MAXIMAL SECTIONS AND CENTRALLY SYMMETRIC BODIES

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Abstract. Let $d \geq 2$ and let $K \subset \mathbb{R}^d$ be a convex body containing the origin 0 in its interior. Let, for each direction ω, the $(d-1)$–volume of the intersection of K and an arbitrary hyperplane with normal ω attain its maximum if the hyperplane contains 0. Then K is symmetric about 0. The proof uses a linear integro–differential operator on S^{d-1}, whose null–space needs to be, and will be determined.

1. Introduction

Let $d \geq 2$ and let $K \subset \mathbb{R}^d$ be a centred convex body. Then the Brunn–Minkowski inequality ([Schneider, 1993]) readily implies that for any hyperplane H and the hyperplane H' parallel to H and passing through 0 we have $V_{d-1}(K \cap H') \geq V_{d-1}(K \cap H)$ (V_{d-1} is the $(d-1)$–volume). Conversely, we prove that if $d \geq 2$ and a convex body $K \subset \mathbb{R}^d$, containing the origin 0 in its interior, has the property that for every hyperplane H the hyperplane H' parallel to H and passing through 0 satisfies $V_{d-1}(K \cap H') \geq V_{d-1}(K \cap H)$, then this convex body K is centred. (Thus this property is a characterization of centredness.) Actually we will prove the analogous implication for star bodies, with positive Lipschitz radial functions, and for k–dimensional sections, where $1 \leq k \leq d-1$. The statement in the case $k = 1$, K a convex body, has been proved by [Hammer, 1954].

This question has been posed by the second named author in another context, see [Martini, 1994] and also [Gardner, 1995], Problem 8.8 (p. 302). An application of this result is contained in [Makai–Martini, 1996]: if each of the measures $\max_{x \in \mathbb{R}^d} V_{d-1}(K \cap (H+x))$ and $V_{d-1}(K \cap H')$ is constant for all hyperplanes H, then K is necessarily a ball.

There is also a physical motivation for the study of the measure $H \mapsto \max_{x \in \mathbb{R}^d} V_{d-1}(K \cap (H+x))$. Several properties of metals, like e.g. electric or heat conductivity, are explained in terms of the so called *Fermi surface* of the metal, that describes the

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position of the free electrons of the metal in the velocity space. The explanation is quantum mechanical, cf. [Mackintosh, 1963], [Shoenberg, 1960]. By Pauli's exclusion principle, the same place in the velocity space can be occupied by at most one free electron of given spin, so by at most two free electrons (of opposite spins). So even in principle at absolute zero temperature all free electrons cannot have velocity 0, but, in case of unit volume, their velocities occupy a certain domain in the velocity space, whose boundary is called the Fermi surface. The free electrons tend to occupy positions in the velocity space so that their total energy is minimized. As a metal crystal lattice cannot have spherical symmetry, the Fermi surface is not a sphere. There are several ways of obtaining information about the Fermi surface, and one is via the above measure $H \mapsto \max_{x\in\mathbb{R}^d} V_{d-1}(K \cap (H+x))$, where K is the domain of the velocity space bounded by the Fermi surface (K is not convex). This measure can be determined by the so called de Haas–van Alphen effect, and in honour of these names this measure is often called the HA–measurement of the body K.

We prove our result by using a new Radon–type transformation, which can be considered as a common generalization of partial differential operators and Radon–type transformations. To the authors' knowledge, this result is the first attempt to the extension of the theory of classical Radon transformations into this direction.

Our integro–differential transformation lets to correspond to a sufficiently nice function $f\colon S^{d-1} \to \mathbb{R}$ the function $R^{(1)}f\colon S^{d-1} \to \mathbb{R}$, where $(R^{(1)}f)(\omega)$ is the integral over the large S^{d-2} of S^{d-1}, with pole ω, of the derivative of f in the direction ω. To prove our theorem we establish that the null–space of this operator $R^{(1)}$ consists of the even functions in the domain of $R^{(1)}$.

For the proof of our theorem we use spherical harmonics and the Funk–Hecke theorem. The third named author has found another proof, by explicit inversion formulas. For not integral geometers it is interesting to observe that the even and odd dimensional inversion formulas for our integro–differential transformation are different in their nature. If the dimension of the base space is even then the inversion formulas are local ones, while if the base space is odd dimensional then the inversion formulas are global ones.

For general analytical background we refer to the books [Tricomi, 1957], [Adams, 1975] and [Ziemer, 1989].

Some words about the proof of Corollary 3.2, that is the main result of the paper, are in order. First the first and second named authors proved (with some heuristics) an infinitesimal variant of this theorem, for bodies near the unit ball. Second, independently, the last named author proved the general case.

