then K is a circle or a paracircle. Otherwise, there is a $k \in K \setminus (\text{conv}\, K')$. Then for $k^* \in \text{int}\,(\text{conv}\, K')$ we have that the open segment (k^*, k), which lies in $\text{int}\, K$, intersects K', which lies in $\text{bd}\, K$. This is a contradiction. (Also cf. the last paragraph of the proof of Lemma 1.9 in [8].) This ends the proof of the lemma. ■

Proof of Theorem 1, **continuation. 3.** Observe that Lemmas 1.6 and 1.7, with their complementary hypotheses, together prove (2) $\Longrightarrow$ (3) in Theorem 1.

The particular case of (2) $\Longrightarrow$ (3), for central symmetry, now follows easily. If the connected components K' and L' are not congruent, one of them is strictly convex. Now let φK and ψL touch each other, so that K' and L' touch each other. Pushing them slightly towards each other, the new intersection has a small diameter, hence is centrally symmetric. It has two points of non-smooothness, the common endpoints of the boundary arcs on K' and L'. The central symmetry should exchange these two common endpoints, hence also these two boundary arcs. However, these two boundary arcs have different curvatures, so this is impossible. This contradiction implies the equality of the curvatures of K' and L'.

Next we turn to the investigation of the implication (3) $\Longrightarrow$ (1) in Theorem 1, under the respective hypotheses.

Lemma 1.8. *Assume* (1) *with* $d = 2$, *and* (3) *of Theorem* 1. *For* $X = H^2$, *let both for* K *and* L, *the infimum of the positive curvatures of its connected boundary components be positive, and let it have at most one boundary component with* 0 *curvature. Then for* $X = S^2$, $\mathbb{R}^2$, H^2, (1) *of Theorem* 1 *holds.*

Proof. **1.** For $X = S^2$, (3) of Theorem 1 clearly implies (1) of Theorem 1. (Observe that for both K and L halfspheres, (1) of Theorem 1 holds vacuously.)

2. For $X = \mathbb{R}^2$, supposing (3) of Theorem 1, we have the following. The closed convex sets in $\mathbb{R}^2$, whose boundaries are disconnected, are just the parallel strips. Furthermore, the closed convex sets in $\mathbb{R}^2$, with connected boundaries, whose boundaries are cycles or straight lines, are just circles or half-planes, resp. Thus, any of K and L can be a circle, a parallel strip, or a half-plane. If one of K and L is a circle, then $(\varphi K) \cap (\psi L)$ is axially symmetric, hence (1) of Theorem 1 holds. If each of K and L is either a parallel strip or a half-plane, then there does not exist $(\varphi K) \cap (\psi L)$, with nonempty interior, and of arbitrarily small diameter. Hence now (1) of Theorem 1 holds vacuously.

3. Let $X = H^2$, and suppose (3) of Theorem 1. Then each of K and L is either a circle, or a paracircle, or has boundary components which are hypercycles or straight lines. The infimum of the positive curvatures of its boundary components is of course the same infimum, taken only for the hypercycle components. By the hypothesis of the lemma, the distances, for which these hypercycles are distance lines, have an infimum $c > 0$, say. Moreover, there may be still at most one straight line component, both for K and for L.

Let, e.g., K_1 and K_2 be two boundary components of K, and let $x_1 \in K_1$, and $x_2 \in K_2$. Let $K' \subset K$ and $L' \subset L$ be defined, as the nonempty closed convex sets (possibly with empty interiors), bounded by all the straight lines for which the boundary components are distance lines, and by the at most one straight line component. In particular, K_1 and K_2 are distance lines for K_1' and K_2', with a non-negative signed distance. Then the segment $[x_1, x_2]$ intersects both K_1' and K_2', at points $x_1' \in K_1'$ and $x_2' \in K_2'$, and for the distances we have $d(x_1, x_2) \geq d(x_1, x_1') + d(x_2', x_2) \geq c$. This means that the distances of the different boundary components both of K, and of L, are bounded from below by c. The same holds vacuously for circles and paracircles. Hence, if $\operatorname{diam}[(\varphi K) \cap (\psi L)] < c$, then $(\varphi K) \cap (\psi L)$ is compact, and is bounded by portions of only one boundary component of φK, and of ψL.

