ON A THEOREM OF I. JUHÁSZ ON THE IMAGE WEIGHT SPECTRUM

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Abstract. Using $C(X)$-techniques, we give a simple proof of a recent theorem of I. Juhász: If X is an infinite compact T_2 space of weight α and $\beta < \alpha$ is any infinite cardinal, then X has a T_2 continuous image Y, of weight β. We observe that the answer to the analogous question about the values of a cardinal function for subspaces of a topological space follows in several cases from well-known results on chains of subspaces.

The following question has been investigated in several papers. Let ϕ be a cardinal function (e.g., cardinality, weight, etc.) and X a topological space. Describe the set of values of ϕ for all closed subspaces, or all subspaces of X. In particular, one can ask if this set contains the interval $[\omega, \phi(X)]$, or how large intervals of cardinals can be omitted by such sets.

For the case of *arbitrary* subspaces cf. [T 78a], [T 78b] and [J 80], Ch. 6, that deal with the following question: If $\{X_\lambda\}$ is an increasing chain of subspaces of a space X, with $\phi(X_\lambda) < \alpha$ for each λ, do we then have $\phi(\bigcup_\lambda X_\lambda) \leq \alpha$. If $\phi(A) \geq \phi(\emptyset) = \omega$ and for $B \supset A$, $|B \setminus A| = 1$ we have $\phi(B) \leq \phi(A)^+$, then a positive answer to this question implies that $\{\phi(Y) \mid Y \subset X\} \supset [\omega, \phi(X)]$: choose $\{X_\lambda\}$ the initial segments of a well-ordering of X. (Namely then, letting $\mu = \min\{\lambda \mid \phi(X_\lambda) \geq \beta\}$ for $\omega < \beta < \phi(X)$, we have $\phi(X_\mu) = \beta$.)

For the case of *closed* subspaces we refer to the papers [J 77], [JW], [JNy], [J 92b], [J 93], and to Ch. 6 of the survey paper [J 84] and pp. 425-426 of the survey paper [J 92a]; these deal with the cases that ϕ is the cardinality or the weight.

Dually, M. G. Tkachenko [T 80], I. Juhász [J 97] and I. Juhász, Z. Szentmiklóssy [JSz] initiated the investigation of the values of cardinal functions ϕ (e.g., weight, density, etc.) of continuous images Y of topological spaces X, these images Y belonging to some classes $\mathcal{C}$ of topological spaces. [J 97] has called the set of values of ϕ attained for these spaces Y the *image spectrum of* ϕ for the space X, with respect to the given class $\mathcal{C}$. In particular, they have investigated the following type of question: If some cardinal function ϕ of some space X equals α, and $\beta < \alpha$ is an infinite cardinal, then does there exist a continuous image Y of X, belonging to a certain class $\mathcal{C}$ of topological spaces, such that $\phi(Y) = \beta$.

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For general information on cardinal functions we refer to the books [J 70], [R], [J 80], [E], and to the survey papers [H], [J 84], [J 92a]. For $C(X)$-techniques we refer to the books [GJ], [S], [BNS].

For a topological space X, $w(X)$ and $d(X)$ denote its weight and density, respectively.

Among others, [J 97] has proved, by methods of set-theoretical topology, the following

Theorem. *(I. Juhász [J 97], Corollary 4.) Let X be an infinite compact T_2 space, of weight $w(X) = \alpha$. Then for any infinite cardinal $\beta < \alpha$ there is an infinite (compact) T_2 continuous image Y of X, such that $w(Y) = \beta$.*

In this note we give a simple proof of this theorem by $C(X)$-techniques. A uniformly discrete subspace of a metric space is one in which the distances of any two distinct points are not less than some positive constant.

Proof of the theorem. Consider the ring $C(X)$ of continuous functions $X \to \mathbb{R}$, endowed with the distance $\rho(f, g) = \max\{|f(x) - g(x)| \mid x \in X\}$. We have $\alpha = w(X) = d(C(X))$, cf. [S], Proposition 7.6.5.

Since $C(X)$ is a metric space, we have $\alpha = d(C(X)) = \sup\{|D| \mid D \subset C(X)$ is uniformly discrete$\}$. Therefore there exists a uniformly discrete subspace D of $C(X)$, of cardinality β. Observe that a compact T_2 space Z is infinite if and only if $C(Z)$, as a real vector space, is infinite dimensional. Therefore we may assume that D does not lie in a finite dimensional subspace of $C(X)$, considered as a real vector space.

Let $R(D)$ denote the closed subring of $C(X)$, containing all constant functions, that is generated by D, i.e., the closure in $C(X)$ of the set of all polynomials of several variables with rational coefficients, and of variables $f \in D$. Then $R(D)$ as a real vector space is infinite dimensional, and with the metric inherited from $C(X)$ it satisfies

$$d(R(D)) = d(D) = |D| = \beta.$$

$R(D)$, being a closed subring of $C(X)$, containing all constant functions, is of the form $R(D) = C(Y) \circ F$, for a surjective map $F : X \to Y$ to a (compact) T_2 space Y ([K], Exercise 7S, (f)). Hence $C(Y)$ is isometric to $R(D)$, implying $d(C(Y)) = d(R(D))$, and $C(Y)$ as a real vector space is infinite dimensional, hence Y is infinite. Then for the infinite (compact) T_2 continuous image Y of X we have

$$w(Y) = d(C(Y)) = d(R(D)) = \beta,$$

ending the proof of the theorem. □

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