2. Preliminaries

As usual, $\mathbb{R}^d$ denotes the d–dimensional Euclidean space which is endowed with the standard inner product and norm $|\cdot|$ structure. We will suppose $d \geq 2$. The origin is denoted by 0 and V_{d-1} is the $(d-1)$–dimensional volume on the hyperplanes.

Let S^{d-1} denote the unit sphere with centre 0; its variable point will be denoted by ω, ξ or η. For $\omega \in S^{d-1}$ and $t \in \mathbb{R}$ let $H(\omega, t)$ be the hyperplane given by the equation $\langle x, \omega \rangle = t$. We write $\omega^\perp$ for $H(\omega, 0)$. Often we will use a polar coordinate system on S^{d-1}, with ("north") pole some $\xi \in S^{d-1}$. That is, any $\omega \in S^{d-1}$ can be written as

$$(1) \qquad \omega = \xi \sin\psi + \eta \cos\psi, \text{ where } \eta \in S^{d-1} \cap \xi^\perp \text{ and } -\pi/2 \le \psi \le \pi/2$$

(thus ψ is the *geographic latitude*, which will be more convenient for us than the costumarily used $\varphi = \pi/2 - \psi$); then we write

$$(2) \qquad \omega = (\eta, \psi) \,.$$

In particular,

$$(3) \qquad (\eta, 0) = \eta \,.$$

A real function defined on S^{d-1} is called *even (odd)*, if for all $\omega \in S^{d-1}$ we have $f(-\omega) = f(\omega)$ $(f(-\omega) = -f(\omega))$.

Let K be a convex body which contains the origin 0 in its interior. Let

$$Q(K, \omega) = \max_{t \in \mathbb{R}} V_{d-1}(K \cap H(\omega, t))$$

be the maximum of the $(d-1)$–dimensional volumes of the intersections of the convex body K and the hyperplanes parallel to $\omega^\perp$. We call $Q(K, \omega)$ the *inner quermass (or HA–measurement) of K relative to the hyperplane $\omega^\perp$.*

A set $A \subset \mathbb{R}^d$ is *centred*, if it is symmetric with respect to the origin.

For $x \in \mathbb{R}^d$, $K \subset \mathbb{R}^d$ we say that K is a *star body with respect to x*, if K is of the form $x + \{\mu\omega \mid \omega \in S^{d-1},\ 0 \le \mu \le \varrho(\omega)\}$, where $\varrho\colon S^{d-1} \to \mathbb{R}$ is a positive continuous function that is called the *radial function of K with respect to x*. For $x = 0$ we just say star body and radial function.

We turn to spherical harmonics, which are higher–dimensional generalizations of the trigonometric functions $\cos(nx)$, $\sin(nx)$ (these are obtained for $d = 2$). Standard references are [Müller, 1966], [Seeley, 1966], [Erdélyi et al., 1953] and, for $d = 3$ in more detail, [Sansone, 1959]; further references, with some geometrical applications, are e.g. [Funk, 1913], Kap. 2, [Alexandroff, 1937], [Petty, 1952], Cor. 1.31, [Ungar, 1954], [Blaschke, 1956], §23, Anhang, [Petty, 1961], §4, [Schneider, 1967], [Schneider, 1969], [Schneider, 1970], [Falconer, 1983], [Gardner, 1995], Appendix C, and also the survey paper [Groemer, 1993] and the books [Schneider, 1993], pp. 428–432, as well as [Groemer, 1996], which contain ample further bibliography. Some further papers in geometry, related to the topic of our paper, are [Funk, 1916], [Lifshitz–Pogorelov, 1954].

A polynomial $f\colon \mathbb{R}^d \to \mathbb{R}$ is *harmonic*, if $\sum_{i=1}^{d} (\partial/\partial x_i)^2 f = 0$. (This is invariant under the choice of an orthonormal base.) For an integer $n \ge 0$ a *spherical harmonic (of degree n)* in d dimensions is the restriction of a homogeneous harmonic polynomial $f\colon \mathbb{R}^d \to \mathbb{R}$ (of degree n) to S^{d-1}. (Since d will be fixed, later we will not refer to the dimension.) The spherical harmonics of degree n form a finite dimensional vector space. Choosing from this subspace an orthonormal base $\{Y_{ni} \mid 1 \le i \le N(d, n)\}$ (orthonormality meant in the space $L^2(S^{d-1})$, for the Lebesgue measure on S^{d-1}), their union for each $n \geq 0$ is a complete orthonormal system in $L^2(S^{d-1})$. Thus each $f \in L^2(S^{d-1})$ has a Fourier expansion $\sum\limits_{n=0}^{\infty} \left(\sum\limits_{i=1}^{N(d,n)} c_{ni} Y_{ni} \right)$. Here we will write $\sum\limits_{i=1}^{N(d,n)} c_{ni} Y_{ni} = Y_n(f)$, thus the Fourier expansion of f is $\sum\limits_{n=0}^{\infty} Y_n(f)$.