Thus $(\varphi K) \cap (\psi L)$ is the intersection of two sets, both being a circle, a paracircle, or a convex domain bounded by a hypercycle, including a half-plane. Recall that a circle, and a paracircle are axially symmetric w.r.t. any straight line passing through their centres. Thus, if both above sets are a circle or a paracircle, then their intersection is axially symmetric. There remain the cases when one set is a convex set bounded by a hypercycle, and the other one is a circle, a paracircle, or a convex set bounded by a hypercycle. In the first case an axis of symmetry of the intersection is the straight line passing through the centre of the circle, and orthogonal to the base line of the hypercycle. In the second case, by compactness of the intersection, the centre of the paracircle cannot lie at an endpoint of the base line. Therefore an axis of symmetry of the intersection is the straight line passing through the centre of the paracircle, and orthogonal to the base line of the hypercycle. In the third case, again by compactness of the intersection, having interior points, the base lines of the hypercycles are ultraparallel. Moreover, the hypercycles lie on those closed sides of their base lines, as the other base line. Therefore, the unique straight line orthogonal to both base lines is an axis of symmetry of the intersection. ■

Last we turn to the investigation of (3) $\not\Longrightarrow$ (2) in Theorem 1, under the respective hypotheses.

Lemma 1.9. *Assume* (1) *with* $d = 2$, *and* (3) *of Theorem* 1 *for* K. *Let* $X = H^2$, *and let* K *be not a circle or a paracircle. Let us prescribe in any way the curvatures of the hypercycle and straight line connected boundary components of* K *(with multiplicity), so that the infimum of the positive curvatures is* 0, *or there are two* 0 *curvatures. Then there exists a* K, *with these prescribed curvatures of its hypercycle and straight line boundary components (with multiplicity), and an* L, *such that* (3) *of Theorem* 1 *holds for* K *and* L, *but* (2) *of Theorem* 1 *does not hold for them.*

Proof. **1.** We begin with an example $K = K_0$ and $L = L_0$, where each of $\mathrm{bd}\, K_0$ and $\mathrm{bd}\, L_0$ consists of two straight lines. We consider the collinear model. Let $k_1, k_2 \subset H^2$ be distinct parallel straight lines, with axis of symmetry k. Let, for $i = 1, 2$, the points $x_i, y_i \in k_i$ be symmetric w.r.t. k, with all six pairwise distances at most ε. Moreover, let the point x_i separate y_i and the common infinite point of k_1 and k_2. Then $x_1x_2y_2y_1$ is a symmetrical quadrangle of diameter at most ε (since $X = H^2$). Moreover, it is the intersection of the closed convex sets K_0, bounded by k_1 and k_2, and L_0, bounded by the straight lines x_1x_2 and y_1y_2.

Let us consider a small generic perturbation $x_1'x_2'y_2'y_1'$ of the quadrangle $x_1x_2y_2y_1$, where x_1' is the small perturbation of x_1 etc., satisfying $x_i', y_i' \in k_i$. Then by this perturbation K_0 goes over to the closed convex set K_0' bounded by the parallel but distinct lines $x_1'y_1' = k_1$ and $x_2'y_2' = k_2$, i.e., $K_0' = K_0$. Moreover, L_0 goes over to the closed convex set L_0' bounded by the ultraparallel straight lines $x_1'x_2'$ and $y_1'y_2'$. Then any non-trivial congruence admitted by the perturbed quadrangle preserves both pairs of opposite sides (separately the parallel and the ultraparallel ones), and preserves the above separation properties. Therefore x_1' (and y_1') has as image either itself, or x_2' (and y_2'). Then the congruence is an identity, or it exchanges x_1', x_2' as well as y_1', y_2'. The second case is only possible if $d(x_1', y_1') = d(x_2', y_2')$. Generically this equality does not hold, so generically a non-trivial congruence admitted by the perturbed quadrangle does not exist. Let us fix some such generic quadrangle $x_1'x_2'y_2'y_1'$, which admits no non-trivial congruence. We may suppose that $\mathrm{diam}\,(x_1'x_2'y_2'y_1') \le 2\varepsilon$.