The spherical harmonics are the eigenfunctions of many linear operators commuting with rotations. For example, the Funk–Hecke theorem ([Seeley, 1966], Theorem 3) says the following. Let F be measurable on $[-1, 1]$, with $\int\limits_{-1}^{1} |F(t)|(1 - t^2)^{(d-3)/2} dt < \infty$. Then any spherical harmonic Y_n of n–th degree is an eigenfunction of the integral operator $f \mapsto g = g(\xi) = \int\limits_{S^{d-1}} F(\langle \xi, \eta \rangle) f(\eta) d\eta$, that is

$$\int\limits_{S^{d-1}} F(\langle \xi, \eta \rangle) Y_n(\eta) d\eta = \lambda_n Y_n(\xi) \,,$$

where the eigenvalue λ_n equals

$$\lambda_n = V_{d-2}(S^{d-2}) C_n(1)^{-1} \int\limits_{-1}^{1} F(t) C_n(t) (1 - t^2)^{(d-3)/2} dt \,.$$

Here V_{d-2} means $(d-2)$–dimensional volume, and $C_n(t) = C_n^{(d-2)/2}(t)$ is the n'th Gegenbauer polynomial, of order $(d-2)/2$, that is a non–zero polynomial of degree n, satisfying for $0 \leq n < m$ the orthogonality relations

$$\int\limits_{-1}^{1} C_n(t) C_m(t) (1 - t^2)^{(d-3)/2} dt = 0 \,.$$

There holds $C_n(1) \neq 0$, [Seeley, 1966], (3). For n odd (even) C_n is an odd (even) function [Erdélyi et al., 1953], §10.9, (16). References to Gegenbauer polynomials are [Erdélyi et al., 1953] and [Tricomi, 1955].

For suitable measures or distributions on $[-1, 1]$ a formula similar to the Funk–Hecke theorem holds, cf. for example Lemma 3.7.

3. Concurrent maximal sections and centredness

Rather than considering convex bodies $K \subset \mathbb{R}^d$, $d \geq 2$, with $0 \in \operatorname{int} K$, we will consider more generally star bodies $K \subset \mathbb{R}^d$, $d \geq 2$, with radial functions $\varrho : S^{d-1} \to \mathbb{R}$ positive and Lipschitz. (We use on S^{d-1} the geodesic metric, and Lipschitz is meant with respect to it.) This is actually a generalization. Namely, it is easily seen that, for a convex body $K \subset \mathbb{R}^d$, with $0 \in \operatorname{int} K$, and with radial function ϱ, where $0 < \varrho_0 \leq \varrho(\omega) \leq \varrho_1$ for $\omega \in S^{d-1}$, we have the following: ϱ

satisfies the Lipschitz condition, with a Lipschitz constant $L(\varrho)$ dominated by a function of ϱ_0 and ϱ_1 only. (Easy 2–dimensional local arguments give the sharp upper bound $L(\varrho) \le \varrho_1^2 \varrho_0^{-1} \sqrt{1 - \varrho_0^2 \varrho_1^{-2}}$.)

Denoting by V_k the k–dimensional volume (=Lebesgue measure), we have the following theorem, that for K a convex body and $k = 1$ has been proved by [Hammer, 1954], Theorem 1.

Theorem 3.1. *Let $d \ge 2$, and let $K \subset \mathbb{R}^d$ be a star body, having a positive Lipschitz radial function $\varrho : S^{d-1} \to \mathbb{R}$. Let $1 \le k \le d-1$ be an integer, and let for any linear k–subspace $L_k \subset \mathbb{R}^d$ the function $y \mapsto V_k(K \cap (L_k + y))$ have a local extremum at $y = 0$. Then K is centred.*

Corollary 3.2. *Let $d \ge 2$ and let $1 \le k \le d-1$ be an integer. Then a convex body $K \subset \mathbb{R}^d$ with $0 \in \text{int}\, K$ is centred if and only if, for any linear k–subspace $L_k \subset \mathbb{R}^d$, the function $y \mapsto V_k(K \cap (L_k + y))$ attains its maximum at $y = 0$.*

To prove this theorem and its corollary we need some lemmas.

Lemma 3.3. *Let $\varrho : [-\pi/2, \pi/2] \to \mathbb{R}$ be a Lipschitz function, with $0 < \varrho_0 \le \varrho(\psi)$ for $\psi \in [-\pi/2, \pi/2]$, and with Lipschitz constant at most $L > 0$. Then there is a number $t_0 = t_0(\varrho_0, L) \in (0, \varrho_0)$, such that the following holds. The function $\psi \mapsto \varrho(\psi) \sin \psi$, where $\psi \in [-\pi/2, \pi/2]$, assumes each value $t \in [-t_0, t_0]$ exactly for one ψ; moreover the inverse function $t \mapsto \psi$, $t \in [-t_0, t_0]$, also is Lipschitz, with Lipschitz constant dominated by a function of ϱ_0, L only.*

Proof. Let $f(\psi) = \varrho(\psi) \sin \psi$. By $t(-\pi/2) \le -\varrho_0 < \varrho_0 \le t(\pi/2)$, each value $t \in [-\varrho_0, \varrho_0]$ is attained for some $\psi \in [-\pi/2, \pi/2]$.

For sufficiently small $t_0 \in (0, \varrho_0)$, let $t_1, t_2 \in [-t_0, t_0]$. Suppose $-\pi/2 \le \psi_1 < \psi_2 \le \pi/2$ and $f(\psi_i) = t_i$. By $t_0 \ge |t_i| = |f(\psi_i)| \ge \varrho_0 |\sin \psi_i|$ we have $|\sin \psi_i| \le t_0/\varrho_0$. We may suppose $t_0/\varrho_0 \le 1/\sqrt{2}$ and $L t_0/\varrho_0 \le \varrho_0/(2\sqrt{2})$. Then we have $f(\psi_2) - f(\psi_1) = t_2 - t_1 = (\varrho(\psi_2) - \varrho(\psi_1)) \sin \psi_2 + \varrho(\psi_1)(\sin \psi_2 - \sin \psi_1) \ge -L(\psi_2 - \psi_1) t_0/\varrho_0 + \varrho_0(\psi_2 - \psi_1)/\sqrt{2} \ge \varrho_0(\psi_2 - \psi_1)/(2\sqrt{2}) > 0$. Therefore each value $t \in [-t_0, t_0]$ is attained by f at most for one $\psi \in [-\pi/2, \pi/2]$, thus for exactly one $\psi = \psi(t) \in [-\pi/2, \pi/2]$; moreover we have $0 < \psi(t_2) - \psi(t_1) = \psi_2 - \psi_1 \le (t_2 - t_1) 2\sqrt{2}/\varrho_0$. □

Corollary 3.4. *Let $d \ge 2$, and let $K \subset \mathbb{R}^d$ be a star body, having a positive Lipschitz radial function $\varrho : S^{d-1} \to \mathbb{R}$, with $0 < \varrho_0 \le \varrho(\xi)$ for $\xi \in S^{d-1}$, and with Lipschitz constant at most $L > 0$. Then, for $\xi \in S^{d-1}$ and $|t| \le t_0(\varrho_0, L)$ (from Lemma 3.3), we have that $K \cap H(\xi, t)$ is a star body in $H(\xi, t) = \{x \in \mathbb{R}^d \mid \langle x, \xi \rangle = t\}$ with respect to $t\xi$.*

Proof. By Lemma 3.3 each ray from $t\xi$, lying in $H(\xi, t)$, intersects bd K exactly once. Furthermore, by compactness of (bd K) $\cap H(\xi, t)$, this intersection point, $t\xi + r(\xi, \eta, t)\eta$, say, depends continuously on the direction vector $\eta \in S^{d-1} \cap \xi^\perp$ of the ray considered. Moreover $\xi t \in$ int K, while far points on these rays lie in $\mathbb{R}^d \setminus K$. This implies that the star body in $H(\xi, t)$, with respect to $t\xi$, and with radial function $\eta \mapsto r(\xi, \eta, t)$ equals $K \cap H(\xi, t)$. □

We recall from (1), (2) and (3) the representation in polar coordinates $\omega = (\eta, \psi)$ for $\omega = \xi \sin \psi + \eta \cos \psi$, where $\eta \in S^{d-1} \cap \xi^\perp$ and $-\pi/2 \le \psi \le \pi/2$. Thus, in the following lemma $(\partial \varrho / \partial \psi)(\eta) = (\partial \varrho / \partial \psi)(\eta, 0)$ means the angular derivative of ϱ at η, along the meridian passing through η.

Lemma 3.5. *Let $d \geq 2$, and let $K \subset \mathbb{R}^d$ be a star body, having a positive Lipschitz radial function $\varrho : S^{d-1} \to \mathbb{R}$. Then, for almost all $\xi \in S^{d-1}$, the function $t \mapsto V_{d-1}(K \cap H(\xi, t))$ is differentiable at $t = 0$, and its derivative at $t = 0$ is equal to*

$$\int\limits_{S^{d-1}\cap\xi^\perp} \varrho(\eta)^{d-3} \frac{\partial\varrho}{\partial\psi}(\eta) d\eta$$

($(\partial\varrho/\partial\psi)(\eta)$ existing almost everywhere on $S^{d-1} \cap \xi^\perp$, and the integral existing).

Proof. By [Whitney, 1957], Ch. IX, Theorem 11A, Lipschitz functions $S^{d-1} \to \mathbb{R}$ are almost everywhere differentiable.