2. Suppose that the set (with multiplicity) of the positive curvatures of the connected hypercycle boundary components K_i of K is prescribed, and has infimum $c = 0$, or there are at least two 0 curvatures. Then we make the following generalization of the above example. We begin with constructing K. *These hypercycles* K_i *are distance lines, with base lines* K_i^*, *and for prescribed distances* c_i (cf. §3). We consider a closed convex set K', bounded in the collinear model by the prescribed many, at least two, but at most countably infinitely many chords K_i^* of the collinear model circle, one for each i. Let, with at most one exception, these chords occur in disjoint pairs having exactly one common endpoint. Hence $\mathrm{int}\, K' \neq \emptyset$. Then we replace these chords K_i^* by the corresponding distance lines K_i, outwards from K'. If

(1) there are two 0 curvatures, then the corresponding chords K_i^* should occur in an above pair, and if

(2) $c = 0$, then there should be above pairs of (K_i^*)'s, for which both distances c_i are arbitrarily small.

We define L as L_0', and we define ψ as identity.

In case (1) there are two boundary components K_i, with $c_i = 0$, hence satisfying $K_i = K_i^*$, with a common infinite point. Recall that any three distinct points of the boundary circle of the model (collinear or conformal) can be taken over to any other three distinct points of the boundary circle, of the same orientation, by (the extension of) some orientation-preserving congruence. Therefore we may choose φ so that it takes these two K_i's to the above k_1, k_2. Also we may suppose that, in the collinear model, the images of k_1 and k_2 enclose a small angle. Moreover, the image of $\psi L = \psi L_0' = L_0'$ lies in a small neighbourhood (meant in $\mathbb{R}^2$, which contains the collinear model circle) of the common infinite point of k_1 and k_2. Then the images of all other K_j's in the collinear model lie far from the common infinite point of k_1 and k_2, therefore $(\varphi K) \cap (\psi L)$ remains unchanged if we delete all these K_j's from $\operatorname{bd} K$. Then $(\varphi K) \cap (\psi L)$ equals the above fixed (generic) quadrangle $x_1'x_2'y_2'y_1'$, which admits no non-trivial congruence. This proves the lemma for case (1).

In case (2) there are two boundary components K_i, with both respective c_i's arbitrarily small, with a common infinite point. Consider the respective base lines K_i^*, and let us choose φ so that it takes these two (K_i^*)'s to the above k_1, k_2. Like in case (1), we may suppose that, in the collinear model, k_1 and k_2 enclose a small angle. Moreover, $\psi L_0 = L_0'$ lies in a small neighbourhood (in $\mathbb{R}^2$, containing the collinear model) of the common infinite point of k_1 and k_2. Like in case (1), deletion of all other K_j's from $\operatorname{bd} K$ lets $(\varphi K) \cap (\psi L)$ unchanged. Then $(\varphi K) \cap (\psi L)$ is an arc-quadrangle. It is bounded by two hypercycle arcs lying on the K_i's, each very close to k_1 and to k_2, resp., and by two segments lying on the lines $x_1'x_2'$ and $y_1'y_2'$. Then $(\varphi K) \cap (\psi L)$ is a very small perturbation of the quadrangle $x_1'x_2'y_2'y_1'$. Hence we may suppose that $\operatorname{diam}[(\varphi K) \cap (\psi L)] \le 3\varepsilon$. Moreover, the quadrangle $x_1'x_2'y_2'y_1'$ admits no non-trivial congruences.