Now we consider the Stiefel–manifold $S_{2,d-2} = \{(\xi, \eta) \mid \xi, \eta \in S^{d-1},\ \langle \xi, \eta \rangle = 0\}$. This is an S^{d-2}–bundle over S^{d-1}, via either projection $\pi_1 : (\xi, \eta) \mapsto \xi$, $\pi_2 : (\xi, \eta) \to \eta$. Applying locally the Fubini theorem to both π_2 and π_1, we gain in turn that $V_{2d-3}(\{(\xi, \eta) \in S_{2,d-2} \mid \varrho \text{ is not differentiable at } \eta\}) = 0$, and $V_{d-1}(\{\xi \in S^{d-1} \mid \{\eta \in S^{d-1} \cap \xi^\perp \mid \varrho \text{ is not differentiable at } \eta\}$ does not have $(d-2)$–Lebesgue measure $0\}) = 0$ (V_i denoting here invariant Lebesgue type measure in the respective spaces). In other words: for almost all $\xi \in S^{d-1}$ we have for almost all $\eta \in S^{d-1} \cap \xi^\perp$ that ϱ is differentiable at η. In the following we assume that ξ has this property.

Using Corollary 3.4 and the notations ϱ_0, L and $r(\xi, \eta, t)$ from its statement and proof we have

$$V_{d-1}(K \cap H(\xi, t)) = \int\limits_{S^{d-1}\cap\xi^\perp} r(\xi, \eta, t)^{d-1} d\eta \Big/ (d-1)$$

for $|t| \leq t_0(\varrho_0, L)$. Using polar coordinates with pole ξ, for $t(\xi, \eta, \psi) = \varrho(\eta, \psi) \sin\psi$ we have $(\partial t/\partial\psi)(\xi, \eta, 0) = \varrho(\eta) \geq \varrho_0 > 0$. Therefore at points $(\eta, 0)$ of differentiability of ϱ we have by $r(\xi, \eta, t(\xi, \eta, \psi)) = \varrho(\eta, \psi) \cos\psi$ that

$$\left(\frac{\partial}{\partial t} r(\xi, \eta, t)^{d-1} \right)\Bigg|_{t=0} =$$

$$(d-1) r(\xi, \eta, 0)^{d-2} \cdot \frac{\partial r}{\partial\psi}(\xi, \eta, 0) \Big/ \frac{\partial t}{\partial\psi}(\xi, \eta, 0) =$$

$$(d-1)\varrho(\eta)^{d-3} \cdot \frac{\partial\varrho}{\partial\psi}(\eta) .$$

Let, furthermore, $\varrho(\xi) \leq \varrho_1$ for $\xi \in S^{d-1}$. Then, for $|t| \leq t_0(\varrho_0, L)$, the function $t \mapsto r(\xi, \eta, t)^{d-1}$ is Lipschitz, with Lipschitz constant dominated by some function of ϱ_0, ϱ_1, L. Namely, it is the composition of the functions $t \mapsto$ [the unique ψ with $t(\xi, \eta, \psi) = t$], $\psi \mapsto \varrho(\eta, \psi) \cos\psi$ and $r \mapsto r^{d-1}$, each of these functions satisfying the analogous statement, by the hypotheses and Lemma 3.3. Therefore for $0 < |t| \leq t_0(\varrho_0, L)$ and $t \to 0$ we have that $(r(\xi, \eta, t)^{d-1} - r(\xi, \eta, 0)^{d-1})/t$ is a continuous function of η, is uniformly bounded, and for points $(\eta, 0)$ of differentiability of ϱ, thus by assumption almost everywhere on $S^{d-1} \cap \xi^\perp$, it converges to $(d-1)\varrho(\eta)^{d-3}(\partial\varrho/\partial\psi)(\eta)$. Then this limit function is measurable, and Lebesgue's dominated convergence theorem (with an integrable majorant some constant function) finishes the proof. □

Let $d \geq 2$. We denote by $Lip(S^{d-1})$ the set of all Lipschitz functions $f\colon S^{d-1} \to \mathbb{R}$. Let $-\pi/2 \leq \psi \leq \pi/2$, $f \in Lip(S^{d-1})$ and $\xi \in S^{d-1}$. Using polar coordinates with pole ξ, we define the integro–differential transform $R_\psi^{(1)} f$ of f by

$$(R_\psi^{(1)} f)(\xi) = \int\limits_{S^{d-1}\cap\xi^\perp} \frac{\partial f}{\partial \psi}(\eta,\psi)d\eta\,, \tag{4}$$

provided that the right–hand side exists. Here, like in 3.5, we use representation in polar coordinates $\omega = (\eta,\psi)$ for $\omega = \xi\sin\psi + \eta\cos\psi$ $(\eta \in S^{d-1}\cap\xi^\perp,\ -\pi/2 \leq \psi \leq \pi/2)$, and $(\partial f/\partial\psi)(\eta,\psi)$ means angular derivative of f at (η,ψ) along the meridian passing through η. For $\psi = 0$ we drop the lower index.

Lemma 3.6. *Let $d \geq 2$, $-\pi/2 \leq \psi \leq \pi/2$ and $f, g \in Lip(S^{d-1})$. Then, for almost all $\xi \in S^{d-1}$ we have for almost all $\eta \in S^{d-1}\cap\xi^\perp$ that $(\partial f/\partial\psi)(\eta,\psi)$ exists, and, for almost all $\xi \in S^{d-1}$, the integral defining $(R_\psi^{(1)} f)(\xi)$ exists. We have $R_\psi^{(1)} f \in L^\infty(S^{d-1})$, and $R_\psi^{(1)}$ is symmetric, i.e., $\int_{S^{d-1}} (R_\psi^{(1)} f)(\xi)g(\xi)d\xi = \int_{S^{d-1}} f(\xi)(R_\psi^{(1)} g)(\xi)d\xi$.*

Proof. The claimed existence of $\partial f/\partial\psi$ for $\psi = \pm\pi/2$ follows from [Whitney, 1957], cited in Lemma 3.5, and otherwise follows like in 3.5, using the manifold $\{(\xi,\omega) \mid \xi,\omega \in S^{d-1},\ \langle\xi,\omega\rangle = \sin\psi\}$. For almost all $\xi \in S^{d-1}$ we have that the function $\eta \mapsto (\partial f/\partial\psi)(\eta,\psi)$ is the almost everywhere limit of the continuous functions

$$\eta \mapsto \left(f(\eta,\psi+\varepsilon) - f(\eta,\psi)\right)/\varepsilon$$

with absolute value below a constant depending of f. Therefore the function $\eta \mapsto (\partial f/\partial\psi)\ (\eta,\psi)$ is integrable and $R_\psi^{(1)} f$ is bounded, for almost all ξ. Also $R_\psi^{(1)} f$ is measurable, since by Lebesgue's dominated convergence theorem it is the almost everywhere limit of the (uniformly bounded set of) continuous functions

$$\xi \mapsto \int\limits_{S^{d-1}\cap\xi^\perp} \left(f(\eta,\psi+\varepsilon) - f(\eta,\psi)\right)\varepsilon^{-1} d\eta\,.$$

Hence $R_\psi^{(1)} f \in L^\infty(S^{d-1})$.

Further, we have

$$\int\limits_{S^{d-1}} (R_\psi^{(1)} f)(\xi)g(\xi)d\xi =$$

$$\lim_{\varepsilon\to 0}\left[\int\limits_{S^{d-1}}\left(\int\limits_{S^{d-1}\cap\xi^\perp} f(\eta,\psi+\varepsilon)d\eta\right)g(\xi)d\xi - \int\limits_{S^{d-1}}\left(\int\limits_{S^{d-1}\cap\xi^\perp} f(\eta,\psi)d\eta\right)g(\xi)d\xi\right]\Big/\ \varepsilon =$$

$$\lim_{\varepsilon\to 0}\left[\int\limits_{S^{d-1}}\left(\int\limits_{S^{d-1}\cap\xi^\perp} g(\eta,\psi+\varepsilon)d\eta\right)f(\xi)d\xi - \int\limits_{S^{d-1}}\left(\int\limits_{S^{d-1}\cap\xi^\perp} g(\eta,\psi)d\eta\right)f(\xi)d\xi\right]\Big/\ \varepsilon =$$

$$\int_{S^{d-1}} (R_\psi^{(1)} g)(\xi) f(\xi) d\xi \,.$$

Here the middle equality follows from Fubini's theorem for the manifold $\{(\xi,\eta) \mid \xi,\eta \in S^{d-1},\ \langle \xi,\eta\rangle = \sin\psi \text{ (or } \sin(\psi+\varepsilon))\}$, respectively for $\psi = \pm\pi/2$ it is evident. (We note that the middle equality — without the limit sign, and for both summands separately — has already been proved by [Schneider, 1969], in another way.) □

We define analogously $R_\psi^{(0)}$ by

$$(R_\psi^{(0)} f)(\xi) = \int_{S^{d-1}\cap\xi^\perp} f(\eta,\psi) d\eta \,.$$

Then we have the following lemma, whose statement for $m = 0$ and $d = 3$ is due to [Radon, 1917], while for $m = 0$ and general d it is due to [Schneider, 1969], cf. formula (5) of his paper. Our proof of the statement for $m = 0$ is different from Schneider's.