Suppose that for both respective c_i's anyhow small $(\varphi K) \cap (\psi L)$ admits a non-trivial congruence. Then in limit (of some subsequence) we would obtain a non-trivial congruence admitted by the quadrangle $x_1'x_2'y_2'y_1'$, contrary to the choice of this quadrangle. Hence, for both c_i's sufficiently small, itself $(\varphi K) \cap (\psi L)$ cannot admit any non-trivial congruences. This proves the lemma for case (2), and the proof of the lemma is finished. ■

Proof of Theorem 1, **continuation. 4.** Now the previous parts of the proof of Theorem 1, and Lemmas 1.2 and 1.6-1.9 prove all statements of Theorem 1. ■

Proof of Theorem 2. For the first three statements we have the evident implications (3) $\Longrightarrow$ (1) $\Longrightarrow$ (2). Now we show the remaining implication (2) $\Longrightarrow$ (3). Let (2) hold. Then by Theorem 1, (2) $\Longrightarrow$ (3), and compactness of K and L we have that K and L are circles. If both K and L are halfspheres, then (3) holds. Else we may apply (2), yielding that the circles K and L are congruent. That is, (3) holds.

For the last five statements we have the evident implications (8) $\Longrightarrow$ (4) and (4) $\Longrightarrow$ (5) $\Longrightarrow$ (7), and (4) $\Longrightarrow$ (6) $\Longrightarrow$ (7). The remaining implication (7) $\Longrightarrow$ (8) follows from Theorem 1, (2) $\Longrightarrow$ (3). ■

Proof of Theorem 3. The implication (2) $\Longrightarrow$ (1) is evident.

For the implication (1) $\Longrightarrow$ (2) we observe that evidently (1) of Theorem 3 implies (2) of Theorem 1, and then we can apply Theorem 1, (2) $\Longrightarrow$ (3). In **2** of the proof of Lemma 1.8, for $X = \mathbb{R}^2$, we have seen that (3) of Theorem 1 implies that any of K and L can be a circle, a parallel strip, or a half-plane. Moreover, if one of K and L is a circle, then $(\varphi K) \cap (\psi L)$ is axially symmetric. For the remaining cases observe that the intersection of two parallel strips is always centrally symmetric, and the intersection of two half-planes is always axially symmetric. However, the intersection of a parallel strip and a half-plane, with nonempty interior, admits in general no non-trivial congruence. Thus (1) $\Longrightarrow$ (2) holds.

The two particular cases, with central, or axial symmetries in (1), follow by easy discussions. ■

Proof of Theorem 4. **1.** The implication (2) $\Longrightarrow$ (1) is evident, so we turn to the proof of (1) $\Longrightarrow$ (2).

2. Observe that (1) of Theorem 4 implies (2) of Theorem 1, and (2) of Theorem 1 implies, by Theorem 1, (3) of Theorem 1. By Theorem 1, for the case of central symmetries in (2) of Theorem 1, the connected components of the boundaries of K and L are congruent. That is, *K and L are either two congruent circles, or two paracircles, or all their boundary components are either congruent hypercycles, or straight lines.* However, in the case of straight lines, their total number is finite, by the hypothesis of the theorem.

The case that K and L are paracircles is clearly impossible. Namely, we may choose φ and ψ so, that $\varphi K = \psi L$, and then their intersection is a paracircle. However, this has exactly one point at infinity, hence is not centrally symmetric.

In the next two lemmas we are going to show that also the case of (finitely many) straight lines, and the case of congruent hypercycles is impossible.

Lemma 4.1. *Assume* (1) *with $d = 2$ and let $X = H^2$. Then the case, when all connected components of the boundaries both of K and L are straight lines, when, by hypothesis, their total number is finite, is impossible.*

Proof. Now it will be convenient to use the collinear model for H^2. Then, in this model, both K and L are bounded by finitely many non-intersecting open chords of the boundary circle of the model. Possibly we have chords with common end-points. Let K_1, or L_1 be some connected component of $\mathrm{bd}\, K$, or $\mathrm{bd}\, L$, resp. We may choose φ and ψ so, that $\varphi K_1 = \psi L_1 = (\varphi K) \cap (\psi L)$, and this line contains the centre of the model. Thus φK and ψL lie on the opposite sides of this straight line. Let us change φ and ψ a bit, so that in the model φK and ψL rotate a little bit about the centre of the model. (Suppose in Remark 1 that φK_1 is the vertical axis, and φK or ψL lies on the right, or left hand side of φK_1, resp. Then in case (1) from Remark 1 we rotate φK in the negative sense and ψL in the positive sense, while in case (2) from Remark 1 conversely.) We will not use new notations for the new orientation preserving congruences, but will retain the old ones φ and ψ.

Let the intersection C of the closed half-circles of the collinear model circle, bounded by φK_1, or ψL_1, and containing φK, or ψL, in their new positions, resp, satisfy the following. It does not contain any end-point of any chord, which in the model represents some boundary component of φK or ψL, except of course one end-point of φK_1, and one end-point of ψL_1. By the finiteness hypothesis, this can be attained, and implies the following. The set C does not intersect the closure in $\mathbb{R}^2$ of any other boundary components of φK, or of ψL (i.e., different from φK_1, or ψL_1, resp.) than those, which satisfy the following properties (1) and (2).
(1) They are in the collinear model chords of the model circle with one common end-point with the chords φK_1, or ψL_1, resp. Moreover, this/these common end-point/s lie in C (i.e., is/are endpoint/s of the circular arc corresponding to C).
(2) From this/these connected component/s of the boundaries only a/ half-line/s is/are in C.

Then $(\varphi K) \cap (\psi L)$ is, in the collinear model, either
(a) a sector of the model circle, or
(b) a triangle, with two sides parallel, and having two finite vertices, or
(c) a quadrangle, with opposite sides parallel.
Case (a) gives a set having exactly one non-smooth boundary point. If it were centrally symmetric, this boundary point would be the centre of symmetry, which is a contradiction. In case (b) we have a set having exactly one point at infinity, hence it is not centrally symmetric, which is a contradiction. In case (c), if there were a centre of symmetry, that would be an inner point of our set. Then one side and its centrally symmetric image side would span ultraparallel straight lines. However, the lines spanned by any two sides of this quadrangle are either intersecting, or parallel. So we have a contradiction in each of the three cases.

This ends the proof of the lemma. ■

Lemma 4.2. *Assume* (1) *with* $d = 2$ *and let* $X = H^2$. *Then the case, when all connected components of the boundaries both of* K *and* L *are congruent hypercycles (degeneration to straight lines not admitted), is impossible.*

Proof. Denote by $l > 0$ the common value of the distance, for which these boundary component hypercycles are distance lines for their base lines.

Again, it will be convenient to consider the collinear model. Both for K and L, remove from it the union of the convex hulls of its boundary components, thus obtaining the closed convex sets K_0 ($\subset K$) and L_0 ($\subset L$). If there are two boundary components of K or L with both infinite points common, then K_0 or L_0 is a straight line, which we consider as doubly counted. Else K_0 or L_0 is a closed convex set with interior points, and its boundary components are the base lines of the boundary components of K or L, resp.

The parallel domain of K_0, or L_0, with distance l, contains K, or L, resp. However, also these parallel domains are contained in K, or L, resp. Namely, if, e.g., $z \in K_0$, then also $z \in K$. If however the distance of a point $z \notin K_0$ from some $x \in K_0$ is at most l, then the segment $[z, x]$ intersects some boundary component $\tilde{K}_{0,1}$ of K_0, say, at a point x'. Then $d(z, x') \leq d(z, x) \leq l$, hence the distance of z from its own projection to $\tilde{K}_{0,1}$ is also at most l. Therefore z lies (not strictly) between $\tilde{K}_{0,1}$ and the respective boundary component $\tilde{K}_1$ of K, hence $z \in \operatorname{cl}\operatorname{conv} \tilde{K}_1 \subset K$. (If there are two such $\tilde{K}_1$'s, then the above statement holds for one of them.) That is, we have (in both cases)

(60) the parallel domain of K_0, or L_0, with distance l, equals K, or L, resp.