Lemma 3.7. *Let $d \geq 2$, $m \in \{0,1\}$, $-\pi/2 \leq \psi \leq \pi/2$, and let $Y_n\colon S^{d-1} \to \mathbb{R}$ be a spherical harmonic of degree n. Then Y_n is an eigenfunction of $R_\psi^{(m)}$, i.e., $R_\psi^{(m)} Y_n = \lambda_n Y_n$, with*

$$\lambda_n = V_{d-2}(S^{d-2}) C_n^{(d-2)/2}(1)^{-1} \left(\frac{d}{d\psi}\right)^m C_n^{(d-2)/2}(\sin\psi) \,.$$

Proof. The Funk–Hecke theorem (§2), applied to $F(t)$, the characteristic function of $[-1, \sin\psi]$ $\quad(-\pi/2 < \psi < \pi/2)$, gives

$$\int_{\{\omega\in S^{d-1} \mid \langle\xi,\omega\rangle \leq \sin\psi\}} Y_n(\omega) d\omega =$$

$$V_{d-2}(S^{d-2}) C_n(1)^{-1} \int_{-1}^{\sin\psi} C_n(t)(1-t^2)^{(d-3)/2} dt \cdot Y_n(\xi) \,.$$

Differentiation with respect to ψ gives

$$\cos^{d-2}\psi \cdot \int_{S^{d-1}\cap\xi^\perp} Y_n(\eta,\psi) d\eta =$$

$$\int_{\{\omega\in S^{d-1} \mid \langle\xi,\omega\rangle = \sin\psi\}} Y_n(\omega) d\omega =$$

$$V_{d-2}(S^{d-2}) C_n(1)^{-1} C_n(\sin\psi) \cos^{d-2}\psi \cdot Y_n(\xi) \,,$$

that yields our statement for $m = 0$ (for $\psi = \pm\pi/2$ passing to limit). From this case differentiation with respect to ψ proves the statement for $m = 1$. □

The statement corresponding to the case $m = 0$, $\psi = 0$ in the following theorem is the Funk integral theorem (for $f \in C(S^{d-1})$), cf. [Funk, 1913], Kap. 2, [Lifshitz–Pogorelov, 1954], [Schneider, 1969], [Helgason, 1980], Ch. 3, §1.B and [Helgason, 1984].

Theorem 3.8. *Let $d \geq 2$ and $-\pi/2 \leq \psi \leq \pi/2$. Then the null–space of the operator $R_\psi^{(1)}\colon Lip(S^{d-1}) \to L^\infty(S^{d-1})$ equals $\{f \in Lip(S^{d-1}) \mid$ the Fourier expansion $\sum_{n=0}^{\infty} Y_n(f)$ of f satisfies that $(d/d\psi)\, C_n^{(d-2)/2}(\sin\psi) \neq 0$ implies $Y_n(f) = 0\}$. In particular, for $\psi = 0$ the null–space of $R^{(1)} = R_0^{(1)}$ equals $\{f \in Lip(S^{d-1}) \mid f$ is even$\}$.*

Proof. For the first statement we proceed analogously to [Alexandroff, 1937], [Petty, 1961], [Schneider, 1969], [Schneider, 1970], [Falconer, 1983]. Let $f \in Lip(S^{d-1})$. Then, by 3.6, we have $R_\psi^{(1)} f \in L^\infty(S^{d-1}) \subset L^2(S^{d-1})$. Moreover, by completeness of spherical harmonics, $R_\psi^{(1)} f = 0$ holds a.e. if and only if for each $n \geq 0$, and each spherical harmonic Y_n of degree n we have $0 = \langle R_\psi^{(1)} f, Y_n\rangle$, where $\langle\ ,\ \rangle$ now denotes scalar product in $L^2(S^{d-1})$. Letting $\sum_{n=0}^{\infty} Y_n(f)$ be the Fourier expansion of f, we have by 3.6 and 3.7 that

$$\langle R_\psi^{(1)} f, Y_n\rangle = \langle f, R_\psi^{(1)} Y_n\rangle = \lambda_n \langle f, Y_n\rangle =$$

$$V_{d-2}(S^{d-2}) C_n(1)^{-1} \frac{d}{d\psi} C_n(\sin\psi) \cdot \langle Y_n(f), Y_n\rangle\,.$$

For fixed n and Y_n arbitrary this is 0 if and only if $(d/d\psi)\, C_n(\sin\psi) \cdot Y_n(f) = 0$. This implies the first statement.

For the second statement first observe that for $\psi = 0$ and f even by 3.6 we have for almost all $\xi \in S^{d-1}$ that for almost all $\eta \in S^{d-1} \cap \xi^\perp$ both $(\partial f/\partial\psi)(\eta, 0)$ and $(\partial f/\partial\psi)(-\eta, 0)$ exist and then have sum 0, thus $(R^{(1)} f)(\xi) = 0$ a.e.