Let φK_1, or ψL_1 denote a boundary component of φK, or ψL, whose base line is denoted by $\varphi K_{0,1}$, or $\psi L_{0,1}$, resp. Let φK_1^* or ψL_1^* denote the closed convex set bounded by φK_1 or ψL_1, resp. Let us suppose that φK and ψL are in such a position, that φK_1^* and ψL_1^* have exactly one infinite point in common (which must be the unique common infinite point of φK_1 and ψL_1). This can be attained by applying some orientation preserving congruences φ, ψ.

(61) $$\text{Let } M := (\varphi K_1^*) \cap (\psi L_1^*).$$

By the conformal model, *the set* M *is bounded by some arcs of* φK_1 *and* ψL_1, *having one common infinite endpoint, which is the only infinite point of* M. Evidently

$\varphi K \subset \varphi K_1^*$, and $\psi L \subset \psi L_1^*$, hence

$$(\varphi K) \cap (\psi L) \subset M. \tag{62}$$

We are going to show that also $(\varphi K) \cap (\psi L) \supset M$. It will suffice to show $M \subset \varphi K$ (the other inclusion is proved analogously).

The straight line $\varphi K_{0,1}$ cuts H^2 into two closed half-planes $\varphi H^{\pm}_{K,0,1}$, with $\varphi K_1 \subset \varphi H^+_{K,0,1}$.

For $z \in M \cap (\varphi H^+_{K,0,1})$ we have $z \in (\varphi K_1^*) \cap (\varphi H^+_{K,0,1}) = \mathrm{cl}\,\mathrm{conv}\,(\varphi K_1) \subset \varphi K$, as stated.

For $z \in M \cap (\varphi H^-_{K,0,1})$ we have $z \in (\psi L_1^*) \cap (\varphi H^-_{K,0,1})$. Now the straight line $\psi L_{0,1}$ cuts H^2 into two closed half-planes $\psi H^{\pm}_{L,0,1}$, with $\psi L_1 \subset \varphi H^+_{L,0,1}$. Then $\varphi H^-_{K,0,1} \subset \psi H^+_{L,0,1}$, hence $z \in (\psi L_1^*) \cap (\psi H^+_{L,0,1}) = \mathrm{cl}\,\mathrm{conv}\,(\psi L_1)$. Therefore for the projection ψq of z to $\psi L_{0,1}$ we have $d(z, \psi q) \le l$. By $z \in \varphi H^-_{K,0,1}$ we have that z and ψq are (not strictly) separated by $\varphi K_{0,1}$, hence the segment $[z, \psi q]$ intersects $\varphi K_{0,1}$, at a point φp, say. Then

$$\mathrm{dist}\,(z, \varphi K_0) \le \mathrm{dist}\,(z, \varphi K_{0,1}) \le d(z, \varphi p) \le d(z, \psi q) \le l, \tag{63}$$

hence, by (60), $z \in \varphi K$, as stated.

Therefore $z \in M$ implies $z \in \varphi K$, i.e., $M \subset \varphi K$. Similarly $M \subset \psi L$, thus $M \subset (\varphi K) \cap (\psi L)$. Then, by (62),

$$(\varphi K) \cap (\psi L) = M. \tag{64}$$

As written above (just below (61)), the set M has just one point at infinity, which implies that it cannot be centrally symmetric. This ends the proof of the lemma. ■

Proof of Theorem 4, **continuation. 3.** Now Theorem 4 follows from the previous parts of the proof of Theorem 4, and from Lemmas 4.1-4.2. ■

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