Then for the second statement it remains to show that conversely $R^{(1)} f = 0$ implies that f is even. Let $R^{(1)} f = 0$ and let n be odd. Then C_n is odd, hence $C_n(0) = 0$, and, since $\{C_0, C_1, ...\}$ is a system of orthogonal polynomials, C_n only has simple zeros ([Erdélyi et al., 1953], p.158), thus $C_n'(0) \neq 0$. Therefore by the first statement $Y_n(f) = 0$, so $f = \sum_{k=0}^{\infty} Y_{2k}(f)$ is even a.e., so by continuity everywhere. □

Proof of Theorem 3.1. It suffices to prove the statement for $k = d-1$. Namely, for any linear $(k+1)$–subspace L_{k+1}, we have that $K \cap L_{k+1}$ also satisfies the hypotheses of the theorem; furthermore, if each $K \cap L_{k+1}$ is centred, then K is as well.

Let therefore $k = d-1$. Then, for any linear $(d-1)$–subspace $\xi^\perp$ $(\xi \in S^{d-1})$ for which the function $t \mapsto V_{d-1}(K \cap H(\xi, t))$ is differentiable at $t = 0$, thus by 3.5 for almost all ξ, we have that the derivative of this function at $t = 0$ is 0. Letting $f = \varrho^{d-2}/(d-2)$ for $d \geq 3$, and $f = \log\varrho$ for $d = 2$, we have $f \in Lip(S^{d-1})$, and, by 3.5, the above derivative at $t = 0$ is equal to $(R^{(1)} f)(\xi)$ for almost all ξ. Then 3.8 yields that f, and thus ϱ, is even, so K is centred. □

Proof of Corollary 3.2. The maximum property implies centredness by Theorem 3.1 and the remarks preceding it, while the converse follows from the Brunn–Minkowski theorem. □

Since Corollary 3.2 directly implies an affirmative answer to Problem 8.8 from [Gardner, 1995] (see also [Martini, 1994]), we give an additional corollary in terms of that problem. For doing this, we add two notions: that of the *intersection body* IK (introduced in [Lutwak, 1988], see also [Gardner, 1995], Definition 8.1.1) and that of the *cross–section body* CK (introduced in [Martini, 1992], see also [Gardner, 1995], Definition 8.3.1) of a convex body $K \subset \mathbb{R}^d$.

Let $d \geq 2$. The body IK, for K with the origin as interior point, is the star body with (necessarily continuous) radial function $\omega \mapsto V_{d-1}(K \cap \omega^\perp)$, where ω runs through S^{d-1}. On the other hand, CK is the star body with (necessarily continuous) radial function $\omega \mapsto Q(K, \omega)$, i.e., the inner $(d-1)$–quermass, or HA–measurement of K defines the boundary of CK.

Moreover, Theorem 1 from [Makai–Martini, 1996] implies that if $d \geq 2,\ 0 \in \text{int}K$ and IK and CK are homothets, then they even coincide.

Together with Corollary 3.2 above, this implies

Corollary 3.9. *Let $d \geq 2$ and let $K \subset \mathbb{R}^d$ be a convex body with $0 \in intK$. If IK and CK are homothets, then K is centred.* □

Essentially the same statement has already been given in [Makai–Martini, 1996], Theorem 2; we have included it here since its proof becomes complete (as already mentioned in [Makai–Martini, 1996]) only by our Corollary 3.2.

More generally, we pose

Problem 3.10. Let $d \geq 2$, and let $K \subset \mathbb{R}^d$ be a convex body with $0 \in \text{int}\ K$. Let $\omega \in S^{d-1}$, and let $1 \leq \ell \leq d-2$. Then, if K is centred, the quermassintegrals $W_\ell((K \cap H(\omega, t)) - t\omega)$ (considered in $\omega^\perp$) attain their maximum for $t = 0$, by the Alexandroff–Fenchel inequalities (see [Schneider, 1993], §6.3). (By the same argument, the same holds for the mixed volumes $V((K \cap H(\omega,t)) - t\omega, \ldots, (K \cap H(\omega, t)) - t\omega, M_1, \ldots, M_\ell)$, where $M_i \subset \omega^\perp$ are fixed centred convex bodies.) Our question is: do such properties characterize centredness of K?

Remark. Let $d \geq 2$. Then for any integer $m \geq 1$ one can define analogously an integro–differential transform $R_\psi^{(m)} f$ for those functions $f\colon S^{d-1} \to \mathbb{R}$, whose all partial derivatives of order at most $m-1$ exist and are Lipschitz functions. Namely, we put

$$(R_\psi^{(m)} f)(\xi) = \int\limits_{S^{d-1} \cap \xi^\perp} \frac{\partial^m f}{\partial \psi^m}(\eta, \psi) d\eta \,,$$

provided that the right–hand side exists — where $-\pi/2 \leq \psi \leq \pi/2$, $\xi \in S^{d-1}$, we use polar coordinates with pole ξ, and differentiation is meant as in (4). Then analogues of 3.6 and 3.7 hold, and a certain analogue of 3.8 holds as well. These we published in a separate paper [Makai–Martini–Ódor 2001].

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